Fixed Point Theory, 18(2017), No. 2, 601-614
DOI 10.24193/fpt-ro.2017.2.48
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINTS OF LOCALLY FUZZY CONTRACTIVE SET-VALUED MAPPINGS IN FUZZY METRIC SPACES

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Abstract. In this paper, we first introduce new notions of the locally fuzzy contraction of set-valued mappings and ε-chainable fuzzy metric space. Using these notions, we deal with some issues of fixed point theory involving the generalization of fuzzy contractive mappings introduced by other authors. We enlarge this class and establish fuzzy versions of some known fixed point theorems(such as the Nadler's set-valued contractive and Edelstein's locally contractive fixed point theorems). The results are supported by examples.
Key Words and Phrases: Locally fuzzy contractive mapping, ε-chainable fuzzy metric space, fw-distance, fixed point.
2010 Mathematics Subject Classification: 37C25, 47H10.

1. Introduction

The celebrated Banach-Caccioppoli contraction mapping principle is one of the simplest and most useful methods for construction of solutions of linear and nonlinear equations and fixed points of dynamical systems. This principle has been generalized in different directions in different spaces by mathematicians over the years. For instance, Nadler [4] proposed the contractive set-valued mapping principle; Suzuki [5] presented a new type of generalization of the Banach contraction principle and did characterize the metric completeness by using the concept of w-distance on a metric space introduced by [8]. The corresponding works we still further refer to [6] and references therein. In particular, Edelstein [7] gave a notion of locally contraction to specify the validity in the case of the Banach condition of single-valued mappings only holding for sufficiently close points. Then, Suzuki and Takahashi [9] extended this locally notion into set-valued mappings and proved the existence of fixed points for such mappings via the w-distance on a determinacy metric space.

The fuzzy contraction of mappings in fuzzy metric spaces recently has received extensive attention. Such as Grabiec [10] proved a fuzzy Banach contraction theorem and Vasuki [11] generalized the results of Grabiec for common fixed point theorem for a sequence of mappings in a fuzzy metric space. Then, many authors devoted to investigate the subject in various different directions. Some instances of these works are in [10]-[27]. The main idea consists to use a fuzzy metric instead of the regular metric, as the topology of the sets under consideration. This problem has investigated from different points of view. In particular, George and Veeramani [1, 2] introduced and studied a notion of fuzzy metric M on a set X with the help of continuous t-norms introduced in [3] and from now on, when we talk about fuzzy metrics we refer to this type. Let us mention that Gregori and Sapena [12] have introduced a kind of contractive mappings in fuzzy metric spaces in the sense of George and Veeramani [1] and proved a fuzzy fixed point theorem which extends classical Banach contraction principle by using a strong condition for completeness, now called the completeness in the sense of Grabiec, or G-completeness. As a complete fuzzy metric space in the usual sense, that is an M-complete fuzzy metric space, need not be G-complete (see [13]), an important problem raised by the paper of Gregori and Sapena is to decide whether a fuzzy contractive sequence is M-Cauchy. Mihet [14] defined a new fuzzy contraction called fuzzy ψ-contraction which enlarges the class of fuzzy contractive mappings of Gregori and Sapena and consider these mappings in fuzzy metric spaces in the sense of Kramosil and Michalek. He has shown that every fuzzy contractive sequence in a large class of fuzzy metric spaces is M-Cauchy and proved a fuzzy Banach contraction theorem for M-complete non-Archimedean fuzzy metric spaces. Moreover, the author posed an open question that whether this fixed point theorem holds if the non-Archimedean fuzzy metric space is replaced by a fuzzy metric space. Vetro [15] introduced a notion of weak non-Archimedean fuzzy metric space and proved common fixed point results for a pair of generalized contractive type mappings. Wang [16] gave a positive answer for the open question. Hong et al. [17, 25] further extended and modified the above fuzzy ψ-contraction for set-valued mappings.

The above mentioned fuzzy contraction is a global idea. It is natural to ask whether it could be modified as in the ordinary metric spaces so as to be valid when the fuzzy ψ-condition is assumed to hold only for sufficiently close points in the sense of a fuzzy metric. For this purpose, in this paper we present the new notion of locally fuzzy contraction of set-valued mappings in fuzzy metric spaces. Moreover, it is significative to introduce the ε-chainable fuzzy metric space for the demand of our fuzzy fixed point theory. We enlarge the class of fuzzy ψ-contractive mappings since it is shown that there exist locally fuzzy ψ- contractive mappings which are not globally fuzzy ψ-contractive in the under section 3. To illustrate the applicability of our ideas, another main purpose of this paper is to obtain some fixed point theorems for locally fuzzy contractive set-valued mappings in fuzzy metric spaces by utilizing the fw-distance introduced by Hong [17]. Our results substantially generalize and extend several comparable results as in [14, 16] and are also regarded as the fuzzy versions of corresponding results in [7, 9].

2. Preliminaries

Let us recall [3] that a continuous t-norms is a binary operation $* : [0,1]\times[0,1] \to [0,1]$ such that $([0,1], \leq, *)$ is an ordered Abelian topological monoid with unit 1. In this sequel, we further assume that $*$ satisfies the condition $a*b \geq ab$ for all $a, b \in [0,1]$.

For examples of t-norm satisfying the above conditions we enumerate $a * b = ab$, $a * b = \min\{a, b\}$ and $a * b = ab/\max\{a, b, \lambda\}$ for $0 < \lambda < 1$, respectively.

Definition 2.1. [1] A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(F1) $M(x, y, t) > 0$,
(F2) $M(x, y, t) = 1$ if and only if $x = y$,
(F3) $M(x, y, t) = M(y, x, t)$,
(F4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and
(F5) $M(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is continuous.

In this sense, M is called a fuzzy metric on X.

The simple but useful facts are that M is a continuous function on $X \times X \times (0, \infty)$ and $M(x, y, \cdot)$ is nondecreasing for all $x, y \in X$[28]. In addition, in the definition of Kramosil and Michalek [29], M is a fuzzy set on $X \times X \times [0, \infty)$ that satisfies (F3) and (F4), and (F1), (F2), (F5) are replaced by (K1), (K2), (K5), respectively, as follows:

(K1) $M(x, y, 0) = 0$;
(K2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(K5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

We will refer to these fuzzy metric spaces as KM fuzzy metric spaces.

In the definition of (KM) fuzzy metric spaces, in further, it is satisfied

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t) \quad \text{for all } x, y, z \in X \text{ and all } t > 0.$$

Then $(X, M, *)$ is said to be a non-Archimedean fuzzy metric space [14].

Let $(X, M, *)$ be a fuzzy metric space. For fixed $t > 0$ and $r \in (0, 1]$, the open ball $B(x, t, r)$ with center $x \in X$ is defined by

$$B(x, t, r) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, t, r) \subset A$. Let $\mathcal{T}$ denote the family of all open subsets of X. Then $\mathcal{T}$ is a topology on X induced by the fuzzy metric M. This topology is metrizable (see [2]). Therefore, A closed subset B of X is equivalent that $x \in B$ if and only if there exists a sequence $\{x_n\} \subset B$ such that $\{x_n\}$ topologically converging to x. In fact, the topologically convergence of sequences can be indicated by the fuzzy metric as follows

Definition 2.2. [1] Let $(X, M, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for any $t > 0$.
(ii) A sequence $\{x_n\}$ in X is called Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for any $m, n \geq n_0$.
(iii) A fuzzy metric space $(X, M, *)$ in which every Cauchy sequence is convergent is said to be complete.

There exist two fuzzy versions of Cauchy sequences and completeness, i.e., besides called M-Cauchy sequence and M-completeness in the sense of Definition 2.2,

G-Cauchy sequence defined by $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t, p > 0$ and corresponding G-completeness introduced by [10]. In [13] the authors have pointed out that a G-Cauchy sequence is not an M-Cauchy in general. It is clear that an M-Cauchy sequence is G-Cauchy and hence a fuzzy metric space is M-complete if it is G-complete. From now on, by Cauchy sequence and completeness we mean an M-Cauchy sequence and M-completeness.

The authors in [17] have introduced the following notion of fw-distance for which not either of the implications $\mathcal{P}(x, y, t) = 1 \Leftrightarrow x = y$ (namely (F2)) necessarily holds and $\mathcal{P}$ is nonsymmetric, i.e., in general, $\mathcal{P}$ does not satisfy (F3).

Definition 2.3. Let $(X, M, *)$ be a fuzzy metric space. A fuzzy set $\mathcal{P}$ on $X \times X \times (0, \infty)$ is said to be a fw-distance if the following are satisfied:

(w1) $\mathcal{P}(x, y, t) * \mathcal{P}(y, z, s) \le \mathcal{P}(x, z, t + s)$ for all $x, y, z \in X$ and all $s, t > 0$;

(w2) for any $x \in X, t \in (0, \infty)$, $\mathcal{P}(x, \cdot, t) : X \to [0, 1]$ is upper semicontinuous and $\mathcal{P}(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is left continuous for $x, y \in X$;

(w3) for any $\varepsilon \in (0, 1)$ and $t > 0$, there exists $\delta \in (0, 1)$ such that $\mathcal{P}(z, x, t/2) \ge 1-\delta$ and $\mathcal{P}(z, y, t/2) \ge 1 - \delta$ imply $M(x, y, t) \ge 1 - \varepsilon$.

Clearly, the fuzzy metric M is a fw-distance on X, but the reverse is not true. fw-distance has some useful properties for their proof we refer to[17].

Proposition 2.4. *Let $(X, M, *)$ be a fuzzy metric space and $\mathcal{P}$ be a fw-distance on X. Then for sequences $\{x_n\}$ and $\{y_n\}$ in X, the function sequences $\{a_n(t)\}$ and $\{b_n(t)\}$ with $a_n, b_n : (0, \infty) \to [0, 1)$ converging to 0 for $t > 0$, and for $x, y, z \in X$ we have the following*

(1) *if, for $t > 0$, $\mathcal{P}(x_n, y, t/2) \ge 1 - a_n(t/2)$ and $\mathcal{P}(x_n, z, t/2) \ge 1 - b_n(t/2)$ for any $n \in \mathbb{N}$, then $y = z$; in particular, if $\mathcal{P}(x, y, t) = 1$ and $\mathcal{P}(x, z, t) = 1$, then $y = z$;*

(2) *if, for $t > 0$, $\mathcal{P}(x_n, y_n, t/2) \ge 1 - a_n(t/2)$ and $\mathcal{P}(x_n, z, t/2) \ge 1 - b_n(t/2)$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*

(3) *if, for $t > 0$, $\mathcal{P}(x_n, x_m, t/2) \ge 1 - a_n(t/2)$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*

(4) *if, for $t > 0$, $\mathcal{P}(y, x_n, t/2) \ge 1 - a_n(t/2)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence;*

(5) *if $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ and $\mathcal{P}(x, y_n, t) \ge \omega$ for some $\omega = \omega(x) \in (0, 1)$, then $\mathcal{P}(x, y, t) \ge \omega$.*

By $CB(X)$ we denote the collection consisting of all nonempty bounded closed subsets of X.

The following collection of functions Ψ is given in [14], i.e., $\psi \in \Psi$ means that $\psi : [0, 1] \to [0, 1]$ is continuous, nondecreasing and $\psi(t) > t$ for each $t \in (0, 1)$.

Definition 2.5. [17] Let $\psi \in \Psi$ and $\mathcal{P}$ be a fw-distance. $T : X \to CB(X)$ is called a fuzzy ψ-p-contractive set-valued mapping if the following implication takes place: for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$, there exists $y_2 \in Tx_2$ such that

$$\mathcal{P}(x_1, x_2, t) > 0 \Rightarrow \mathcal{P}(y_1, y_2, t) \ge \psi(\mathcal{P}(x_1, x_2, t)), \quad \forall t > 0.$$

At the end of this section, we list the following lemmas regarding fw-distance which play a key role in the proof of the main result of this paper.

Lemma 2.6. [17] *Let X be a fuzzy metric space with the fuzzy metric M, let $\mathcal{P}$ be a fw-distance on X, and let $\mathcal{Q}$ be a function from $X \times X \times (0, +\infty)$ into $[0, 1]$ satisfying (w1), (w2) in Definition 2.3. Suppose that $\mathcal{Q}(x, y, t) \leq \mathcal{P}(x, y, t)$ for every $x, y \in X, t \in (0, +\infty)$. Then $\mathcal{Q}$ is also a fw-distance on X. In particular, if $\mathcal{Q}$ satisfies (w1), (w2) in Definition 2.3 and $\mathcal{Q}(x, y, t) \leq M(x, y, t)$ for every $x, y \in X, t \in (0, +\infty)$, then $\mathcal{Q}$ is a fw-distance on X.*

A point $x \in X$ is said to be a fixed point of the set-valued mapping T if $x \in Tx$.

Lemma 2.7. [17] *Let $(X, M, *)$ be a complete fuzzy metric space and T be a fuzzy ψ-p-contractive set-valued mapping from X into $CB(X)$. If there exists $x \in X$ such that $\mathcal{P}(x, y, t) > 0$ for some $y \in Tx$ and any $t > 0$, then T has at least a fixed point $x_0 \in X$. Moreover, if $\mathcal{P}(x_0, x_0, t) > 0$ then $\mathcal{P}(x_0, x_0, t) = 1$ for all $t > 0$.*

3. Locally fuzzy contractions

In what follows, we always assume that $(X, M, *)$ is a fuzzy metric space unless otherwise specified. Let $\mathcal{P}$ be a fw-distance on $X \times X \times (0, \infty)$ and $\psi \in \Psi$.

Motivated by [28], we define a function on $CB(X) \times CB(X) \times (0, \infty)$ as follows

$$H_M(A, B, t) = \min\{\delta_M(A, B, t), \delta_M(B, A, t)\}$$

for any $A, B \in CB(X)$ and $t > 0$, where $\delta_M(C, D, t) = \inf_{c \in C} H^M(c, D, t)$ with $H^M(c, D, t) = \sup_{d \in D} M(c, d, t)$ for $C, D \in CB(X)$. On the family of compact subsets of X, in [28] the authors have shown that H_M satisfies the conditions (F1)-(F5) given as in Definition 2.1. Clearly, $H_M(\{x\}, \{y\}, t) = M(x, y, t)$ for all $x, y \in X$ and $t > 0$.

For $x, y \in X$ and $t > 0$, a finite sequence $\{u_0, u_1, \ldots, u_k\} \subset X$ is called a fuzzy ε-chain in X linking x and y if $u_0 = x, u_k = y$ and there exists a positive number $\varepsilon \in (0, 1]$ such that $u_{i+1} \in B(u_i, t, \varepsilon)$ for $i = 0, 1, \ldots, k - 1$.

For the sake of convenience, we introduce the following necessary notations

$U_k = \{u_0, u_1, \ldots, u_k\} \subset X$.

$\mathscr{U}_\varepsilon(x, y, t) = \{U_k : U_k$ is an ε-chain in X linking x and $y\}$ with $t > 0$.

$S(t, k) = \left\{S_k = (s_0, s_1, \ldots, s_{k-1}) : s_i > 0, \sum_{i=0}^{k-1} s_i \leq t\right\}$.

$\mathscr{S}_\varepsilon(x, y, t) = \{(S_k, U_k) : S_k \in S(t, k), u_{i+1} \in B(u_i, s_i, \varepsilon(s_i)), u_0 = x, u_k = y\}$, where $t > 0$ and $\varepsilon : (0, \infty) \to (0, 1]$ is a function.

$\prod_{i=0}^{k-1} * M(u_i, u_{i+1}, s_i) = M(u_0, u_1, s_0) * M(u_1, u_2, s_1) * \cdots * M(u_{k-1}, u_k, s_{k-1})$ with $t > 0$.

Let $\varepsilon : (0, \infty) \to (0, 1]$ be a nondecreasing function. Then, $\mathscr{U}_{\varepsilon(s)}(x, y, s) \subset \mathscr{U}_{\varepsilon(t)}(x, y, t)$ ($\mathscr{S}_\varepsilon(x, y, s) \subset \mathscr{S}_\varepsilon(x, y, t)$) if $0 < s \leq t$, $U_k \in \mathscr{U}_{\varepsilon(t)}(x, y, t)$ if $(S_k, U_k) \in \mathscr{S}_\varepsilon(x, y, t)$ since $B(u_i, s_i, \varepsilon(s_i)) \subset B(u_i, t, \varepsilon(t))$.

Correspond to the chainable property of regular metric space, we introduce the following fuzzy counterpart which will be useful in obtaining our fixed point results.

Definition 3.1. Let $\varepsilon : (0, \infty) \to (0, 1]$ be a nondecreasing function. $(X, M, *)$ is called weakly fuzzy ε-chainable if there exist $x, y \in X$ such that $\mathscr{U}_{\varepsilon(t)}(x, y, t)$ with

$t > 0$ is nonempty. If ε is independent of t, then $(X, M, *)$ is called uniformly fuzzy ε-chainable.

$(X, M, *)$ is evidently weakly fuzzy ε-chainable if it is uniformly fuzzy ε-chainable.

Example 3.2. Let $X = [0, \infty)$ and $a * b = ab$ for $a, b \in [0, 1]$. For any $x, y \in X$, denote $d(x, y) = |x - y|$ and define

$$M_d(x, y, t) =: M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Let $\varepsilon \in (0, 1)$. Then, for any $x, y \in X$ with $x < y$ and $t > 0$, there exists $k \in \mathbb{N}$ such that $U_k = \{u_0, u_1, \ldots, u_k\}$ with $x = u_0 < u_1 < \cdots < u_k = y$ satisfying $u_{i+1} - u_i \le \frac{t\varepsilon}{1-\varepsilon}$. This implies that $u_{i+1} \in B(u_i, t, \varepsilon)$ for $i = 0, 1, \ldots, k-1$, that is, for any $x, y \in X$ there exists an ε-chain in X linking x and y and hence $(X, M_d, *)$ is uniformly fuzzy ε-chainable.

For any given $x, y \in X$ and $t > 0$, let $U_k = \{u_0 = x, u_1, \ldots, u_{k-1}, u_k = y\}$ be a partition of the line segment $|x - y|$ such that every segment $|u_{i+1} - u_i| \le \frac{t\varepsilon}{k(1-\varepsilon)}$. Putting $s_i = \frac{t}{k}$ for $i = 0, 1, \ldots, k - 1$, then we have

$$s_i = \frac{t}{k} \ge \frac{1 - \varepsilon}{\varepsilon}|u_{i+1} - u_i|, \quad i = 0, 1, \ldots, k - 1.$$

This guarantees that $u_{i+1} \in B(u_i, s_i, \varepsilon) \subset B(u_i, t, \varepsilon)$ for $i = 0, 1, \ldots, k - 1$ and $(S_k, U_k) \in \mathscr{S}_\varepsilon(x, y, t)$ with $S_k = (s_0, s_1, \ldots, s_{k-1})$ and $U_k = \{u_0, u_1, \ldots, u_k\}$. To wit, $\mathscr{S}_\varepsilon(x, y, t)$ is nonempty.

Lemma 3.3. *Let $(X, M, *)$ be a fuzzy metric space and $\varepsilon : (0, \infty) \to (0, 1)$ a function. Then the function $\mathcal{F} : X \times X \times (0, +\infty) \to [0, 1]$ defined by*

$$\mathcal{F}(x, y, t) = \begin{cases} \sup\left\{\prod_{i=0}^{k-1} *M(u_i, u_{i+1}, s_i) : (S_k, U_k) \in \mathscr{S}_\varepsilon(x, y, t)\right\}, & \mathscr{S}_\varepsilon(x, y, t) \ne \emptyset, \\ 0, & \mathscr{S}_\varepsilon(x, y, t) = \emptyset \end{cases}$$

for $t > 0$ is a fw-distance on X.

Proof. Note that $\mathcal{F}$ is well-defined. Moreover, from the hypothesis $a * b \ge ab$ it follows that $\mathcal{F}(x, y, t) > 0$ if $\mathscr{S}_\varepsilon(x, y, t) \ne \emptyset$ for $(x, y, t) \in X \times X \times (0, \infty)$. In order to check (w1), we set that $x, y, z \in X$ and $s, t > 0$ are arbitrary. We consider two cases.

Case 1. $\mathcal{F}(x, z, t + s) > 0$. If $\mathcal{F}(x, y, t) > 0$ and $\mathcal{F}(y, z, s) > 0$, then, for $\eta \in (0, \min\{\mathcal{F}(x, y, t),\ \mathcal{F}(y, z, s)\})$ there exist $(S_k, U_k) \in \mathscr{S}_\varepsilon(x, y, t)$ and $(T_l, V_l) \in \mathscr{S}_\varepsilon(y, z, s)$ with $U_k = (u_0, u_1, \ldots, u_k)$, $V_l = (v_0, v_1, \ldots, v_l)$, $S_k = (s_0, s_1, \ldots, s_{k-1}) \in S(t, k)$ and $T_l = (\tau_0, \tau_1, \ldots, \tau_{l-1}) \in S(s, l)$ such that

$$\prod_{i=0}^{k-1} *M(u_i, u_{i+1}, s_i) \ge \mathcal{F}(x, y, t) - \eta, \quad \prod_{j=0}^{l-1} *M(v_j, v_{j+1}, \tau_j) \ge \mathcal{F}(y, z, s) - \eta.$$

Let

$$S'_{k+l} = (s_0, s_1, \ldots, s_{k-1}, \tau_0, \tau_1, \ldots, \tau_{l-1}), \quad U'_{k+l} = \{u_0, u_1, \ldots, u_k, v_1, v_2, \ldots, v_l\}.$$

It is easy to see that $S'_{k+l} \in S(t+s, k+l)$ and $(S_{k+l}, U'_{k+l}) \in \mathscr{S}_\varepsilon(x, z, t+s)$. Hence

$$\begin{aligned}\mathcal{F}(x,z,t+s) &\geq \prod_{i=0}^{k-1} *M(u_i, u_{i+1}, s_i) * \prod_{j=0}^{l-1} *M(v_j, v_{j+1}, \tau_j)\\ &\geq (\mathcal{F}(x,y,t) - \eta) * (\mathcal{F}(y,z,s) - \eta).\end{aligned}$$

Since $\eta > 0$ is arbitrary, we have $\mathcal{F}(x,z,t+s) \geq \mathcal{F}(x,y,t) * \mathcal{F}(y,z,s)$. Hence, in this case, (w1) is satisfied. In addition, this implies that $\mathcal{F}(x,z,s+t) > 0$ whenever $\mathcal{F}(x,y,t)$ and $\mathcal{F}(y,z,s)$ both are positive.

Otherwise, at least one of $\mathcal{F}(x,y,t)$ and $\mathcal{F}(y,z,s)$ is zero, say, $\mathcal{F}(x,y,t) = 0$. Then the conclusion obviously holds.

Case 2. $\mathcal{F}(x,z,t+s) = 0$. In this case, there is at least one of $\mathcal{P}(x,y,t)$ and $\mathcal{F}(y,z,s)$ is zero. In a similar way, we get $\mathcal{F}(x,z,t+s) \geq \mathcal{F}(x,y,t) * \mathcal{F}(y,z,s)$. Consequently, (w1) holds.

It is sufficient for checking (w2) to prove that $\mathcal{F}(x,\cdot,t)$ is upper semicontinuous for given $x \in X$ and $t > 0$ since $\mathcal{F}(x,y,\cdot)$ is obviously continuous. To this end, we take $y \in X$ and assume that $\{y_n\}$ is a sequence in X with $y_n \to y$. Thus, for any $0 < \epsilon \leq \varepsilon(t)$ and $t' > t$, there exists $n_0 \in \mathbb{N}$ such that $M(y, y_n, t'-t) \geq 1-\epsilon$ and $M(y, y_n, t) \geq 1-\epsilon$ for every $n \geq n_0$. Suppose that there exist infinite many $n \in \mathbb{N}$ such that $\mathcal{F}(x, y_n, t) > 0$. For any fixed $n \geq n_0$ and $\mu \in (0, \mathcal{F}(x, y_n, t))$, there exist $S_k \in S(t,k)$ and $(S_k, U_k) \in \mathscr{S}_\varepsilon(x, y_n, t)$ such that

$$\prod_{i=0}^{k-1} *M(u_i, u_{i+1}, s_i) \geq \mathcal{F}(x, y_n, t) - \mu.$$

Let $\hat{S}_{k+1} = (s_0, s_1, \ldots, s_{k-1}, t'-t) = (S_k, t'-t), \hat{U}_{k+1} = \{u_0, u_1, \ldots, u_k, y\} = (U_k, y)$. Then $\hat{S}_{k+1} \in S(t', k+1)$, $(\hat{S}_{k+1}\hat{U}_{k+1}) \in \mathscr{S}_\varepsilon(x, y, t')$ and hance

$$\mathcal{F}(x,y,t') \geq \prod_{i=0}^{k-1} *M(u_i, u_{i+1}, s_i) * M(y_n, y, t'-t) \geq (\mathcal{F}(x, y_n, t) - \mu) * M(y_n, y, t'-t).$$

Now, by the arbitrariness of μ, we obtain

$$\mathcal{F}(x,y,t') \geq (\mathcal{F}(x,y_n,t)) * M(y_n, y, t'-t) \geq (\mathcal{F}(x,y_n,t)) * (1-\epsilon).$$

In virtue of the arbitrariness of ϵ, we have $\mathcal{F}(x,y,t') \geq \limsup_{n\to\infty} \mathcal{F}(x,y_n,t)$. In view of the continuity of $\mathcal{F}(x,y,\cdot)$, we obtain

$$\mathcal{F}(x,y,t) \geq \limsup_{n\to\infty} \mathcal{F}(x,y_n,t).$$

If $\mathcal{F}(x,y_n,t) = 0$ except finite many $n \in \mathbb{N}$, the above inequality is obviously valid. Hence, we deduce that $\mathcal{F}(x,\cdot,t)$ is upper semicontinuous as desired.

Finally, for any $x, y \in X$ and $t > 0$, from Definition 2.1-(F4), it follows

$$\prod_{i=0}^{k-1} *M(u_i, u_{i+1}, s_i) \leq M(x,y,t)$$

for every $S_k = (s_0, s_1, \ldots, s_{k-1}) \in S(t,k)$ and $(S_k, U_k) \in \mathscr{S}_\varepsilon(x,y,t)$ with $U_k = \{u_0, u_1, \ldots, u_k\}$. This implies $\mathcal{F}(x,y,t) \le M(x,y,t)$ for all $(x,y,t) \in X \times X \times (0,\infty)$. By Lemma 2.6, $\mathcal{F}$ is a fw-distance. □

Following the line in arguments of Lemma 3.3, we can check the following

Lemma 3.4. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space. If there exists a nondecreasing function $\varepsilon : (0,\infty) \to (0,1]$ such that X is weakly fuzzy ε-chainable, then the function $\mathcal{F} : X \times X \times (0,+\infty) \to [0,1]$ defined by*

$$\mathcal{F}(x,y,t) = \begin{cases} \sup\left\{\prod_{i=0}^{k-1} *M(u_i, u_{i+1}, t) : U_k \in \mathscr{U}_{\varepsilon(t)}(x,y,t)\right\}, & \mathscr{U}_{\varepsilon(t)}(x,y,t) \neq \emptyset, \\ 0, & \mathscr{U}_{\varepsilon(t)}(x,y,t) = \emptyset \end{cases}$$

for $t > 0$ is a fw-distance on X.

Definition 3.5. T is called a locally fuzzy (ψ,ε)-contractive set-valued mapping if $\psi \in \Psi$, $\varepsilon : (0,\infty) \to [0,1)$ is a function and, for any $x \in X$ and $t > 0$, the following inequality holds:

$$H_M(Tx, Ty, t) \ge \psi(M(x,y,t))$$

whenever $y \in B(x,t,\varepsilon(t))$. If ε is independent of t, then T is called locally fuzzy (ψ,ε)-uniformly contractive. In particular, T is said to be (globally) fuzzy ψ-contractive when $\varepsilon \equiv 1$.

T is called locally fuzzy (ψ,ε)-strongly contractive if T is locally fuzzy (ψ,ε)-contractive and there exist $x \in X, y \in Tx$ such that $\mathscr{S}_\varepsilon(x,y,t) \neq \emptyset$ for any $t > 0$.

T is called locally fuzzy (ψ,ε)-SU contractive if T is locally fuzzy (ψ,ε)-strongly contractive and locally fuzzy (ψ,ε)-uniformly contractive.

Remark 3.6. Definition 3.5 is also valid when $(X, M, *)$ is a KM fuzzy metric space since $M(x,y,t) > 0$ whenever $y \in B(x,t,\varepsilon)$.

Let X be a complex plane and the analytic function $f : X \to X$ map every connected and compact subset C of X into itself. If $|f'(x)| < 1$ for every $x \in C$, note that f' is continuous on C, we can find $\lambda > 0$ such that $|f'(x)| \le \lambda < 1$ for every $x \in C$. For fixed $x \in C$, there exists a neighborhood $U(x,r) \subset X$ with $0 < r < 1$ such that $|f'(y)| \le \lambda$ for all $y \in U(x,r)$. Since $\bigcup_{x\in C} U(x,r)$ is an open covering of C, from the compactness of C there exist finite many neighborhoods, say, $\{U(x_i,r_i)\}_{i=1}^n$ such that $C \subset \bigcup_{i=1}^n U(x_i,r_i)$. Let $\varepsilon = \min_{1\le i\le n}\{r_i\}$. Then $0 < \varepsilon < 1$ and for any $x \in C, y \in X$ with $|x-y| < \varepsilon$, there exists $1 \le i_0 \le n$ such that $x, y \in U(x_{i_0}, r_{i_0})$ and hence

$$|f(x) - f(y)| \le \int_x^y |f'(z)|dz \le \lambda|x-y|.$$

This implies that f is locally Banach contractive on C. In fact, it is quite easy to exhibit that the mapping T which admits a locally Banach contraction on a regular metric space X is equivalent to locally fuzzy contraction on X endowed a suitable fuzzy metric.

Example 3.7. Let $a * b = \min\{a,b\}$ for $a, b \in [0,1]$ and X be the complex plane endowed with the fuzzy metric M_d. If the analytic function $f : X \to X$ which maps every connected and compact subset C of X into itself satisfies $|f'(x)| < 1$ for every

$x \in C$, then there exists $\psi \in \Psi$ such that f is locally fuzzy (ψ, η)-strongly contractive on C, where η is a function from $(0, \infty)$ into $(0, 1]$ with $\eta(t) = \varepsilon/(t+\varepsilon)$. In particular, there exists $\varepsilon \in (0, 1)$ such that f is locally fuzzy (ψ, ε)-SU contractive on C under the fuzzy metric $\widetilde{M}(x, y, t) = 1/(1 + |x - y|)$.

Proof. It is easy to see that $(X, M_d, *)$ is a KM fuzzy metric space. In addition, for any $x \in C$ and $t > 0$, $y \in B(x, t, \eta(t))$ if and only if $|x - y| \leq \varepsilon$ and this yields $|f(x) - f(y)| \leq \lambda|x - y|$. Moreover, it is equivalent to

$$M_d(f(x), f(y), t) \geq \frac{M_d(x, y, t)}{M_d(x, y, t) + \lambda(1 - M_d(x, y, t))}.$$

We now set $\psi(t) = \frac{t}{t+\lambda(1-t)}$. Then, $\psi \in \Psi$ and $M_d(f(x), f(y), t) \geq \psi(M_d(x, y, t))$ for all $x \in C$ and $y \in B(x, t, \eta(t))$. In particular, f is locally fuzzy (ψ, η)-uniformly contractive on C with $\eta = \varepsilon/(1 + \varepsilon)$ if we consider the fuzzy metric $\widetilde{M}$.

In addition, following the line in arguments of Example 3.2, for $\varepsilon > 0$ we see $\mathscr{S}_\varepsilon(x, f(x), t) \neq \emptyset$ for all $x \in C$ and $t > 0$, that is, f is (ψ, ε)-strongly contractive on C. Consequently, f is (ψ, η)-SU contractive on C under the fuzzy metric $\widetilde{M}$. □

The author in [7] exhibited spaces which admit (uniformly) locally Banach contractive mappings are not globally Banach contractive. Moreover, we see that the locally fuzzy (ψ, ε)-(uniformly) contraction of mappings is equivalent to the Banach contraction under consideration of Example 3.7 involving $\psi(t) = \frac{t}{t+\lambda(1-t)}$ and $\lambda \in (0, 1)$. Therefore, there exist mappings such that they are even locally fuzzy (ψ, ε)-uniformly contractive single-valued mappings but not globally fuzzy ψ-contractive.

4. Fixed point theorems

As an application of the locally fuzzy contraction, in this section we approach to state and prove the fuzzy fixed point theorems which are one of our main results.

Theorem 4.1. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. If there exists a nondecreasing function $\varepsilon : (0, \infty) \to (0, 1]$ such that X is weakly fuzzy ε-chainable, the set-valued mapping T from X into $CB(X)$ is locally fuzzy (ψ, ε)-contractive with $\psi \in \Psi$ and the condition*

(i) $\Psi^*_\psi = \{\phi \in \Psi : \phi(a) * \phi(b) \geq \phi(a * b)$ *for any* $a, b \in [0, 1]$ *and* $\phi(a) < \psi(a)$ *for* $a \in [0, 1)\} \neq \emptyset$

holds, then T has at least a fixed point $x_0 \in X$. Moreover, there exists a fw-distance $\mathcal{P}$ such that $\mathcal{P}(x_0, x_0, t) > 0$ implies $\mathcal{P}(x_0, x_0, t) = 1$ for $t > 0$.

Proof. Define a fuzzy function as follows

$$\mathcal{P}(x, y, t) = \begin{cases} \sup\left\{\prod_{i=0}^{k-1} *M(u_i, u_{i+1}, t) : U_k \in \mathscr{U}_{\varepsilon(t)}(x, y, t)\right\}, & \mathscr{U}_{\varepsilon(t)}(x, y, t) \neq \emptyset, \\ 0, & \mathscr{U}_{\varepsilon(t)}(x, y, t) = \emptyset. \end{cases}$$

$\mathcal{P}$ is well-defined since X is weakly fuzzy ε-chainable. Moreover, $\mathcal{P}(x, y, t) \leq M(x, y, t)$ for all $x, y \in X$ and $t > 0$ on the non-Archimedean fuzzy metric space. Lemma 3.4 guarantees that $\mathcal{P}$ is a fw-distance. Moreover, there exist $x, y \in X$ such that $\mathcal{P}(x, y, t) > 0$ for any $t \in (0, \infty)$. We next verify that T is fuzzy ϕ-p-contractive for

$\phi \in \Psi_\psi^*$. Let $x_1, x_2 \in X, y_1 \in Tx_1$. If $\mathcal{P}(x_1, x_2, t) > 0$ for arbitrarily given $t > 0$ then, for $\eta \in (0, \mathcal{P}(x_1, x_2, t))$, there exists $U_k \in \mathscr{U}_{\varepsilon(t)}(x, y, t)$ with $U_k = \{u_0, u_1, \ldots, u_k\}$ such that

$$\prod_{i=0}^{k-1} *M(u_i, u_{i+1}, t) \geq \mathcal{P}(x_1, x_2, t) - \eta.$$

Take advantage of $x_1 = u_0$ and $u_1 \in B(u_0, t, \varepsilon(t))$, together with the hypothesis of the locally contraction of T, we have

$$H_M(Tu_0, Tu_1, t) \geq \psi(M(u_0, u_1, t)) > \phi(M(u_0, u_1, t)).$$

Put $v_0 = y_1 \in Tx_1 = Tu_0$, by the definition of H_M it is easy to see that there exists $v_1 \in Tu_1$ such that

$$M(v_0, v_1, t) \geq \phi(M(u_0, u_1, t)) > \phi(1 - \varepsilon(t)) > 1 - \varepsilon(t).$$

Analogously, by means of $u_2 \in B(u_1, t, \varepsilon(t))$ and the locally fuzzy (ψ, ε)-contraction of T, for $v_1 \in Tu_1$, combining the definition of H_M, we can find $v_2 \in Tu_2$ such that

$$M(v_1, v_2, t) \geq \phi(M(u_1, u_2, t)) > \phi(1 - \varepsilon(t)) > 1 - \varepsilon(t).$$

Repeating the process, we can obtain a point sequence $\{v_0, v_1, \ldots, v_k\} \in \mathscr{U}_{\varepsilon(t)}(y_1, v_k, t)$ such that $v_i \in Tu_i$ for $i = 0, 1, \ldots, k$ and

$$M(v_i, v_{i+1}, t) \geq \phi(M(u_i, u_{i+1}, t)) > \phi(1 - \varepsilon(t)) > 1 - \varepsilon(t)$$

for each $i = 0, 1, \ldots, k-1$. Putting $y_2 = v_k$, since $y_2 \in Tu_k = Tx_2$, we have

$$\begin{aligned} \mathcal{P}(y_1, y_2, t) &\geq \prod_{i=0}^{k-1} *M(v_i, v_{i+1}, t) \geq \prod_{i=0}^{k-1} *\phi(M(u_i, u_{i+1}, t)) \\ &\geq \phi\left(\prod_{i=0}^{k-1} *M(u_i, u_{i+1}, t)\right) \geq \phi(\mathcal{P}(x_1, x_2, t) - \eta). \end{aligned}$$

In virtue of the continuity of ϕ and the arbitrariness of η, we have $\mathcal{P}(y_1, y_2, t) \geq \phi(\mathcal{P}(x_1, x_2, t))$, which implies that T is a fuzzy ϕ-p-contractive set-valued mapping. In virtue of Lemma 2.7, we arrive at the desired results. This Proof is complete. □

In what follows, we appropriately relax the restriction of fuzzy spaces, that is, the non-Archimedean fuzzy metric space is replaced by a fuzzy metric space. In this case, we present the following

Theorem 4.2. *Let $\varepsilon : (0, \infty) \to (0, 1]$ be a nondecreasing function and $(X, M, *)$ be complete. Suppose that (i) and the following condition hold:*

(ii) *The set-valued mapping T from X into $CB(X)$ is locally fuzzy (ψ, ε)-strongly contractive.*

Then T has at least a fixed point $x_0 \in X$. Moreover, there exists a fw-distance $\mathcal{P}$ such that $\mathcal{P}(x_0, x_0, t) > 0$ implies $\mathcal{P}(x_0, x_0, t) = 1$ for $t > 0$.

Proof. Define a fw-distance $\mathcal{P} = \mathcal{F}$ on $X \times X \times (0, \infty)$ with $\mathcal{F}$ given as in Lemma 3.3. We first observe that from the hypothesis (ii) there exist $x \in X$ and some $y \in Tx$ such that $\mathcal{P}(x, y, t) > 0$ for any $t > 0$.

Choosing a function $\phi \in \Psi_\psi^*$ such that $\phi(a) < \psi(a)$ for each $a \in [0,1)$, we next prove that T is fuzzy ϕ-p-contractive. Let $x_1, x_2 \in X, y_1 \in Tx_1$. Suppose $\mathcal{P}(x_1,x_2,t) > 0$. Then, for arbitrarily given $t > 0$ and $\eta \in (0, \mathcal{P}(x_1,x_2,t))$, there exists $(S_k,U_k) \in \mathscr{S}_\varepsilon(x_1,x_2,t)$ with $S_k = (s_0,s_1,\dots,s_{k-1}), U_k = \{u_0,u_1,\dots,u_k\}$ such that

$$\prod_{i=0}^{k-1} {*M(u_i,u_{i+1},s_i)} \geq \mathcal{P}(x_1,x_2,t) - \eta.$$

Take advantage of $x_1 = u_0$ and $u_1 \in B(u_0,s_0,\varepsilon(s_0))$, together with the hypothesis (ii), we have

$$H_M(Tu_0,Tu_1,s_0) \geq \psi(M(u_0,u_1,s_0)) > \phi(M(u_0,u_1,s_0)).$$

Put $v_0 = y_1 \in Tx_1 = Tu_0$, by the definition of H_M it is easy to see that there exists $v_1 \in Tu_1$ such that

$$M(v_0,v_1,t) \geq M(v_0,v_1,s_0) \geq \phi(M(u_0,u_1,s_0)) > \phi(1-\varepsilon(s_0)) > 1-\varepsilon(s_0).$$

Analogously, by means of $u_2 \in B(u_1,s_1,\varepsilon(s_1))$ and the locally fuzzy (ψ,ε)-contraction of T, for $v_1 \in Tu_1$, combining the definition of H_M, we can find $v_2 \in Tu_2$ such that

$$M(v_1,v_2,t) \geq M(v_1,v_2,s_1) \geq \phi(M(u_1,u_2,s_1)) > \phi(1-\varepsilon(s_1)) > 1-\varepsilon(s_1).$$

Repeating the process, we can obtain an element $(S_k,V_k) \in \mathscr{S}_\varepsilon(y_1,v_k,t)$ with $V_k = \{v_0,v_1,\dots,v_k\}$ such that $v_i \in Tu_i$ with $u_i \in B(u_{i-1},s_{i-1},\varepsilon(s_{i-1}))$ for $i = 1,2,\dots,k$ and

$$M(v_i,v_{i+1},t) \geq M(v_i,v_{i+1},s_i) \geq \phi(M(u_i,u_{i+1},s_i)) > \phi(1-\varepsilon(s_i)) > 1-\varepsilon(s_i)$$

for each $i = 0,1,\dots,k-1$. Putting $y_2 = v_k$, since $y_2 \in Tu_k = Tx_2$, we have

$$\begin{aligned} \mathcal{P}(y_1,y_2,t) &\geq \prod_{i=0}^{k-1} {*M(v_i,v_{i+1},s_i)} \geq \prod_{i=0}^{k-1} {*\phi(M(u_i,u_{i+1},s_i))} \\ &\geq \phi\left(\prod_{i=0}^{k-1} {*M(u_i,u_{i+1},s_i)}\right) \geq \phi(\mathcal{P}(x_1,x_2,t)-\eta). \end{aligned}$$

In virtue of the continuity of ϕ and the arbitrariness of η, we have $\mathcal{P}(y_1,y_2,t) \geq \phi(\mathcal{P}(x_1,x_2,t))$, which implies that T is a fuzzy ϕ-p-contractive set-valued mapping. Lemma 2.7 now guarantees the existence of fixed point as desired and this proof is complete. □

Example 4.3. Let $X = (0,1]$, $a*b = ab$ for any $a,b \in [0,1]$ and $M(x,y,t) = \frac{\min\{x,y\}}{\max\{x,y\}}$. Define the set-valued mapping $T : X \to 2^X$ as follows

$$Tx = \begin{cases} \{\sqrt{x}\}, & x \in \left(0, \frac{1}{2}\right], \\ \left[\frac{1}{\sqrt{2}}, \sqrt{x}\right], & x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Conclusion. T is locally fuzzy $(\psi, 1/2)$-uniformly contractive and hence has a fixed point in X.

Proof. Using similar arguments to ones in [[30],Theorem 16], one can show that $(X, M, *)$ is complete. Obviously, X is a fuzzy (1/2)-chainable and non-Archimedean fuzzy metric space. Let $\psi(t) = \sqrt{t}$ and $\phi(t) = \sqrt[4]{t^3}$ for $t \in [0, 1]$. Then $\psi \in \Psi$ and $\phi \in \Psi^*_\psi$, i.e., $\Psi^*_\psi \neq \emptyset$. For any $x \in X$ and $t > 0$, we distinguish two case:

Case 1. $x \in \left(0, \frac{1}{2}\right]$ and $y \in B\left(x, t, \frac{1}{2}\right)$. If $y \in \left(0, \frac{1}{2}\right]$ and $x \leq y$, then

$$H_M(Tx, Ty, t) = M(Tx, Ty, t) = \frac{\sqrt{x}}{\sqrt{y}} = \psi(M(x, y, t)).$$

The proof is similar for $x > y$. If $y \in \left(\frac{1}{2}, 1\right]$, then $x < y$ and

$$H_M(Tx, Ty, t) = H_M\left(\sqrt{x}, \left[\frac{1}{\sqrt{2}}, \sqrt{y}\right), t\right) \geq \frac{\sqrt{x}}{\sqrt{y}} = \psi(M(x, y, t)).$$

Therefore, the locally fuzzy $(\psi, 1/2)$-uniformly contractive condition is satisfied.

Case 2. $x \in \left(\frac{1}{2}, 1\right]$ and $y \in B\left(x, t, \frac{1}{2}\right)$. If $y \in \left(0, \frac{1}{2}\right]$, the proof is the same as in Case 1. If $y \in \left(\frac{1}{2}, 1\right]$ and $x \leq y$, then

$$H_M(Tx, Ty, t) = H_M\left(\left[\frac{1}{\sqrt{2}}, \sqrt{x}\right), \left[\frac{1}{\sqrt{2}}, \sqrt{y}\right), t\right) = \frac{\sqrt{x}}{\sqrt{y}} = \psi(M(x, y, t)).$$

The proof is similar for $y < x$. Therefore, the locally fuzzy $(\psi, 1/2)$-uniformly contractive condition is satisfied.

We now get that T is the locally fuzzy $(\psi, 1/2)$-uniformly contractive. Consequently, Theorem 4.1 guarantees that T has at least a fixed point in X. □

As direct consequences of Theorem 4.1 or Theorem 4.2, we obtain the following corollaries which are the fuzzy versions of Nadler's fixed point theorem [4] for set-valued mappings and Edelstein's fixed point theorem [7] on an ε-chainable regular metric space, respectively.

Corollary 4.4. *Let $(X, M, *)$ be complete and $T : X \to CB(X)$ be a fuzzy ψ-contractive set-valued mapping with $\psi \in \Psi$. Remain valid for (i) in Theorem 4.1. Then T has a fixed point in X.*

Proof. Theorem 4.1 immediately derive Corollary 4.4. Thereinafter, we will utilize Theorem 4.2 to prove the desired result. We observe that X is 1-strongly chainable and T is locally fuzzy $(\psi, 1)$-uniformly contractive and hence it is locally fuzzy $(\psi, 1)$-SU contractive since $\mathscr{S}_1(x, y, t) \neq \emptyset$ for every $x, y \in X$ with $x \neq y$ and $t > 0$, i.e., (ii) of Theorem 4.2 holds. Using Theorem 4.2, we obtain the desired result. □

Corollary 4.5. *Let $\varepsilon : (0, \infty) \to (0, 1]$ be a nondecreasing function, $(X, M, *)$ be complete and the single-valued mapping T from X into itself be locally fuzzy (ψ, ε)-contractive. If one of the following conditions is satisfied, then T has an unique fixed point.*

(i) *T is locally fuzzy (ψ, ε)-strongly contractive. Moreover, for any two fixed points x, y of T and $t > 0$ there exists a fw-distance $\mathcal{P}$ such that $\mathcal{P}(x, y, t) > 0$.*

(ii) *$(X, M, *)$ is a weakly fuzzy ε-chainable and non-Archimedean fuzzy metric space.*

Proof. By Theorems 4.1 or 4.2, there exists $x_0 \in X$ with $Tx_0 = x_0$ and there exists a fw-distance $\mathcal{P}$ such that $\mathcal{P}(x_0, x_0, t) = 1$ for all $t > 0$.

(i) Let $y_0 = Ty_0$ and $\mathcal{P} = \mathcal{F}$ with $\mathcal{F}$ as in Lemma 3.3. Then $\mathcal{P}(x_0, y_0, t) > 0$ for all $t > 0$. By a similar way to the proof of Theorem 4.2, we obtain that $\mathcal{P}(x_0, y_0, t) = \mathcal{P}(Tx_0, Ty_0, t) \geq \psi(\mathcal{P}(x_0, y_0, t))$. If $\mathcal{P}(x_0, y_0, t) < 1$ then $\mathcal{P}(Tx_0, Ty_0, t) > \mathcal{P}(x_0, y_0, t) > 0$. This contradiction implies $\mathcal{P}(x_0, y_0, t) = 1$. Now $\mathcal{P}(x_0, x_0, t) = 1$ and Proposition 2.4-(1) infer $x_0 = y_0$.

(ii) Let the fw-distance $\mathcal{P}$ be given as in the proof of Theorem 4.1. Then $\mathcal{P}(x, y, t) > 0$ for all $(x, y, t) \in X \times X \times (0, \infty)$. The rest of this proof is analogous to (i). □

5. Conclusion

In this work we have dealt with a more general class of fuzzy contractive set-valued mappings, i.e. the locally fuzzy contractive class, the idea of which roots in literatures [7, 9]. We have considered such mapping in a fuzzy metric space and proved that every set-valued mapping with some locally fuzzy contraction has a fixed point (the fixed point uniquely exists in the sense of single-valued mappings).

Note that a globally fuzzy ψ-contractive mapping can be regarded as a locally fuzzy $(\psi, 1)$-SU contractive since, in this sense, we have $\mathscr{S}_1(x, y, t) \neq \emptyset$ for every $x, y \in X$ with $x \neq y$ and $t > 0$. Hence, Corollary 4.5(i) is an essential extension and improvement of several comparable results in [14, 16].

In addition, without considering the fuzzy or crisp metric issue, differ from the cited literatures, in Theorem 4.2 we assume that the set-valued mapping T is locally fuzzy (ψ, ε)-strongly contractive without the assumption that X is fuzzy ε-chainable. Fortunately, from Example 3.2 it is not hard to see the existence of the locally fuzzy (ψ, ε)-strongly contractive mapping. However, under the hypothesis of the fuzzy ε-chainability, we obtain no result if "non-Archimedean fuzzy metric space" is replaced by "fuzzy metric space" in Theorem 4.1.

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Received: December 17, 2014; Accepted: May 15, 2015.