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BOUNDARY VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

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Abstract. In this paper, we investigate the existence of solution for boundary value problem with fractional differential equations with nonlocal conditions involving the standard Riemann-Liouville fractional derivative of order $r \in (1, 2]$.

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1. Introduction

This paper deals with the existence of solutions for boundary value problems (BVP for short), of a class of fractional order differential equation. In Section 3 we consider the boundary value problem with nonlocal conditions

$$D^r y(t) = f(t, y), \text{ for a.e. } t \in J = [0, T], \quad 1 < r \leq 2, \tag{1.1}$$

$$y(0) = 0, \ y(T) = g(y), \tag{1.2}$$

where D^r is the Riemann-Liouville fractional derivative, $(E, |\cdot|)$ denotes a Banach space $f : J \times E \to E$ is a continuous function and $g : E \to E$ is a continuous function.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, and so on. There has been a significant development in fractional differential equations in recent years; see the monographs of Hilfer [28], Kilbas *et al.* [30], Delbosco*et al.* [16], Miller*et al.* [34], Heymans *et al.* [27], Podlubny [39, 40], Kaufman *et al.* [29], Momani and Hadid [37], and the papers by Agarwal *et al.* [1], Bai *et al.* [5, 6], Benchohra *et al.* [9, 10, 11]. In this paper, we present existence results for the problem (1.1)-(1.2), when we apply the method associated with the technique of measure of noncompactness and the fixed point theorem of Mönch type. This technique was mainly initiated in the monograph of Banas and Goebel [7] and subsequently developed and used in many papers; see, for example, Banas ans Sadarangani [8], Guo *et al* [25], Lakshimikanthan and Leela [32],Mönch [35], and Szufla [41].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $C(J,E)$ be the Banach space of all continuous functions from J into E with the norm

$$\|y\| = \sup\{|y(t)| : 0 \le t \le T\},$$

and we let $L^1(J,E)$ denote the Banach space of functions $y : J \longrightarrow E$ which are Bochner integrable with norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

Let $L^\infty(J,E)$ be the Banach space of functions $y : J \to E$ which are bounded equipped with the norm

$$\|y\|_{L^\infty} = \inf\{c > 0 \,:\, \|y(t)\| \le c : \text{a.e}\, t \in J\}.$$

Let $AC^1(J,E)$ is the space of functions $y : J \to E$, which are absolutely continuous whose first derivative, y', is absolutely continuous.

$$V(t) = \{\vartheta(t) \,:\vartheta \in V\},\, t \in J,$$
$$V(J) = \{\vartheta(t) \,:\vartheta \in V\},\, t \in J.$$

Definition 2.1. ([31, 39]). The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^r h(t) = \int_a^t \frac{(t-s)^{r-1}}{\Gamma(r)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^r h(t) = h(t) * \varphi_r(t)$, where $\varphi_r(t) = \dfrac{t^{r-1}}{\Gamma(r)}$ for $t > 0$, and $\varphi_r(t) = 0$ for $t \le 0$, and $\varphi_r \to \delta(t)$ as $r \to 0$, where δ is the delta function.

Definition 2.2. ([31, 39]). For a function h given on the interval $[a,b]$, the r Riemann-Liouville fractional-order derivative of h, is defined by

$$(D_{a+}^r h)(t) \quad = \quad \frac{1}{\Gamma(n-r)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-r-1} h(s)ds.$$

Here $n = [r] + 1$ and $[r]$ denotes the integer part of r.

For convenience,we first recall the definition of the Kuratowski measure of noncompactness, and summarize the main properties of this measure.

Definition 2.3. ([4, 7]) Let E be a Banach space and let Ω_E be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0,\infty)$ defined by

$$\alpha(B) = \inf\{\epsilon > 0, : B \subset \bigcup_{j=1}^{m} \text{and } diam(B_j) \leq \epsilon\}\,; \text{here}\, B \in \Omega_E.$$

Properties.

(1) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (M is relatively compact).
(2) $\alpha(B) = \alpha(\overline{B})$
(3) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$
(4) $\alpha(A+B) \leq \alpha(A) + \alpha(B)$.
(5) $\alpha(cB) = c\alpha(B)\,; c \in \mathbb{R}$.
(6) $\alpha(conB) = \alpha(B)$.

Here $\overline{B}$ and $conB$ denote the closure and the convex hull of the bounded set B, respectively.

The details of α and its properties can be found in [4, 7].

Definition 2.4. A multivalued map $F : J \times E \to E$ is said to be Carathéodory if

(1) $t \to F(t,u)$ is measurable for each $u \in E$.
(2) $u \to F(t,u)$ is upper semicontinuous for almost all $t \in J$.

Let us now recall the Mönch's fixed point theorem and the important lemma.

Theorem 2.5. ([35], [3]) *Let D be a bounded, closed and convex subset of a Banach space E such that $0 \in D$, and let N be a continuous mapping of D into itself, if the implication*

$$V = \overline{co}N(V) \; orV = N(V) \cup \{0\} \Longrightarrow \alpha(V) = 0. \tag{2.1}$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.6. ([41]) *Let D be a bounded, closed and convex subset of a Banach space $C(J,E)$, G a continuous function on $J \times J$, and a function $f : J \times E \to E$ satisfies the Carathéodory conditions, and there exists $p \in L^1(J,\mathbb{R}_+)$ such that for each $t \in J$ and each bounded set $B \subset E$ one has*

$$\lim_{k\to 0^+} \alpha(f(J_{t,k} \times B)) \leq p(t)\alpha(B);\; where\; J_{t,k} = [t-k,t] \cap J. \tag{2.2}$$

If V is an equicontinuous subset of D, then

$$\alpha(\{\int_J G(s,t)f(s,y(s))ds \,: y \in V\}) \leq \int_J \|G(t,s)\|p(s)\alpha(V(s))ds. \tag{2.3}$$

3. Main Results

Let us start by defining what we mean by a solution of the problem (1.1)–(1.2).

Definition 3.1. A function $y \in C([0,T],E)$ is said to be a solution of (1.1)–(1.2) if y satisfies the equation $D^r y(t) = f(t, y(t))$ on J, and the condition $y(0) = 0,\ y(T) = g(y)$.

For the existence of solutions for the problem (1.1)–(1.2), we need the following auxiliary lemma.

Lemma 3.2. [6] *Let $r > 0$, and $h \in C(0,T) \cap L(0,T)$ then*

$$I^\alpha D^\alpha h(t) = h(t) + c_1 t^{r-1} + c_2 t^{r-2} + \ldots + c_n t^{r-n}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, where n is the smallest integer greater than or equal to r.

Lemma 3.3. *Let $1 < r < 2$ and let $h : [0,T] \to \mathbb{R}$ be continuous. A function $y \in C([0,T],E)$ is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds \\ &+ \frac{t^{r-1}}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} h(s) ds - \frac{t^{r-1}}{T^{r-1}} g(y) \end{aligned} \tag{3.1}$$

if and only if y is a solution of the fractional BVP

$$D^r y(t) = h(t), \ \ t \in [0,T], \tag{3.2}$$

$$y(0) = 0, \ y(T) = g(y). \tag{3.3}$$

Proof. Assume y satisfies (3.2), then Lemma 3.2 implies that

$$y(t) = c_1 t^{r-1} + c_2 t^{r-2} + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds.$$

From (3.3), a simple calculation gives

$$c_2 = 0,$$

and

$$c_1 = \frac{1}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} h(s) ds + \frac{1}{T^{r-1}} g(y).$$

Hence we get equation (3.1). Inversely, it is clear that if y satisfies the integral equation (3.1), then equations (3.2)-(3.3) hold.

Theorem 3.4. *Assume the following hypotheses hold:*

(H1) *The function $f : J \times E \longrightarrow E$ satisfy the Carathéodory conditions.*

(H2) *There exists $p \in L^\infty(J, \mathbb{R}_+)$, such that*

$$\|f(t,y)\| \le p(t)\|y\| \text{ for a.e. } t \in J \text{ and each } y \in E.$$

(H3) *There exists constant $k^* > 0$ such that*

$$\|g(y)\|| \le k^*\|y\| \text{ for each } y \in E.$$

(H4) *For almost each* $t \in J$ *and each bounded set* $B \subset E$ *we have*

$$\lim_{k \to 0^+} \alpha(f(J_{t,k} \times B)) \leq p(t)\alpha(B).$$

(H5) *For almost each bounded set* $B \subset E$ *we have*

$$\alpha(g(B)) \leq k^* \alpha(B).$$

Then the BVP (1.1)–(1.2) has at least one solution on $C(J, B)$, *provided that*

$$\frac{T^r + T^{2r}}{\Gamma(r)} \|p\|_{L^\infty} + \frac{k^* T^r}{\Gamma(r)} < 1, \tag{3.4}$$

Proof. Transform the problem (1.1)–(1.2) into a fixed point problem. Consider the operator

$$\begin{aligned} (Ny)(t) &= \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s, y(s)) ds \\ &+ \frac{t^{r-1}}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} f(s, y(s)) ds - \frac{t^{r-1}}{T^{r-1}} g(y) \end{aligned}$$

Remark 3.5. Clearly, from Lemma 3.3, the fixed points of N are solutions to (1.1)–(1.2).

Let $R > 0$ and consider the set

$$D_R = \{y \in C(J, E) : \|y\|\|_\infty \leq R\}.$$

We shall show that N satisfies the assumptions of the Mönch's fixed point theorem. The proof will be given in several steps.

Step 1. N *is continuous.*

Let $|y_n|$ be a sequence such that $y_n \to y$ in $C(J, E)$. Then, for each $t \in J$,

$$\begin{aligned} |(Ny_n)(t) - (Ny)(t)| &\leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y_n(s) - f(s, y(s))\| ds \\ &+ \frac{t^{r-1}}{T^{r-1}\Gamma(r)} \int_0^T (T-s)^{r-1} \|f(s, y_n(s) - f(s, y(s))\| ds. \end{aligned}$$

Let $\rho > 0$ be such that

$$\|y_n\|_\infty \leq \rho, \ \|y\|_\infty \leq \rho.$$

By (H2)-(H3) we have

$$\|f(s, y_n(s) - f(s, y(s))\| \leq 2\rho p(s) := \sigma(s); \ \sigma \in L^1(J, \mathbb{R}_+).$$

Since f, is Carathéodory functions, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2. N *maps* D_R *into itself.*

For each $y \in D_R$, by (H2) and (3.4) we have for each $t \in J$

$$\begin{aligned}\|N(y)(t)\| &\leq \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\|f(s,y(s))\|ds \\ &+ \frac{t^{r-1}}{T^{r-1}\Gamma(r)}\int_0^T (T-s)^{r-1}\|f(s,y(s))\|ds + \frac{t^{r-1}}{T^{r-1}}\|g(y)\| \\ &\leq \frac{T^r + T^{2r}}{\Gamma(r)}\|p\|_{L^\infty} + \frac{k^* T^r}{\Gamma(r)} \\ &\leq R.\end{aligned}$$

Step 3. *$N(D_R)$ is bounded and equicontinuous.*

By Step 2, it is obvious that $N(D_R) \subset C(J,E)$ is bounded.
For the equicontinuity of $N(D_R)$. Let t_1, $t_2 \in J$, $t_1 < t_2$, and $y \in D_R$. we have

$$\begin{aligned}|(Ny)(t_2) - (Ny)(t_1)| &= \Big\|\frac{1}{\Gamma(r)}\int_0^{t_1}[(t_2-s)^{r-1} - (t_1-s)^{r-1}]f(s,y(s))ds \\ &+ \frac{1}{\Gamma(r)}\int_{t_1}^{t_2}(t_2-s)^{r-1}f(s,y(s))ds \\ &+ \frac{(t_1-t_2)^{r-1}}{T^{r-1}\Gamma(r)}\int_0^T (T-s)^{r-1}|f(s,y(s))|ds \\ &+ \frac{(t_1-t_2)^{r-1}}{T^{r-1}}g(y)\Big\| \\ &\leq \frac{p(t)}{\Gamma(r)}\int_0^{t_1}[(t_1-s)^{r-1} - (t_2-s)^{r-1}]ds \\ &+ \frac{p(t)}{\Gamma(r)}\int_{t_1}^{t_2}(t_2-s)^{r-1}ds \\ &+ \frac{p(t)(t_2-t_1)^{r-1}}{T^{r-1}\Gamma(r)}\int_0^T (T-s)^{r-1}ds + \frac{k^*(t_1-t_2)^{r-1}}{T^{r-1}} \\ &\leq \frac{p(t)}{\Gamma(r+1)}[(t_2-t_1)^r + t_1^r - t_2^r] + \frac{p(t)}{\Gamma(r+1)}(t_2-t_1)^r \\ &+ \frac{p(t)(t_2-t_1)^{r-1}}{T^{r-1}\Gamma(r)} + \frac{k^*(t_1-t_2)^{r-1}}{T^{r-1}} \\ &\leq \frac{p(t)}{\Gamma(r+1)}(t_2-t_1)^r + \frac{p(t)}{\Gamma(r+1)}(t_1^r - t_2^r) \\ &+ \frac{p(t)(t_2-t_1)^{r-1}}{T^{r-1}\Gamma(r)} + \frac{k^*(t_1-t_2)^{r-1}}{T^{r-1}} \end{aligned} \tag{3.5}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.
Now let V be a subset of D_R such that $V \subset \overline{co}(N(V) \cup \{0\})$.
v is bounded and equicontinuous, and therefore the function $\vartheta \to \vartheta = \alpha(V(t))$ is continuous on J. By (H3),Lemma2.6, and the properties of the measure α we have

for each $t \in J$

$$\begin{aligned}
\vartheta(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\
&\leq \alpha(N(V)(t)) \\
&\leq \int_0^t \frac{T^r + T^{2r}}{\Gamma(r)} p(s)\alpha(V(s))ds + \frac{k^* T^r}{\Gamma(r)} \alpha(V(t)) \\
&\leq \|\vartheta\|_{L^\infty} \left[\frac{T^r + T^{2r}}{\Gamma(r)} \|p\|_{L^\infty} + \frac{k^* T^r}{\Gamma(r)} \right].
\end{aligned}$$

This means that

$$\|\vartheta\|_{L^\infty} \left(1 - \frac{T^r + T^{2r}}{\Gamma(r)} \|p\|_{L^\infty} + \frac{k^* T^r}{\Gamma(r)} \right) \leq 0.$$

By (3.4) it follows that $\|\vartheta\|_\infty = 0$, that is, $\vartheta = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_R. Applying now Theorem 3.4 we conclude that N has a fixed point which is a solution of the problem (1.1)–(1.2).

4. An example

As an application of the main results, we consider the fractional differential equation

$$D^r y(t) = \frac{2}{19 + e^t} |y(t)|, \text{ for a.e. } t \in J = [0, 1], \quad 1 < r \leq 2, \tag{4.1}$$

$$y(0) = 0, \ y(1) = \sum_{i=1}^{n} c_i y(t_i), \tag{4.2}$$

where $0 < t_1 < t_2 < ... < t_n < 1$, $c_i, \ i = 1, ..., n$ are given positive constants with

$$\sum_{i=1}^{n} c_i < \frac{4}{5}.$$

Set

$$f(t, x) = \frac{2}{19 + e^t} x, \ (t, x) \in J \times [0, \infty),$$

Clearly, conditions (H1), (H2) hold with

$$p(t) = \frac{2}{19 + e^t}, \ k^*(t) = \frac{4}{5}.$$

Condition (3.4)is satisfied with $T = 1$. Indeed

$$\frac{T^r + T^{2r}}{\Gamma(r)} \|p\|_{L^\infty} + \frac{k^* T^r}{\Gamma(r)} \quad \leq \quad \frac{9}{10\Gamma(r)} < 1,$$

which is satisfied for each $r \in (1, 2]$. Then by Theorem 3.4 the problem (4.1)-(4.2) has a solution on $[0, 1]$.

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