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POSITIVE SOLUTIONS FOR EVEN-ORDER MULTI-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract. In this paper, we consider the nonlinear even-order m-point boundary value problems on time scales. We establish the criteria for the existence of at least one and three positive solutions for higher order nonlinear m-point boundary value problems on time scales by using Krasnosel'skii's fixed point theorem and Leggett-Williams' fixed point theorem, respectively.
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1. Introduction

In this paper, we study the existence of multiple positive solutions for even order m-point boundary value problem (BVP) on time scales:

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f(t, y(t)), \ \ t \in [t_1, t_2] \subset \mathbb{T}, \ \ n \in \mathbb{N} \\ \alpha y^{\Delta^{2i}}(t_1) - \beta y^{\Delta^{2i+1}}(t_1) = \sum\limits_{p=1}^{m-2} a_p y^{\Delta^{2i+1}}(\xi_p), \\ \gamma y^{\Delta^{2i}}(t_2) + \delta y^{\Delta^{2i+1}}(t_2) = \sum\limits_{p=1}^{m-2} b_p y^{\Delta^{2i+1}}(\xi_p), \end{cases} \tag{1.1}$$

for $0 \le i \le n-1$, where $m \ge 3$, $\alpha > 1$, β, γ, $\delta > 0$, $t_1 < \xi_1 < \ldots < \xi_{m-2} < t_2$ and a_p, $b_p \ge 0$ are given constants. We assume that $f : [t_1, t_2] \times [0, \infty) \to [0, \infty)$ is continuous.

Throughout this paper we suppose $\mathbb{T}$ is any time scale (nonempty closed subset of $\mathbb{R}$) and $[t_1, t_2]$ is a subset of $\mathbb{T}$ such that $[t_1, t_2] = \{t \in \mathbb{T} : t_1 \le t \le t_2\}$. The study of dynamic equations on time scales goes back to its founder Hilger [5] and is a rapidly expanding area of research. It has been created in order to unify continuous and discrete analysis and it allows a simultaneous treatment of differential and difference equations, extending those theories to so-called dynamic equations. Some basic definitions and theorems on time scales can be found in the books [1, 2], which are excellent references for calculus of time scales.

The multi-point boundary value problems have applications in a variety of different areas of applied mathematics and physics. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem as in [14]; also, many problems in the theory of elastic stability can be handled by multi-point problems as in [15]. In 1987, Il'in and Moiseev [6] studied the existence of solutions for a linear multi-point boundary value problem. Since then, by applying the cone theory techniques, more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [3, 8, 9, 16] and their references. Higher order multi-point boundary value problems have attracted the attention of many researchers in recent years (see [7, 10, 11, 13, 17, 18, 19] and the references therein).

Yaslan [20] studied the following higher order m-point BVP on time scales:

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f(t, y(t)), \ \ t \in [t_1, t_m] \subset \mathbb{T}, \ \ n \in \mathbb{N} \\ y^{\Delta^{2i+1}}(t_m) = 0, \ \alpha y^{\Delta^{2i}}(t_1) - \beta y^{\Delta^{2i+1}}(t_1) = \sum\limits_{k=2}^{m-1} y^{\Delta^{2i+1}}(t_k). \end{cases} \tag{1.2}$$

Conditions for the existence of at least one, two and three positive solutions were obtained by using four functional fixed point theorem, Avery-Henderson fixed point theorem and five functional fixed point theorem, respectively.

In this paper, motivated by the above results, first, we provide some preliminary lemmas which are key tools for our main results. Second, we obtained the existence of at least one positive solution for the BVP (1.1) by using the Krasnosel'skii fixed point theorem. Finally, we use the Leggett-Williams fixed-point theorem to show the existence of at least three positive solutions to the BVP (1.1).

To the best of our knowledge, the existence results for positive solutions of the BVP (1.1) have not been studied previously. The results are even new for the difference equations and differential equations as well as for dynamic equations on general time scales.

We assume that the following conditions are satisfied:

(H1) $\frac{\beta}{\alpha} > t_2$.

(H2) If $m \geq 3$, then $\gamma \sum\limits_{k=1}^{m-2} a_k \geq \alpha \sum\limits_{k=1}^{m-2} b_k$ and if $m > 3$, then

$$\alpha\delta > \gamma \sum_{k=1}^{j-1} a_k \geq \alpha \sum_{k=1}^{j-1} b_k > \beta\gamma \text{ where } 2 \leq j \leq m-2.$$

(H3) $\alpha\delta > \alpha \sum\limits_{p=1}^{m-2} b_p + \gamma \sum\limits_{p=1}^{m-2} a_p$.

2. Preliminaries

To state the main results of this paper, we will need the following lemmas.

Lemma 2.1. *If* $K := \alpha\gamma(t_2 - t_1) + \alpha\delta - \alpha \sum_{p=1}^{m-2} b_p + \gamma\beta + \gamma \sum_{p=1}^{m-2} a_p$, *then Green's function for the boundary value problem*

$$\begin{cases} -y^{\Delta^2}(t) = 0, \ \ t \in [t_1, t_2], \\ \alpha y(t_1) - \beta y^{\Delta}(t_1) = \sum\limits_{p=1}^{m-2} a_p y^{\Delta}(\xi_p), \\ \gamma y(t_2) + \delta y^{\Delta}(t_2) = \sum\limits_{p=1}^{m-2} b_p y^{\Delta}(\xi_p) \end{cases}$$

is given by

$$G(t,s) = \frac{1}{K} \begin{cases} (\alpha(s-t_1)+\beta)(\gamma(t_2-t)+\delta-\sum\limits_{p=1}^{m-2} b_p), t_1 \le s \le \xi_1,\ t \ge s; \\ (\alpha(t-t_1)+\beta)(\gamma(t_2-s)+\delta-\sum\limits_{p=1}^{m-2} b_p) + \gamma \sum\limits_{p=1}^{m-2} a_p(t-s), \\ t_1 \le s \le \xi_1,\ t \le s; \\ (\alpha(s-t_1)+\beta+\sum\limits_{k=1}^{j-1} a_k)(\gamma(t_2-t)+\delta-\sum\limits_{p=j}^{m-2} b_p) \\ +\sum\limits_{k=1}^{j-1} b_k(\alpha(t-s)+\sum\limits_{p=j}^{m-2} a_p),\ \xi_{j-1} \le s \le \xi_j,\ t \ge s,\ 2 \le j \le m-2; \\ (\alpha(t-t_1)+\beta+\sum\limits_{k=1}^{j-1} a_k)(\gamma(t_2-s)+\delta-\sum\limits_{p=j}^{m-2} b_p) \\ +\sum\limits_{p=j}^{m-2} a_p(\gamma(t-s)+\sum\limits_{k=1}^{j-1} b_k),\ \xi_{j-1} \le s \le \xi_j,\ t \le s,\ 2 \le j \le m-2; \\ (\alpha(s-t_1)+\beta+\sum\limits_{k=1}^{m-2} a_k)(\gamma(t_2-t)+\delta) + \alpha \sum\limits_{k=1}^{m-2} b_k(t-s), \\ \xi_{m-2} \le s \le t_2,\ t \ge s; \\ (\alpha(t-t_1)+\beta+\sum\limits_{k=1}^{m-2} a_k)(\gamma(t_2-s)+\delta),\ \xi_{m-2} \le s \le t_2,\ t \le s. \end{cases} \tag{2.1}$$

Proof. A direct calculation gives that if $h \in C[t_1, t_2]$, then the following boundary value problem

$$\begin{cases} -y^{\Delta^2}(t) = h(t), \ \ t \in [t_1, t_2], \\ \alpha y(t_1) - \beta y^{\Delta}(t_1) = \sum\limits_{p=1}^{m-2} a_p y^{\Delta}(\xi_p), \\ \gamma y(t_2) + \delta y^{\Delta}(t_2) = \sum\limits_{p=1}^{m-2} b_p y^{\Delta}(\xi_p) \end{cases}$$

has the unique solution

$$\begin{aligned} y(t) &= -\int_{t_1}^{t}(t-s)h(s)\Delta + \frac{t}{K}\Big\{\alpha\int_{t_1}^{t_2}\left(\gamma(t_2-s)+\delta\right)h(s)\Delta s \\ &+ \sum_{p=1}^{m-2}(\gamma a_p-\alpha b_p)\int_{t_1}^{\xi_p}h(s)\Delta s\Big\} \\ &+ \frac{1}{K}\Big\{\left(\beta+\sum_{p=1}^{m-2}a_p-\alpha t_1\right)\int_{t_1}^{t_2}(\gamma(t_2-s)+\delta)h(s)\Delta s \\ &+ \left(\alpha t_1-(\beta+\sum_{p=1}^{m-2}a_p)\right)\sum_{p=1}^{m-2}b_p\int_{t_1}^{\xi_p}h(s)\Delta s \\ &+ \left(\frac{\gamma(\beta+\sum\limits_{p=1}^{m-2}a_p)}{\alpha}-\frac{K}{\alpha}-\gamma t_1\right)\sum_{p=1}^{m-2}a_p\int_{t_1}^{\xi_p}h(s)\Delta s\Big\}. \end{aligned}$$

Hence, we obtain (2.1). □

Lemma 2.2. *The Green's function $G(t,s)$ in* (2.1) *satisfies*

$$0 < G(t,s) \le G(s,s)$$

for $(t,s)\in[t_1,t_2]\times[t_1,t_2]$.

Proof. From (H1), (H2), (H3) and (2.1), $G(t,s)>0$.

Now we will show that $G(t,s)\le G(s,s)$.

(i) Let $s\in[t_1,\xi_1]$ and $t\ge s$. Since $G(t,s)$ is decreasing in t, we have $G(t,s)\le G(s,s)$.
(ii) Let $s\in[t_1,\xi_1]$ and $t\le s$. Since $G(t,s)$ is increasing in t, we get $G(t,s)\le G(s,s)$.
(iii) Take $s\in(\xi_{j-1},\xi_j]$, $2\le j\le m-2$ and $t\ge s$. From (H2), $G(t,s)$ is decreasing in t. So, we obtain $G(t,s)\le G(s,s)$.
(iv) Take $s\in(\xi_{j-1},\xi_j]$, $2\le j\le m-2$ and $t\le s$. Since $G(t,s)$ is increasing in t, we have $G(t,s)\le G(s,s)$.
(v) Let $s\in(\xi_{m-2},\ t_2]$ and $t\ge s$. From (H2), $G(t,s)$ is decreasing in t. So, we get $G(t,s)\le G(s,s)$.
(vi) Let $s\in(\xi_{m-2},\ t_2]$ and $t\le s$. Since $G(t,s)$ is increasing in t, we obtain $G(t,s)\le G(s,s)$. □

Lemma 2.3. *Green's function $G(t,s)$ in* (2.1) *satisfies*

$$\min_{t\in[t_1,t_2]} G(t,s) \ge z\|G(\cdot,s)\|,$$

with

$$z = \min\{z_1,\, z_2,\, z_3,\, z_4,\, z_5,\, z_6\} \tag{2.2}$$

where

$$z_1 = \frac{\delta - \sum_{p=1}^{m-2} b_p}{\gamma(t_2-t_1) + \delta - \sum_{p=1}^{m-2} b_p}, \quad z_2 = \frac{(\alpha\delta - \alpha\sum_{p=1}^{m-2} b_p - \gamma\sum_{p=1}^{m-2} a_p)(t_2-t_1)}{\gamma(t_2-t_1) + \delta - \sum_{p=1}^{m-2} b_p)(\alpha(t_2-t_1)+\beta}$$

$$z_3 = \frac{\delta - \sum_{p=2}^{m-2} b_p}{\gamma(t_2-t_1) + \delta - \sum_{p=2}^{m-2} b_p}, \quad z_4 = \frac{\frac{\beta}{\alpha} - t_2}{\alpha(t_2-t_1) + \beta + \sum_{k=1}^{m-3} a_k}, \quad z_5 = \frac{\delta}{\gamma(t_2-t_1)+\delta}$$

$$z_6 = \frac{\frac{\beta}{\alpha} - t_2}{\alpha(t_2-t_1) + \beta + \sum_{k=1}^{m-2} a_k}$$

and $\|\cdot\|$ *is defined by* $\|x\| = \max_{t\in[t_1,t_2]} |x(t)|$.

Proof.

(i) Take $s \in [t_1, \xi_1]$ and $t \geq s$. Since $G(t,s)$ is decreasing in t and $0 < z_1 < 1$, we have $\min_{t\in[t_1,t_2]} G(t,s) = G(t_2,s)$ and $\min_{t\in[t_1,t_2]} G(t,s) \geq z_1 G(s,s) = z_1\|G(\cdot,s)\|$.

(ii) Take $s \in [t_1, \xi_1]$ and $t \leq s$. Since $G(t,s)$ is increasing in t and $0 < z_2 < 1$, we get $\min_{t\in[t_1,t_2]} G(t,s) = G(t_1,s)$ and $\min_{t\in[t_1,t_2]} G(t,s) \geq z_2 G(s,s) = z_2\|G(\cdot,s)\|$.

(iii) Let $s \in (\xi_{j-1}, \xi_j]$, $2 \leq j \leq m-2$ and $t \geq s$. From (H2), $G(t,s)$ is decreasing in t and it is clear that $0 < z_3 < 1$. So, we find $\min_{t\in[t_1,t_2]} G(t,s) = G(t_2,s)$ and $\min_{t\in[t_1,t_2]} G(t,s) \geq z_3 G(s,s) = z_3\|G(\cdot,s)\|$.

(iv) Let $s \in (\xi_{j-1}, \xi_j]$, $2 \leq j \leq m-2$ and $t \leq s$. Since $G(t,s)$ is increasing in t and $0 < z_4 < 1$, we obtain $\min_{t\in[t_1,t_2]} G(t,s) = G(t_1,s)$ and $\min_{t\in[t_1,t_2]} G(t,s) \geq z_4 G(s,s) = z_4\|G(\cdot,s)\|$.

(v) Take $s \in (\xi_{m-2},\, t_2]$ and $t \geq s$. From (H2), $G(t,s)$ is decreasing in t. It is clear that $0 < z_5 < 1$. So, we have $\min_{t\in[t_1,t_2]} G(t,s) = G(t_2,s)$ and $\min_{t\in[t_1,t_2]} G(t,s) \geq z_5 G(s,s) = z_5\|G(\cdot,s)\|$.

(vi) Take $s \in (\xi_{m-2},\, t_2]$ and $t \leq s$. Since $G(t,s)$ is increasing in t and $0 < z_6 < 1$, we get $\min_{t\in[t_1,t_2]} G(t,s) = G(t_1,s)$ and $\min_{t\in[t_1,t_2]} G(t,s) \geq z_6 G(s,s) = z_6\|G(\cdot,s)\|$.

Thus, $\min_{t\in[t_1,t_2]} G(t,s) \geq z\|G(\cdot,s)\|$ where $z = \min\{z_1,\, z_2,\, z_3,\, z_4,\, z_5,\, z_6\}$. □

If we let $G_1(t,s) := G(t,s)$ for G as in (2.1), then we can recursively define

$$G_j(t,s) = \int_{t_1}^{t_2} G_{j-1}(t,r)G(r,s)\Delta r$$

for $2 \le j \le n$ and $G_n(t,s)$ is Green's function for the homogeneous problem

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = 0, \ \ t \in [t_1, t_2], \\ \alpha y^{\Delta^{2i}}(t_1) - \beta y^{\Delta^{2i+1}}(t_1) = \sum\limits_{p=1}^{m-2} a_p y^{\Delta^{2i+1}}(\xi_p), \\ \gamma y^{\Delta^{2i}}(t_2) + \delta y^{\Delta^{2i+1}}(t_2) = \sum\limits_{p=1}^{m-2} b_p y^{\Delta^{2i+1}}(\xi_p), \end{cases}$$

where $m \ge 3$ and $0 \le i \le n-1$.

Lemma 2.4. *The Green's function $G_n(t,s)$ satisfies the following inequalities*

$$0 \le G_n(t,s) \le L^{n-1}\|G(\cdot,s)\|, \quad (t,s) \in [t_1,t_2] \times [t_1,t_2]$$

and

$$G_n(t,s) \ge z^n L^{n-1}\|G(\cdot,s)\|, \quad (t,s) \in [t_1,t_2] \times [t_1,t_2]$$

where z is given in (2.2) *and*

$$L = \int_{t_1}^{t_2} \|G(\cdot,s)\|\Delta s > 0. \tag{2.3}$$

Proof. Use induction on n and Lemma 2.3. □

Let E denote the Banach space $C[t_1,t_2]$ with the norm $\|y\| = \max\limits_{t\in[t_1,t_2]} |y(t)|$. Define the cone $P \subset E$ by

$$P = \{y \in E : y(t) \ge 0, \min_{t\in[t_1,t_2]} y(t) \ge z^n\|y\|\} \tag{2.4}$$

where z and L are given in (2.2) and (2.3), respectively.

(1.1) is equivalent to the nonlinear integral equation

$$y(t) = \int_{t_1}^{t_2} G_n(t,s) f(s, y(s))\Delta s. \tag{2.5}$$

We can define the operator $A : P \to E$ by

$$Ay(t) = \int_{t_1}^{t_2} G_n(t,s) f(s, y(s))\Delta s, \tag{2.6}$$

where $y \in P$. Therefore solving (2.5) in P is equivalent to finding fixed points of the operator A.

Lemma 2.5. *If the conditions (H1), (H2), (H3) hold, then $A : P \to P$ is completely continuous.*

Proof. If $y \in P$, then $Ay(t) \geq 0$ on $[t_1, t_2]$ and by using Lemma 2.4,

$$\begin{aligned} \min_{t\in[t_1,t_2]} Ay(t) &= \min_{t\in[t_1,t_2]} \int_{t_1}^{t_2} G_n(t,s) f(s, y(s))\Delta s \\ &\geq z^n \int_{t_1}^{t_2} \max_{t\in[t_1,t_2]} G_n(t,s) f(s,y(s))\Delta s \\ &= z^n \|Ay\|. \end{aligned}$$

Thus $Ay \in P$ and therefore $AP \subset P$.

Recall that an operator (nonlinear, in general) acting in a Banach space is called completely continuous if it is continuous and transforms every bounded set into relatively compact set. From Lemma 2.4 and the continuity of f, it is clear that A is continuous. Now, we take arbitrary bounded set $Y \subset P$ and show that its image $A(Y)$ is relatively compact in P. Since the continuity of f, there exists a constant $c > 0$ such that

$$f(s, y(s)) < c \tag{2.7}$$

for all $s \in [t_1, t_2]$ and $y \in Y$. By using Lemma 2.4 and (2.7), $A(Y)$ is equibounded. Also, $A(Y)$ is equicontinuous from (2.7) and the continuity of $G_n(t, s)$. Applying the Arzela-Ascoli theorem, we obtain $A(Y)$ is relatively compact. Hence, $A : P \to P$ is completely continuous. □

In order to follow the main results of this paper easily, now we state the fixed point theorems which we applied to prove main theorems.

Theorem 2.6. *(*[4]*) (Krasnosel'skii Fixed Point Theorem) Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$;

or

(ii) $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$

hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 2.7. *(*[12]*) (Leggett-Williams Fixed Point Theorem) Let P be a cone in a real Banach space E. Set*

$$P_r := \{x \in P : \|x\| < r\}$$

$$P(\psi, a, b) := \{x \in P : a \leq \psi(x),\ \|x\| \leq b\}.$$

Suppose $A : \overline{P_r} \to \overline{P_r}$ be a completely continuous operator and ψ be a nonnegative continuous concave functional on P with $\psi(u) \leq \|u\|$ for all $u \in \overline{P_r}$. If there exists $0 < p < q < l \leq r$ such that the following condition hold,

(i) *$\{u \in P(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ and $\psi(Au) > q$ for all $u \in P(\psi, q, l)$;*
(ii) *$\|Au\| < p$ for $\|u\| \leq p$;*
(iii) *$\psi(Au) > q$ for $u \in P(\psi, q, r)$ with $\|Au\| > l$,*

then A has at least three fixed points u_1, u_2 and u_3 in $\overline{P_r}$ satisfying

$$\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\| \text{ with } \psi(u_3) < q.$$

3. Main results

Now, we will give the sufficient conditions to have at least one positive solution for the BVP (1.1). Krasnosel'skii Fixed Point Theorem will be used to prove the next theorem.

Theorem 3.1. *Suppose (H1), (H2) and (H3) hold. In addition let there exist numbers $0 < r < R < \infty$ such that the function f satisfies the following conditions:*

(i) $f(t,y) \le \frac{1}{L^n} y(t)$ *for* $(t,y) \in [t_1,t_2] \times [0,r]$,

(ii) $f(t,y) \ge \frac{1}{z^{2n}L^n} y(t)$ *for* $(t,y) \in [t_1,t_2] \times [R,\infty)$.

Then the BVP (1.1) has at least one positive solution.

Proof. Define the open bounded subsets of E by $\Omega_1 = \{y \in P : \|y\| < r\}$ and $\Omega_2 = \left\{y \in P : \|y\| < \frac{R}{z^n}\right\}$. From Lemma 2.5, $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is completely continuous operator.

We now verify one of the conditions of Theorem 2.6.

If $y \in P \cap \partial\Omega_1$, then $\|y\| = r$. Therefore, by using the hypothesis (i) and Lemma 2.4, we have

$$\begin{aligned} Ay(t) &= \int_{t_1}^{t_2} G_n(t,s) f(s,y(s))\Delta s \\ &\le \frac{1}{L^n}\int_{t_1}^{t_2} G_n(t,s) y(s)\Delta s \\ &\le \frac{1}{L}\|y\| \int_{t_1}^{t_2} \|G(\cdot,s)\|\Delta s \\ &= \|y\|. \end{aligned}$$

Thus, we obtain $\|Ay\| \le \|y\|$ for $y \in P \cap \partial\Omega_1$.
On the other hand, $y \in P \cap \partial\Omega_2$ implies

$$y(t) \ge z^n\|y\| = R$$

for $t \in [t_1, t_2]$ and we get

$$\begin{aligned} Ay(t) &= \int_{t_1}^{t_2} G_n(t,s) f(s, y(s)) \Delta s \\ &\geq \frac{1}{z^{2n} L^n} \int_{t_1}^{t_2} G_n(t,s) y(s) \Delta s \\ &\geq \frac{1}{z^n L^n} \|y\| \int_{t_1}^{t_2} G_n(t,s) \Delta s \\ &\geq \|y\|, \end{aligned}$$

from (ii) and Lemma 2.4. Hence, $\|Ay\| \geq \|y\|$ for $y \in P \cap \partial\Omega_2$.

By the first part of Theorem 2.6, A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$, such that $r \leq \|y\| \leq \frac{R}{z^n}$. Therefore BVP (1.1) has at least one positive solution. □

Now we will use the Leggett-Williams fixed point theorem to prove the next theorem.

Theorem 3.2. *Assume that (H1), (H2) and (H3) hold. Suppose that there exist numbers* $0 < p < q < \frac{q}{z^n} \leq r$ *such that the function* f *satisfies the following conditions:*

(i) $f(t,y) \leq \frac{r}{L^n}$ *for* $(t,y) \in [t_1, t_2] \times [0, r]$,

(ii) $f(t,y) < \frac{p}{L^n}$ *for* $(t,y) \in [t_1, t_2] \times [0, p]$,

(iii) $f(t,y) > \frac{q}{z^n L^n}$ *for* $(t,y) \in [t_1, t_2] \times [q, \frac{q}{z^n}]$,

where z *and* L *are as in (2.2) and (2.3), respectively. Then the BVP (1.1) has at least three positive solutions* y_1, y_2 *and* y_3 *satisfying*

$$\max_{t \in [t_1, t_2]} y_1(t) < p, \quad \min_{t \in [t_1, t_2]} y_2(t) > q,$$
$$\max_{t \in [t_1, t_2]} y_3(t) > p \text{ with } \min_{t \in [t_1, t_2]} y_3(t) < q.$$

Proof. Define the nonnegative, continuous, concave functional $\psi : P \to [0, \infty)$ to be $\psi(y) = \min_{t \in [t_1, t_2]} y(t)$ and the cone P as in (2.4). For all $y \in P$, we have $\psi(y) \leq \|y\|$. If $y \in \overline{P_r}$, then $0 \leq y(t) \leq r$ for all $t \in [t_1, t_2]$. We get,

$$\begin{aligned} \|Ay\| &= \max_{t \in [t_1, t_2]} \int_{t_1}^{t_2} G_n(t,s) f(s, y(s)) \Delta s \\ &\leq L^{n-1} \int_{t_1}^{t_2} \|G(\cdot, s)\| f(s, y(s)) \Delta s \\ &\leq r \end{aligned}$$

by hypothesis (i) and Lemma 2.4. This proves that $A : \overline{P_r} \to \overline{P_r}$. From Lemma 2.5, $A : \overline{P_r} \to \overline{P_r}$ is completely continuous.

Since $z < 1$, $y(t) = \frac{q}{z^n} \in P(\psi, q, \frac{q}{z^n})$ and $\psi(\frac{q}{z^n}) > q$. Then

$$\{y \in P(\psi, q, \frac{q}{z^n}) : \psi(y) > q\} \neq \emptyset.$$

On the other hand, for all $y \in P(\psi, q, \frac{q}{z^n})$ and $t \in [t_1, t_2]$, we have $q \leq y(t) \leq \frac{q}{z^n}$. Using assumption (iii) and Lemma 2.4, we find

$$\begin{aligned} \psi(Ay) &= \min_{t\in[t_1,t_2]} \int_{t_1}^{t_2} G_n(t,s) f(s, y(s))\Delta s \\ &\geq z^n L^{n-1} \int_{t_1}^{t_2} \|G(\cdot, s)\| f(s, y(s))\Delta s \\ &> q. \end{aligned}$$

Thus condition (i) of Theorem 2.7 holds.

For $\|y\| \leq p$, we have $0 \leq y(t) \leq p$ for $t \in [t_1, t_2]$. Then from assumption (ii) and Lemma 2.4, we obtain

$$\begin{aligned} \|Ay\| &= \max_{t\in[t_1,t_2]} \int_{t_1}^{t_2} G_n(t,s) f(s, y(s))\Delta s \\ &\leq L^{n-1} \int_{t_1}^{t_2} \|G(\cdot, s)\| f(s, y(s))\Delta s \\ &< p. \end{aligned}$$

Consequently, condition (ii) of Theorem 2.7 is satisfied.

Finally we will check that condition (iii) of Theorem 2.7. We suppose that $y \in P(\psi, q, r)$ with $\|Ay\| > \frac{q}{z^n}$. Then using Lemma 2.4 we obtain

$$\psi(Ay) = \min_{t\in[t_1,t_2]} Ay(t) \geq z^n \|Ay\| > q.$$

Since all conditions of the Leggett-Williams fixed point theorem are satisfied. The BVP (1.1) has at least three positive solutions y_1, y_2 and y_3 such that

$$\begin{aligned} &\max_{t\in[t_1,t_2]} y_1(t) < p, \ \min_{t\in[t_1,t_2]} y_2(t) > q, \\ &\max_{t\in[t_1,t_2]} y_3(t) > p \text{ with } \min_{t\in[t_1,t_2]} y_3(t) < q. \end{aligned}$$

□

Example 3.3. Let $\mathbb{T} = \{(\frac{2}{3})^n : n \in \mathbb{N}_0\} \cup \{0\} \cup [2, 3]$. Taking $n = 1$, $m = 3$, $t_1 = \frac{2}{3}$, $\xi_1 = 1$, $t_2 = 3$, $\alpha = \gamma = \delta = 3$, $\beta = 12$, $a_1 = b_1 = 1$, we consider the following boundary value problem:

$$\begin{cases} -y^{\Delta^2}(t) = f(t, y), \ \ t \in [\frac{2}{3}, 3] \subset \mathbb{T}, \\ 3y(\frac{2}{3}) - 12y^{\Delta}(\frac{2}{3}) = y^{\Delta}(1), \\ 3y(3) + 3y^{\Delta}(3) = 3y^{\Delta}(1). \end{cases} \tag{3.1}$$

Then we get $L = \frac{291}{2} = 145.5$, $z = 0.041$.

If we take $f(t,y) = \frac{1000y^3}{y^2+1}$, $r = 10^{-6}$ and $R = 1$, then $0 < r < R < \infty$ and all the conditions in Theorem 3.1 are satisfied. Thus, by Theorem 3.1, the BVP (3.1) has at least one positive solution y such that $10^{-6} \leq \max\limits_{t\in[2/3,3]} y(t) \leq 1/(0.041)$.

If we take $f(t,y) = \frac{1000y^2}{y^2+1}$, $p = 7.10^{-7}, q = 3.10^{-4}$ and $r = 146000$, then $0 < p < q < \frac{q}{z} < r$ and all the conditions in Theorem 3.2 are fulfilled. Hence, by Theorem 3.2, the BVP (3.1) has at least three positive solutions y_1, y_2 and y_3 satisfying

$$\max_{t\in[\frac{2}{3},3]} y_1(t) < p, \ \min_{t\in[\frac{2}{3},3]} y_2(t) > q$$
$$\max_{t\in[\frac{2}{3},3]} y_3(t) > p \text{ with } \min_{t\in[\frac{2}{3},3]} y_3(t) < q.$$

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