Fixed Point Theory, 18(2017), No. 2, 569-578
DOI 10.24193/fpt-ro.2017.2.45
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON ORTHOGONAL SETS AND BANACH FIXED POINT THEOREM

MADJID ESHAGHI GORDJI*, MARYAM RAMEANI*, MANUEL DE LA SEN**
AND YEOL JE CHO***

*Department of Mathematics, Semnan University
P.O. Box 35195-363, Semnan, Iran
E-mail: meshaghi@semnan.ac.ir; madjid.eshaghi@gmail.com; mar.ram.math@gmail.com

**Institute of Research and Development of Processes University of Basque Country
Campus of Leioa (Bizkaia)-Aptdo, 644-Bilbao, 48080-Bilbao, Spain
E-mail: manuel.delasen@ehu.es

***Department of Mathematics Education and the RINS
Gyeongsang National University, Jinju 660-701, Korea
and
Center for General Education, China Medical University Taichung 40402, Taiwan
E-mail: yjcho@gnu.ac.kr

Abstract. We introduce the notion of the orthogonal sets and give a real generalization of Banach' fixed point theorem. As an application, we find the existence of solution for a first-order ordinary differential equation.
Key Words and Phrases: Orthogonal set, fixed point, differential equation, Picard operator.
2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

The purpose of this paper is to introduce the notion of orthogonality of sets which contains the notion of orthogonality in normed spaces (see [1] and [2]). By using this concept, we discuss an analogue of [3] in orthogonal sets. The main result of [3] is the following theorem:

Theorem 1.1. *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower bound and an upper bound. Furthermore, let d be a metric on X such that (X, d) is a complete metric space. If F is a continuous, monotone mapping from X into X such that*

- *there exists $k \in (0,1)$ with $d(F(x), F(y)) \leq kd(x,y), \forall x \geq y$,*
- *there exists $x_0 \leq F(x_0)$ or $x_0 \geq F(x_0)$.*

Then F is a Picard operator (briefly, PO), that is, F has a unique fixed point x^* and $\lim_{n\to\infty} F^n(x) = x^*$ for each $x \in X$.

*** Corresponding author.

In this paper, we introduce the notion of the orthogonal sets and then we give an extension of Banach' fixed point theorem. We give an example which says that our main theorem is a real generalization of Banach's fixed point theorem. Finally, we study the existence and uniqueness of solution for a first-order ordinary differential equation. Banach' fixed point theorem and other fixed point theorems do not work to prove this problem. There is also a set of examples in the paper which describe the usefulness of orthogonality binary relations and orthogonal sets to describe.

2. Orthogonal sets

We start our work with the following definition, which can be consider the main definition of our paper.

Definition 2.1. Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be an binary relation. If $\perp$ satisfies the following condition:

$$\exists x_0 : (\forall y, y \perp x_0) \text{ or } (\forall y, x_0 \perp y),$$

then it is called an *orthogonal set* (briefly *O-set*). We denote this O-set by $(X, \perp)$.

As an illustration, let us consider the following examples:

Example 2.2. Let X be the set of all peoples in the word. We define $x \perp y$ if x can give blood to y. According to the following table, if x_0 is a person such that his (her) blood type is $O-$, then we have $x_0 \perp y$ for all $y \in X$. This means that $(X, \perp)$ is an O-set. In this O-set, x_0 (in definition) is not unique.

Note that, in the above example, x_0 may be a person with blood type $AB+$. In this case, we have $y \perp x_0$ for all $y \in X$.

Type	You can give blood to	You can receive blood from
A+	A+ AB+	A+ A– O+ O–
O+	O+ A+ B+ AB+	O+ O–
B+	B+ AB+	B+ B– O+ O–
AB+	AB+	Everyone
A–	A+ A– AB+ AB–	A– O–
O–	Everyone	O–
B–	B+ B– AB+ AB–	B– O–
AB–	AB+ AB–	AB– B– O– A–

Example 2.3. In graph theory, a wheel graph W_n is a graph with n vertices for each $n \geq 4$, formed by connecting a single vertex to all vertices of an $(n-1)$-cycle. Let X be the set of all vertices of W_n for each $n \geq 4$. Define $a \perp b$ if there is a connection from a to b. Then $(X, \perp)$ is an O-set.

Example 2.4. Let $X = \mathbb{Z}$. Define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence $(X, \perp)$ is an O-set.

Example 2.5. Let us make a famous fractal called the *Sierpinski Triangle*.
Sierpinski's triangle starts as a shaded triangle of equal lengths in page $\mathbb{R} \times \mathbb{R}$ with vertices $(-1, 0), (1, 0)$ and $(0, \sqrt{3})$. We split the triangle into four equal triangles by connecting the centers of each side together and remove this central triangle. We then repeat this process on the 3 newly created smaller triangles. This process is repeated several times on each newly created smaller triangle to arrive at the displayed picture. A Sierpinski's triangle is created by infinitely repeating this construction process. Let X be the set of all (infinite) removed triangles in above process. For all $a, b \in X$, we define $a \perp b$ if

$$\inf\{y : (x, y) \in a \text{ for some } x \in \mathbb{R}\} \leq \inf\{y : (x, y) \in b \text{ for some } x \in \mathbb{R}\}.$$

Then $(X, \perp)$ is an O-set.

Example 2.6. Let $X = [2, \infty)$ and define $x \perp y$ if $x \leq y$. Then, by putting $x_0 = 2$, $(X, \perp)$ is an O-set.

Example 2.7. Let $X = [0, \infty)$ and define $x \perp y$ if $xy \in \{x, y\}$. Then, by setting $x_0 = 0$ or $x_0 = 1$, $(X, \perp)$ is an O-set.

Example 2.8. Let (X, d) be a metric space and $T : X \to X$ be a Picard operator, that is, there exists $x^* \in X$ such that $\lim_{n\to\infty} T^n(y) = x^*$ for all $y \in X$. We define $x \perp y$ if

$$\lim_{n\to\infty} d(x, T^n(y)) = 0.$$

Then $(X, \perp)$ is an O-set.

By the following example, again we can see that x_0 is not necessary unique:

Example 2.9. Suppose $\mathcal{M}(n)$ is the set of all $n \times n$ matrices and Q is a positive definite matrix. Define the relation $\perp$ on $\mathcal{M}(n)$ by

$$A \perp B \iff \exists X \in \mathcal{M}(n) : AX = B.$$

It is easy to see that $I \perp B$, $B \perp 0$ and $Q^{\frac{1}{2}} \perp B$ for all $B \in \mathcal{M}(n)$.

In the last example, the orthogonal relation is reflexive and transitive, but it is not antisymmetry. Now, we would like to give an orthogonal relation which is a symmetry.

Example 2.10. For $C \in \mathcal{M}(n)$, consider the orthogonal relation $\perp_C$ on $\mathcal{M}(n)$ which respect to C given by

$$A \perp_C B \iff tr(ABC) = tr(CBA).$$

Note that $C \perp_C B$ for all $B \in \mathcal{M}(n)$.

Finally, we have the following example for O-sets.

Example 2.11. Let X be a inner product space with the inner product $\langle \cdot, \cdot \rangle$. Define $x \perp y$ if $\langle x, y \rangle = 0$. It is easy to see that $0 \perp x$ for all $x \in X$. Hence $(X, \perp)$ is an O-set.

3. The main theorems

In this section, we prove the main theorem of the present paper. To this end, we need the following definitions:

Definition 3.1. Let $(X, \perp)$ be O-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an *orthogonal sequence* (briefly, *O-sequence*) if

$$(\forall n, x_n \perp x_{n+1}) \ \ or \ \ (\forall n, \forall x_{n+1} \perp x_n).$$

Definition 3.2. Let $(X, \perp, d)$ be an orthogonal metric space ($(X, \perp)$ is an O-set and (X, d) is a metric space). Then $f : X \to X$ is said to be *orthogonally continuous* (or $\perp-$*continuous*) in $a \in X$ if, for each O-sequence $\{a_n\}_{n\in\mathbb{N}}$ in X with $a_n \to a$, we have $f(a_n) \to f(a)$. Also, f is said to be $\perp-$*continuous* on X if f is $\perp-$continuous in each $a \in X$.

It is easy to see that every continuous mapping is $\perp-$continuous. The following example shows that the converse is not true.

Example 3.3. Let $X = \mathbb{R}$ and suppose that $x \perp y$ if

$$x, y \in \left(n + \frac{1}{3}, n + \frac{2}{3}\right)$$

for some $n \in \mathbb{Z}$ or

$$x = 0.$$

It is easy to see that $(X, \perp)$ is an O-set. Define $f : X \to X$ by $f(x) = [x]$. Then f is $\perp-$continuous on X. Because if $\{x_k\}$ is an arbitrary O-sequence in X such that $\{x_k\}$ converges to $x \in X$, then the following cases hold:

Case 1: If $x_k = 0$ for all k, then $x = 0$ and $f(x_k) = 0 = f(x)$.

Case 2: If $x_{k_0} \neq 0$ for some k_0, then there exists $m \in \mathbb{Z}$ such that $x_k \in (m+\frac{1}{3}, m+\frac{2}{3})$ for all $k \geq k_0$. Thus $x \in [m + \frac{1}{3}, m + \frac{2}{3}]$ and $f(x_k) = m = f(x)$.

This means that f is $\perp-$continuous on X while it is not continuous on X.

We can not prove the following problem about the continuity of functions on inner product spaces:

Problem 3.4. Let X be an inner product space with the inner product $\langle \cdot, \cdot \rangle$. Define $x \perp y$ if $\langle x, y \rangle = 0$. Let $f : X \to X$ be $\perp-$continuous on X. Is f continuous on X?

We cannot prove the above problem for $X = \mathbb{R}^n$ and its inner product.

Definition 3.5. Let $(X, \perp, d)$ be an orthogonal set with the metric d. Then X is said to be *orthogonally complete* (briefly, *O-complete*) if every Cauchy O-sequence is convergent.

It is easy to see that every complete metric space is O-complete and the converse is not true. In the next example, X is O-complete and it is not complete.

Example 3.6. Let $X = [0, 1)$ and suppose that

$$x \perp y \iff \begin{cases} x \leq y \leq \frac{1}{2}, \\ or \ x = 0. \end{cases}$$

Then $(X, \perp)$ is an O-set. Clearly, X with the Euclidian metric is not complete metric space, but it is O-complete. In fact, if $\{x_k\}$ is an arbitrary Cauchy O-sequence in X, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} = 0$ for all $n \geq 1$ or there exists a monotone subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} \leq \frac{1}{2}$ for all $n \geq 1$. It follows that $\{x_{k_n}\}$ converges to a point $x \in [0, \frac{1}{2}] \subseteq X$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent.

Definition 3.7. Let $(X, \perp, d)$ be an orthogonal metric space and $0 < \lambda < 1$. A mapping $f : X \to X$ is called an *orthogonal contraction* (briefly, $\perp$-*contraction*) with Lipschitz constant λ if, for all $x, y \in X$ with $x \perp y$,

$$d(fx, fy) \leq \lambda d(x, y).$$

It is easy to show that every contraction is $\perp$-contraction, but the converse is not true. See the following examples:

Example 3.8. Let $X = [0, 10)$ and the metric on X be the Euclidian metric. Define $x \perp y$ if $xy \leq (x \vee y)$ where $x \vee y = x$ *or* y. Let $f : X \to X$ be a mapping defined by

$$f(x) = \begin{cases} \frac{x}{2}, & x \leq 2, \\ 0, & x > 2. \end{cases}$$

Let $x \perp y$ and $xy \leq x$, then the following cases are satisfied:

Case 1: If $x = 0$ and $y \leq 2$, then $f(x) = 0$ and $f(y) = \frac{y}{2}$.

Case 2: If $x = 0$ and $y > 2$, then $f(x) = f(y) = 0$.

Case 3: If $y \leq 1$ and $x \leq 2$, then $f(y) = \frac{y}{2}$ and $f(x) = \frac{x}{2}$.

Case 4: If $y \leq 1$ and $x > 2$, then $x - y > y$, $f(y) = \frac{y}{2}$ and $f(x) = 0$.

This implies that $|f(x) - f(y)| \leq \frac{1}{2}|x - y|$ and hence f is an $\perp$-contraction. But f is not a contraction. To see this, for each $c < 1$,

$$|f(3) - f(2)| = 1 > c = c|3 - 2|.$$

Example 3.9. Let $X = [0, 1)$ and let the metric on X be the euclidian metric. Define $x \perp y$ if $xy \in \{x, y\}$ for all $x, y \in X$. Let $f : X \to X$ be a mapping defined by

$$f(x) = \begin{cases} \frac{x}{2}, & x \in \mathbb{Q} \cap X, \\ 0, & x \in \mathbb{Q}^c \cap X. \end{cases}$$

Then it is easy to show that f is an $\perp-$contraction on X, but it is not a contraction.

Definition 3.10. Let $(X, \perp)$ be an O-set. A mapping $f : X \to X$ is said to be $\perp-$*preserving* if $f(x) \perp f(y)$ if $x \perp y$. Also, $f : X \to X$ is said to be *weakly* $\perp$-*preserving* if $f(x) \perp f(y)$ or $f(y) \perp f(x)$ if $x \perp y$.

It is easy to see that every $\perp$-preserving mapping is weakly $\perp-$preserving. But the converse is not true. For instance, let $(X, \perp)$ be the O-set in Example 2.2. Let $o_1 \in X$ be a person with blood type $O-$, $a_1 \in X$ be a person with blood type $A+$. Define a mapping $f : X \to X$ by

$$f(x) = \begin{cases} a_1, & x = o_1 \\ o_1, & x \in X - \{o_1\}. \end{cases}$$

Let $o_2 \in X - \{o_1\}$ be a person with blood type $O-$. Then we have $o_1 \perp o_2$, but we have not $f(o_1) \perp f(o_2)$. This means that f is not $\perp$-preserving. It is easy to see that f is weakly $\perp$-preserving.

Now, we are ready to prove the main theorem of this paper which can be consider as a real extension of Banach contraction principle.

Theorem 3.11. *Let $(X, \perp, d)$ be an O-complete metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be $\perp$-continuous, $\perp$-contraction with Lipschitz constant λ and $\perp$-preserving. Then f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is, $\lim f^n(x) = x^*$ for all $x \in X$.*

Proof. By the definition of orthogonality, there exists $x_0 \in X$ such that

$$(\forall y \in X, x_0 \perp y \quad) or(\quad \forall y \in X, y \perp x_0).$$

It follows that $x_0 \perp f(x_0)$ or $f(x_0) \perp x_0$. Let

$$x_1 := f(x_0),\ x_2 = f(x_1) = f^2(x_0),\ \cdots,\ x_{n+1} = f(x_n) = f^{n+1}(x_0)$$

for all $n \in \mathbb{N}$. Since f is $\perp$-preserving, $\{x_n\}_{n \in \mathbb{N}}$ is an O-sequence. On the other hand, f is an $\perp$-contraction. Then we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$$

for all $n \in \mathbb{N}$. If $m, n \in \mathbb{N}$ and $n \leq m$, then

$$\begin{aligned} d(x_n, x_m) &\leq (d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m)) \\ &\leq (\lambda^n d(x_0, x_1) + \cdots + \lambda^{m-1} d(x_0, x_1)) \\ &\leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1). \end{aligned}$$

So, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Therefore, $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy O-sequence. Since X is O-complete, there exists $x^* \in X$ such that $x_n \to x^*$. On the other hand, f is $\perp$-continuous and then $f(x_n) \to f(x^*)$ and $f(x^*) = f(\lim_n (f(x_n)) = \lim_n x_{n+1} = x^*$. Hence x^* is a fixed point of f.

To prove the uniqueness property of fixed point, let $y^* \in X$ be a fixed point of f. Then we have $f^n(x^*) = x^*$ and $f^n(y^*) = y^*$ for all $n \in \mathbb{N}$. By the choice of x_0 in the first part of proof, we have

$$[x_0 \perp x^* \ \text{ and } \ x_0 \perp y^*] \ \text{ or } \ [x^* \perp x_0 \ \text{ and } \ y^* \perp x_0].$$

Since f is $\perp$-preserving, we have

$$[f^n(x_0) \perp f^n(x^*) \ \text{ and } \ f^n(x_0) \perp f^n(y^*)]$$

or

$$[f^n(x^*) \perp f^n(x_0) \ \text{ and } \ f^n(y^*) \perp f^n(x_0)]$$

for all $n \in \mathbb{N}$. Therefore, by the triangle inequality, we have

$$\begin{aligned} d(x^*, y^*) &= d(f^n(x^*), f^n(y^*)) \\ &\leq d(f^n(x^*), f^n(x_0)) + d(f^n(x_0), f^n(y^*)) \\ &\leq \lambda^n d(x^*, x_0) + \lambda^n d(x_0, y^*) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Thus it follows that $x^* = y^*$.

Finally, let $x \in X$ be arbitrary. Similarly, we have

$$[x_0 \bot x^* \text{ and } x_0 \bot x] \text{ or } [x^* \bot x_0 \text{ and } x \bot x_0]$$

and

$$[f^n(x_0) \bot f^n(x^*) \text{ and } f^n(x_0) \bot f^n(x)]$$

or

$$[f^n(x^*) \bot f^n(x_0) \text{ and } f^n(x) \bot f^n(x_0)]$$

for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d(x^*, f^n(x)) &= d(f^n(x^*), f^n(x)) \\ &\leq d(f^n(x^*), f^n(x_0)) + d(f^n(x_0), f^n(x)) \\ &\leq \lambda^n d(x^*, x_0) + \lambda^n d(x_0, x) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. This completes the proof. □

Now, we show that our theorem is a real extension of Banach's contraction principle.

Corollary 3.12. (Banach's contraction principle) *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that, for some $\lambda \in (0, 1]$,*

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$. Then f has a unique fixed point in X.

Proof. Suppose that

$$x \bot y \iff d(f(x), f(y)) \leq d(x, y).$$

Fix $x_0 \in X$. Since f is a contraction, for all $y \in X$, $x_0 \bot y$. Hence $(X, \bot)$ is an O-set. It is obviously that X is O-complete and f is an $\bot$-contraction, $\bot$-continuous and $\bot$-preserving. By applying Theorem 3.11, f has a unique fixed point in X. □

The following examples show that our theorem is a real extension of Banach's fixed point theorem:

Example 3.13. Suppose that $X = [0, 10)$, $\bot$, d and $f : X \to X$ are defined as in the Example 3.8. It is easy to see that X is O-complete (not complete), f is $\bot$-continuous (not continuous on X), an $\bot$-contraction and $\bot$-preserving on X. By our theorem, f has a unique fixed point in X. However, f is not a contraction on X and so, by Banach's contraction principle, we cannot find any fixed point of f in X.

Example 3.14. Suppose that $X = [0, 1)$, $\perp$, d and $f : X \to X$ are defined as in the Example 3.9. We can see that X is O-complete (not complete), f is $\perp$-continuous (not continuous on X), an $\perp$-contraction and $\perp$-preserving on X. Applying Theorem 3.11, f has a unique fixed point in X, but, by Banach's contraction principle, we cannot find any fixed point of f in X.

4. Applications to ordinary differential equations

Our purpose here is to apply Theorem 3.11 to prove the existence of a solution for the following differential equation:

$$\begin{cases} u'(t) = f(t, u(t)), \ a.e. \ t \in I = [0, T], \\ u(0) = a, \ a \geq 1, \end{cases} \tag{4.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:

(c1) $f(s, x) \geq 0$ for all $x \geq 0$ and $s \in I$,

(c2) there exists $\alpha \in L^1(I)$ such that

$$|f(s, x) - f(s, y)| \leq \alpha(s)|x - y|$$

for all $t \in I$ and $x, y \geq 0$ with $xy \geq (x \vee y)$, where $x \vee y = x$ or y.

Note that $f : I \times \mathbb{R} \to \mathbb{R}$ is not necessarily Lipschitz from the given condition (c2). For example, the function

$$f(s, x) = \begin{cases} sx, \ x \leq \frac{1}{2}, \\ 0, \ x > \frac{1}{2} \end{cases}$$

satisfies the conditions (c1) and (c2) while f is not continuous and monotone. Also, for $s \neq 0$,

$$\left| f\left(s, \frac{1}{2}\right) - f\left(s, \frac{2}{3}\right) \right| = s\frac{1}{2} > s\frac{1}{6} = s\left|\frac{1}{2} - \frac{2}{3}\right|.$$

Theorem 4.1. *Under these conditions, for all $T > 0$, the differential equation 4.1 has a unique positive solution.*

Proof. Let $X = \{u \in C(I, \mathbb{R}) : u(t) > 0, \forall t \in I\}$. We consider the following orthogonality relation in X:

$$x \perp y \iff x(t)y(t) \geq (x(t) \vee y(t))$$

for all $t \in I$. Let $A(t) = \int_0^t |\alpha(s)|ds$. Then $A'(t) = |\alpha(t)|$ for almost every $t \in I$. Define

$$\|x\|_A = \sup_{t \in I} e^{-A(t)}|x(t)|, \quad d(x, y) := \|x - y\|_A$$

for all $x, y \in X$. It is easy to see that (X, d) is a metric space.

Now, we show that X is O-complete (not necessarily complete). Take a Cauchy O-sequence $\{x_n\}$ in X. It is easy to show that $\{x_n\}$ is convergent to a point $x \in C(I)$. It is enough to show the $x \in X$. Fix $t \in I$. The definition of $\perp$ implies that

$$x_n(t) \ x_{n+1}(t) \geq (x_n(t) \vee x_{n+1}(t))$$

for each $n \in \mathbb{N}$. Since $x_n(t) > 0$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{n_k}\}$ in $\{x_n\}$ for which $x_{n_k}(t) \geq 1$ for each $k \in \mathbb{N}$. The convergence of this sequence of real numbers to $x(t)$ implies that $x(t) \geq 1$. But since $t \in I$ is arbitrary, it follows that $x \geq 1$ and hence $x \in X$. Define a mapping $\mathcal{F} : X \to X$ by

$$\mathcal{F}u(t) = \int_0^t f(s, u(s))ds + a.$$

Note that the fixed points of $\mathcal{F}$ are the solutions of (4.1). To complete the proof, we need the following three steps:

Step 1: $\mathcal{F}$ is $\perp-$preserving. In fact, for all $x, y \in X$ with $x \perp y$ and $t \in I$,

$$\mathcal{F}x(t) = \int_0^t f(s, x(s))ds + a \geq 1,$$

which implies that $\mathcal{F}x(t)\mathcal{F}y(t) \geq \mathcal{F}x(t)$ and so $\mathcal{F}x \perp \mathcal{F}y$.

Step 2: $\mathcal{F}$ is $\perp-$contraction. In fact, for all $x, y \in X$ with $x \perp y$ and $t \in I$, the condition (c2) implies that

$$\begin{aligned} e^{-A(t)}|\mathcal{F}x(t) - \mathcal{F}y(t)| &\leq e^{-A(t)} \int_0^t |f(s, x(s)) - f(s, y(s))|ds \\ &\leq e^{-A(t)} \int_0^t |\alpha(s)|e^{A(s)}e^{-A(s)}\ |x(s) - y(s)|ds \\ &\leq e^{-A(t)} \Big(\int_0^t |\alpha(s)|e^{A(s)}ds \Big)\ \|x - y\|_A \\ &\leq e^{-A(t)}(e^{A(t)} - 1)\ \|x - y\|_A \\ &\leq (1 - e^{-\|\alpha\|_1})\ \|x - y\|_A \end{aligned}$$

and so

$$\|\mathcal{F}x - \mathcal{F}y\|_A \leq (1 - e^{-\|\alpha\|_1})\ \|x - y\|_A.$$

Since $1 - e^{-\|\alpha\|_1} < 1$, $\mathcal{F}$ is an $\perp-$contraction.

Step 3: $\mathcal{F}$ is $\perp-$continuous. In fact, let $\{x_n\} \subseteq X$ be an O-sequence converging to a point $x \in X$. By using the first part of the proof, we can see that $x(t) \geq 1$ for all $t \in I$ and hence $x_n \perp x$ for all $n \in \mathbb{N}$. Applying the condition (c2), we get

$$\begin{aligned} e^{-A(t)}|\mathcal{F}x_n(t) - \mathcal{F}x(t)| &\leq e^{-A(t)} \int_0^t |f(s, x_n(s)) - f(s, x(s))|ds \\ &\leq (1 - e^{-\|\alpha\|_1})\|x_n - x\|_A \end{aligned}$$

for all $n \in \mathbb{N}$ and $t \in I$. Hence we have

$$\|\mathcal{F}x_n - \mathcal{F}x\|_A \leq (1 - e^{-\|\alpha\|_1})\|x_n - x\|_A$$

for all $n \in \mathbb{N}$. It follows that $\mathcal{F}x_n \to \mathcal{F}x$.

The uniqueness of the solution follows from Theorem 3.11. This completes the proof. □

Acknowledgements. The authors would like to extend our thanks to the referee for his (her) valuable comments and suggestions for improving the results in this paper. The fourth author was supported by Basic Science Research Program through the

National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and future Planning (2014R1A2A2A01002100).

References

[1] J. Diestel, *Geometry of Banach Spaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[2] C.R. Diminnie, *A new orthogonality relation for normed linear spaces*, Math. Nachr., **114**(1983), 197–203.

[3] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations,* Proc. Amer. Math. Soc., **132**(2004), 1435–1443.

Received: October 20, 2014; Accepted: December 11, 2014.