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A NOTE ON KUHLMANN'S FIXED POINT THEOREMS

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Abstract. We show here how recently proven fixed point theorems by Kuhlmann and Kuhlmann can be derived from classical fixed point theorems from order theory.

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Kuhlmann and Kuhlmann have recently proven [4] a new type of fixed point theorems, that they refer to as the fixed point theorems for ball spaces. Their results are stated in a very general language of spaces which are neither metric nor even topological, but instead utilize the notion of so called balls, that is an arbitrary family of some subsets. The authors then proceed to apply their theorems to some valuation theoretic considerations. In this miniature note we investigate how their results relate to other familiar fixed point theorems known from the order theory.

Let X be a nonempty set. A **ball space** is a pair $(X, \mathcal{B})$, where $\mathcal{B}$ is a fixed family of nonempty subsets of X called **balls.** A **nest of balls** is any nonempty chain $\mathcal{N} \subset \mathcal{B}$ ordered by inclusion. If $f : X \to X$ is a mapping, a ball $B \in \mathcal{B}$ is called f-contracting if it is either a singleton consisting of a fixed point of f, or if $f(B) \subsetneq B$. The results in [4] are the following ones:

Theorem 1.1 ([4, Theorem 1]) If $(X, \mathcal{B})$ is a ball space and $f : X \to X$ a mapping such that:

(1) there exists an f-contracting ball,
(2) $f(B)$ contains an f-contracting ball, if $B \in \mathcal{B}$ is an f-contracting ball,
(3) for every nest of f-contracting balls $\mathcal{N}$, the intersection $\bigcap \mathcal{N}$ contains an f-contracting ball,

then f has a fixed point.

Theorem 1.2. ([4, Theorem 2]) If $(X, \mathcal{B})$ is a ball space and $f : X \to X$ a mapping such that:

(1) X is an f-contracting ball,
(2) $f(B)$ is an f-contracting ball, if $B \in \mathcal{B}$ is an f-contracting ball,
(3) for every nest of f-contracting balls $\mathcal{N}$, the intersection $\bigcap \mathcal{N}$ is an f-contracting ball,

then f has a unique fixed point.

Recall that a partial order is inductive if every chain (including the empty one) has an upper bound, and is chain-complete if every chain has a supremum. By convention, every element of a partial order is both an upper and lower bound of the empty subset, and hence the supremum of the empty chain exists if and only if there exists a bottom element $\perp$. Thus the conditions (1) and (3) of Theorem 1.1 guarantee that, for a fixed f-contracting ball B, the set of all f-contracting balls contained in B and reversely ordered by inclusion is inductive. Similarly, the conditions (1) and (3) of Theorem 1.2 imply that the set of all f-contracting balls is chain-complete.

We recall the following fixed point theorem that was proven by Tasković in [7], and can be also seen as a variant of the old result due to Bourbaki [1] and Witt [8]:

Theorem 1.3. Let $(P, \leq)$ be a nonempty inductive partial order, let $f : P \to P$ be a mapping such that for the set

$$\operatorname{Sub} f(P) = f(P) \cup \{a \in P : a = \operatorname{ub} C \text{ for some chain } C \text{ in } P\},$$

where $\operatorname{ub} C$ is an upper bound of C, the following condition is satisfied:

$$\forall a \in \operatorname{Sub} f(P) \ [f(a) \geq a].$$

Then f has a fixed point.

We shall show:

Theorem 1.4. Theorem 1.3 implies Theorem 1.1.

Proof. Let $(X, \mathcal{B})$ be a ball space and $f : X \to X$ a mapping satisfying the hypotheses of Theorem 1.1. By condition (1) of Theorem 1.1, there exists an f-contracting ball $B_0 \in \mathcal{B}$. Let $\mathcal{B}_0$ be the subset of $\mathcal{B}$ consisting of all f-contracting balls contained in B_0. Then, clearly, $(\mathcal{B}_0, \supseteq)$ is a nonempty partially ordered set. As noted before, $(\mathcal{B}_0, \supseteq)$ is inductive. Moreover, by condition (2) of Theorem 1.1, we may use the axiom of choice to construct the choice function F that assigns to a given $B \in \mathcal{B}_0$ an f-contracting ball contained in $f(B)$. Then, clearly, $F(B) \subseteq B$, for all $B \in \mathcal{B}_0$, and, consequently, F satisfies the hypotheses of Theorem 1.3 and hence has a fixed point $B' \in \mathcal{B}_0$. Thus $B' = F(B') \subseteq f(B') \subseteq B'$, so that $f(B') = B'$. By the definition of an f-contracting ball it follows that B' is a singleton consisting of a fixed point of f.

Recall that a partial order is complete if every directed set (including the empty one) has a supremum. By convention it follows that every complete partial order has a bottom element $\perp$. It can be shown using a certain version of Iwamura's Lemma [5, Theorem 1] that every chain-complete partial order is a complete partial order ([2, p. 33], [5, Corollary 2]). Thus the set of all f-contracting balls in Theorem 1.2 reversely ordered by inclusion is complete.

The Bourbaki-Witt Fixed Point Theorem can be also stated in the following form:

Theorem 1.5. Let $(P, \leq)$ be a nonempty complete partial order, let $f : P \to P$ be monotone. Then f has the least fixed point.

Giving a proper reference to the above stated result seems a bit problematic, as the history of fixed point theorems appears to be long and tangled. The Bourbaki-Witt Fixed Point Theorem seems to be the first that is substantially similar to the theorem stated here; although it requires f to be progressive, the proof is easily modified for the case where f is monotone. See [5, Theorem 9] for details. We also remark that the celebrated Tarski-Knaster Fixed Point Theorem for complete lattices [3], [6] is an easy consequence of Theorem 1.5 – again, see [5] for details.

With these remarks out of our way, we are now able to prove:

Theorem 1.6. Theorem 1.5 implies Theorem 1.2.

Proof. Let $(X, \mathcal{B})$ be a ball space and $f : X \to X$ a function satisfying the hypotheses of Theorem 1.2. Let $\mathcal{B}_0$ be the subset of $\mathcal{B}$ consisting of all f-contracting balls contained in B_0. By condition (1) of Theorem 1.2, $X \in \mathcal{B}_0$, so that $(\mathcal{B}_0, \supseteq)$ is a nonempty partially ordered set which, by previous remarks, is complete. Define the function $F : \mathcal{B}_0 \to \mathcal{B}_0$ by $F(B) = f(B)$, for all $B \in \mathcal{B}_0$. By condition (2) of Theorem 1.2, F is well-defined, and since, for $B_1 \supseteq B_2$, $B_1, B_2 \in \mathcal{B}_0$, we clearly have $f(B_1) \supseteq f(B_2)$, F is monotone. Therefore, by Theorem 1.5, F has the least fixed point B'. By the definition of an f-contracting ball it follows that B' is a singleton consisting of a fixed point of f, say x_0. If x_1 was another fixed point of f, then $B'' = \{x_1\} \in \mathcal{B}_0$ is a fixed point of F, so that $B' \supseteq B''$ forcing $x_0 = x_1$.

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