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FIXED POINT THEOREMS FOR SET-VALUED MAPPINGS AND VARIATIONAL PRINCIPLES IN UNIFORM SPACES WITH w-DISTANCES

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Abstract. By making use of w-distances, for set-valued mappings defined on uniform spaces, we generalize some classical results in the existing literature of nonlinear analysis, such as, Bishop-Phelps and Caristi's fixed point theorems, Ekeland's ϵ-variational principle and the nonconvex minimization theorem according to Takahashi. Our version of Caristi's fixed point theorem is used to prove existence of fixed points on uniform spaces for some contractions such as weak, Chatterjea and Kannan contractions defined by means of w-distances. The results introduced in this paper generalize others existing in the literature of nonlinear analysis.
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1. Introduction

Kada *et al.* in [14] introduced the concept of w-distance on a metric space. After that, a number of papers have been written by different authors, thereon we mention the references [17, 20, 21, 22, 23, 25]. The main interest by using this concept seems to be the generalization of some classical results in the existing literature of nonlinear analysis such as, Caristi's fixed point theorem [6], Ekeland's ϵ-variational principle [9, 10] and the nonconvex minimization theorem according to Takahashi [24]. Du in [8] proved the equivalence of these remarkable statements for w-distances on metric spaces. Alternatively, some articles such as [20, 23], have been devoted for characterizing metric completeness by means of w-distances and others, such as [16, 17, 18], have utilized this concept for proving existence of fixed points for set-valued mappings. All results of works mentioned so far are situated in the context of metric spaces.

In a natural way, w-distances defined on metric space can be extended to uniform spaces and the main aim of this paper is to extend some results for mappings defined on metric spaces to mappings defined on uniform spaces, by making use of the mentioned extension of w-distance. First, we prove Weston [26] and [3] Bishop-Phelps type theorem, which enable us to obtain new versions, on uniform spaces with w-distances, of Caristi's fixed point theorem [6], Ekeland's variational principle [10] and the nonconvex minimization theorem according to Takahashi [24]. Other authors have introduced related results on uniform spaces, which are different to ours, since their statements are not based in w-distances. For instance, Brøndsted in [5] extended the Bishop-Phelps type theorem in [3] to uniform spaces and a generalization of this result was given by Mizoguchi in [19], where also a Caristi's theorem type is presented for mappings defined on uniform spaces. Ekeland's variational principles in locally convex and uniform spaces have been treated by Hamel in [11] and [12], respectively. It is worth mentioning that in [1] some fixed point theorems are stated for mappings defined on uniform spaces and by means of a generalization of the w-distance concept. However, their assumptions are too restrictive due to strong continuity conditions are imposed on these mappings.

Our version of Caristi's fixed point theorem is used to prove existence of fixed points on uniform spaces for some contractions appearing frequently in the literature of nonlinear analysis, which in our case are defined by means of w-distances. Indeed, some fixed point theorems for set-valued mappings satisfying a Banach orbital condition with respect to a w-distance are proved. Among these types of extensions are those based on weak contractions as in Berinde [2] and the Chatterjea and Kannan contractions defined in [7] and [15], respectively. The results introduced in this paper generalize, inter alia, some of those presented by Shioji *et al.* [20] and Takahashi *et al.* [25].

Including this introduction, the paper is divided in four sections. In Section 2, some preliminary facts are stated in order to prove in the next section the main results of this paper. Indeed, Section 3 is devoted to state and prove Weston, Bishop-Phelps and Caristi type theorems, which are based on w-distances defined on uniform spaces. Also, in this section, versions of Ekeland's ϵ-variational principle and the nonconvex minimization theorem according to Takahashi, are stated. Finally, Section 4 is devoted to fixed point theorems for different set-valued contractions satisfying a Banach orbital condition with respect to a w-distance.

2. Preliminaries

In this section, X stands for a nonempty set, $f : X \to (-\infty, \infty]$ is a proper function bounded below and $p : X \times X \to [0, \infty)$ denotes a function satisfying the following two conditions:

(w1) $p(x, y) \leq p(x, z) + p(z, y)$, for any $x, y, z \in X$, and
(w2) if $p(u, u) = p(u, v) = 0$, then $u = v$.

Let $\mathrm{dom}(f) = \{y \in X : f(y) < \infty\}$. A relation $\preceq_f$ is defined on $\mathrm{dom}(f)$ as follows: $x \preceq_f y$, if and only if, $x = y$ or $f(y) + p(x, y) \leq f(x)$.

Proposition 2.1. *The relation* $\preceq_f$ *is an order relation on* $\mathrm{dom}(f)$.

Proof. It is clear that $\preceq_f$ is reflexive. Suppose $x \preceq_f y$ and $y \preceq_f x$ with $x \neq y$. We have

$$f(y) + p(x, y) \leq f(x) \leq f(y) - p(y, x)$$

and hence $0 \leq p(x, x) \leq p(x, y) + p(y, x) = 0$. Consequently, $p(x, x) = p(x, y) = 0$ and from (w2), $x = y$. Thus, $\preceq_f$ is antisymmetric. Next, suppose $x \preceq_f z$ and $z \preceq_f y$. If $x = z$ or $z = y$, it is clear that $x \preceq_f y$. Hence we assume $x \neq z$ and $z \neq y$. I.e. $f(z) + p(x, z) \leq f(x)$ and $f(y) + p(z, y) \leq f(z)$. From this we obtain, $f(y) + p(z, y) + p(x, z) \leq f(z) + p(x, z) \leq f(x)$ and consequently,

$$\begin{aligned} f(y) + p(x, y) &\leq f(y) + p(z, y) + p(x, z) \\ &\leq f(z) + p(x, z) \\ &\leq f(x). \end{aligned}$$

Therefore, the proof is complete. □

Remark 2.1. *Let f^* be the restriction of f to* $\mathrm{dom}(f)$. *Then, f^* is non-increasing.*

In what follows, we assume $\mathcal{U}$ is a uniformity defining a Hausdorff topology for X. Moreover, the function p is said to be a w-distance on X, if additionally to (w1) and (w2) the following two conditions hold:

(w3) for each $x \in X$, $p(x, \cdot)$ is lower semicontinuous, and
(w4) for each $U \in \mathcal{U}$, there exists $\delta > 0$ such that $p(z, x) < \delta$ and $p(z, y) < \delta$ imply $(x, y) \in U$.

Remark 2.2. *Notice that condition (w4) implies condition (w2).*

Let p be a w-distance on X. A p-Cauchy sequence in X is, naturally, defined as a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ satisfying $p(x_m, x_n) < \epsilon$, whenever $m, n \geq N$. We say X is p-complete whenever for any p-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$.

Lemma 2.1. *Let p be a w-distance on X, $\{x_n\}_{n\in\mathbb{N}}$ be a p-Cauchy sequence in X and $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be the filterbase in X defined as $B_n = \{x_m; m \geq n\}$. Then, $\mathcal{B}$ is a Cauchy filterbase.*

Proof. Let $U \in \mathcal{U}$ and $\delta > 0$ such that $p(z, u) < \delta$ and $p(z, v) < \delta$ imply $(u, v) \in U$. Since $\{x_n\}_{n\in\mathbb{N}}$ is a p-Cauchy sequence, there exists $N \in \mathbb{N}$ such that $p(x_m, x_n) < \delta$ whenever $m, n \geq N$. Hence, $B_N \times B_N \subseteq U$ due to $p(x_N, x_n) < \delta$ and $p(x_N, x_m) < \delta$ for $m, n \geq N$. Thus, $\mathcal{B}$ is a Cauchy filterbase in X, which completes the proof. □

Proposition 2.2 below states that the completeness respect to the uniformity of the space X is stronger that the p-completeness above defined.

Proposition 2.2. *Let $(X,\mathcal{U})$ be a complete Hausdorff uniform space. Then, for any w-distance p on X, X is p-complete.*

Proof. Let p be a w-distance on X, $\{x_n\}_{n\in\mathbb{N}}$ be a p-Cauchy sequence in X and $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be the filterbase defined as $B_n = \{x_m; m \geq n\}$. From Lemma 2.1, $\mathcal{B}$ is a Cauchy filterbase and since X is complete, there exists $x \in X$ such that $\mathcal{B}$ converges to x. Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $p(x_m, x_n) < \epsilon$ whenever $m, n \geq N$. From the lower semicontinuity of $p(x_m, \cdot)$, we have $p(x_m, x) \leq \epsilon$, for each $m \geq N$. Therefore, X is p-complete, which concludes the proof. $\square$

Proposition 2.3 below shows that p-completeness is preserved for closed subspaces.

Proposition 2.3. *Let $(X,\mathcal{U})$ be a p-complete Hausdorff uniform space and F a closed subset of X. Then, F is p-complete.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a p-Cauchy sequence in F. From assumption and Proposition 2.2, there exists $x \in X$ such that $p(x_n, x) \to 0$. Suppose $x \notin F$. Hence there exists $U \in \mathcal{U}$ such that $U[x] \cap F = \emptyset$, where $U[x] = \{y \in X : (x, y) \in U\}$. From (w4), there exists $\delta > 0$ such that $p(z, u) < \delta$ and $p(z, v) < \delta$ implies $(u, v) \in U$. Choose $N \in \mathbb{N}$ such that $p(x_N, x_m) < \delta$ and $p(x_m, x) < \delta$ whenever $m \geq N$. Hence $p(x_N, x_N) < \delta$ and $p(x_N, x) < \delta$. Consequently, $(x, x_N) \in U$, which is a contradiction due to $x_N \in F$. Therefore, $x \in F$ and the proof is complete. $\square$

Given a uniformity $\mathcal{U}$ on X, in Proposition 2.4 below we show that there exists a natural nontrivial w-distance compatible with this uniformity.

Proposition 2.4. *Let $(X,\mathcal{U})$ be a Hausdorff uniform space. Then, there exists a metric d which is a w-distance on X.*

Proof. From Theorem 1, Chapter IX in [4], there exists a family of pseudometrics $\{d_\lambda\}_{\lambda\in\Lambda}$ defining the uniformity $\mathcal{U}$. Let $d : X \times X \to \mathbb{R}$ be the metric defined as

$$d(x, y) = \sup_{\lambda\in\Lambda} d_\lambda(x, y) \wedge 1.$$

Since for each $x \in X$, $d_\lambda(x, \cdot) \wedge 1$ is continuous, we have $d(x, \cdot)$ is lower semicontinuous. Moreover, for each $U \in \mathcal{U}$, there exist $\lambda_1, \ldots, \lambda_r \in \Lambda$ and $\delta > 0$ such that $\{(x, y) \in X \times X : \max_{1\leq i\leq r} d_{\lambda_i}(x, y) < 2\delta\} \subseteq U$. Consequently $d(z, x) < \delta$ and $d(z, y) < \delta$ imply $(x, y) \in U$. Therefore, d is a w-distance and the proof is complete. $\square$

Remark 2.3. *Given a w-distance p on X and a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we say f is sequentially lower semicontinuous at $x \in X$, if and only if, for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} p(x_n, x) = 0$, one has $f(x) \leq \liminf f(x_n)$. It is easy to provide examples of w-distances showing that there is no general relationship between the lower semicontinuity of f with respect to the topology of X and the sequential semicontinuity with respect to the w-distance.*

3. Main results

Let $\mathcal{LS}(X)$ be the space of all lower semicontinuous proper functions defined on X and bounded below, i.e., $f \in \mathcal{LS}(X)$, if and only if, $f : X \to \mathbb{R} \cup \{+\infty\}$ is bounded below, $\mathrm{dom}(f)$ is nonempty and for each $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) > \alpha\}$ is open.

Let p be a w-distance on X and $f \in \mathcal{LS}(X)$. By a p-point for f, we mean a point $x^* \in X$ such that,

$$f(y) + p(x^*, y) > f(x^*), \quad \text{for all } y \neq x^*.$$

In the setting of the w-distances, we state the following Weston [26] type theorem, which in turn is an extension of the Bishop-Phelps lemma [3] and, in a some sense, extends the lemma by Mizoguchi in [19].

Theorem 3.1. *Let p be a w-distance on X and suppose $(X,\mathcal{U})$ is a p-complete Hausdorff uniform space. Then, the following two equivalent conditions hold:*

(3.1.1) *for each $f \in \mathcal{LS}(X)$ and $x_0 \in \operatorname{dom}(f)$ there exists a maximal element $x^* \in X$ such that $x_0 \preceq_f x^*$, and*

(3.1.2) *for each $f \in \mathcal{LS}(X)$ and $x_0 \in \operatorname{dom}(f)$ there exists a p-point $x^* \in X$ for f such that $x_0 \preceq_f x^*$.*

Reciprocally, if p is symmetric and these conditions hold, then X is p-complete.

Proof. It is clear that conditions (3.1.1) and (3.1.2) are equivalent.

Suppose $(X,\mathcal{U})$ is p-complete, fix $x_0 \in \operatorname{dom}(f)$ and let $S(x) = \{y \in X : x \preceq_f y\}$. If for some $x \in S(x_0)$, $f(y) + p(x,y) > f(x)$ for all $y \in S(x) \setminus \{x\}$, then x is maximal and we can take $x^* = x$. Next, if for each $x \in S(x_0)$, there exists $y \in S(x) \setminus \{x\}$ such that $f(y) + p(x,y) \leq f(x)$, we define recursively an increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\operatorname{dom}(f)$ by

$$x_n \in S(x_{n-1}) \setminus \{x_{n-1}\} \quad \text{with} \quad f(x_n) < L_n + 1/n,$$

where $L_n = \inf\{f(y) : y \in S(x_{n-1}) \setminus \{x_{n-1}\}\}$. Consequently, for each $n, p \in \mathbb{N} \setminus \{0\}$ and $y \in S(x_{n+p-1}) \setminus \{x_{n+p-1}\} \subseteq S(x_{n-1}) \setminus \{x_{n-1}\}$, we have

$$p(x_n, y) \leq f(x_n) - f(y) \leq f(x_n) - L_n < 1/n.$$

In particular, $p(x_n, x_{n+p}) < 1/n$ and hence $\{x_n\}_{n\in\mathbb{N}}$ is a p-Cauchy sequence. Thus there exists $x^* \in X$ such that $\lim_{n\to\infty} p(x_n, x^*) = 0$ and since for each $x \in X$, $S(x) = \{y \in X : f(y) + p(x,y) \leq f(x)\} \cup \{x\}$, the lower semicontinuity of $f + p(x,\cdot)$ implies $S(x)$ is a closed set. By Proposition 2.3, for any $n \in \mathbb{N}$, $x^* \in S(x_n)$ and thus $x_0 \preceq_f x_n \preceq_f x^*$. Suppose $y \in \operatorname{dom}(f)$ satisfies $x^* \preceq_f y$. Hence, $x^* = y$ or for each $n \in \mathbb{N}$, $p(x_n, y) \leq f(x_n) - f(y) < 1/n$. Consequently, $x^* = y$ or $\lim_{n\to\infty} p(x_n, y) = 0$. But from (w4), the limit respect to p is unique and thus $x^* = y$. Therefore $x^* \in \operatorname{dom}(f)$ is a maximal element satisfying $x_0 \preceq_f x^*$, which proves (3.1.1).

Suppose p is symmetric and that (3.1.2) holds. Let $\{x_n\}_{n\in\mathbb{N}}$ be a p-Cauchy sequence in X and observe that, by the symmetry of p, for each $x \in X$ and $m, n \in \mathbb{N}$,

$$|p(x, x_m) - p(x, x_n)| \leq p(x_m, x_n).$$

Hence, since there exists $\lim_{n\to\infty} p(x, x_n)$, we can define $f : X \to \mathbb{R}$ such that $f(x) = 2\lim_{n\to\infty} p(x, x_n)$. Moreover, for each $\alpha \in \mathbb{R}$,

$$\{x \in X : f(x) > \alpha\} = \bigcup_{n=0}^{\infty} \bigcup_{\gamma>\alpha} \bigcup_{m=n}^{\infty} \{x \in X : p(x, x_m) > \gamma\}$$

and since for each $m \in \mathbb{N}$, $p(\cdot, x_m) \in \mathcal{LS}(X)$, we have $f \in \mathcal{LS}(X)$. From (3.1.2) there exists $x^* \in X$ a p-point for f, which implies

$$f(x_n) + p(x^*, x_n) \geq f(x^*), \quad \text{for all } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} f(x_n) = 0$, we have $f(x^*) \geq 2f(x^*)$ and $f(x^*) = 0$. The symmetry of p implies that $\lim_{n\to\infty} p(x_n, x^*) = 0$. Therefore, X is p-complete, which concludes the proof. □

Remark 3.1. *Notice that the maximal element $x^* \in X$, in the preceding theorem, satisfies $p(x^*, x^*) = 0$, if and only if, there exists $y \in X$ such that $f(y) + p(x^*, y) \leq f(x^*)$.*

In the sequel, 2^X stands for the family of all nonempty subsets of X and in what follows of this section, p and $(X, \mathcal{U})$ stand for a w-distance on X and a p-complete Hausdorff uniform space, respectively.

A Caristi type theorem is stated as follows.

Theorem 3.2. *Let $T : X \to 2^X$ be a set-valued mapping and $f \in \mathcal{LS}(X)$ such that the following condition holds:*

(3.2.1) *for each $x \in X$, there exists $y \in Tx$ such that $f(y) + p(x, y) \leq f(x)$.*

Then, there exists $x^* \in X$ such that $f(x^*) < \infty$, $p(x^*, x^*) = 0$ and $x^* \in Tx^*$.

Proof. From (3.1.1), $\preceq_f$ has a maximal element $x^* \in \mathrm{dom}(f)$ and (3.2.1) implies that there exists $y \in Tx^*$ such that

$$f(y) + p(x^*, y) \leq f(x^*) < \infty. \tag{3.1}$$

Hence, $x^* \preceq_f y$ and the maximality of x^* implies $y = x^*$. Consequently, condition (3.1) becomes $f(x^*)+p(x^*, x^*) \leq f(x^*) < \infty$. Therefore, $p(x^*, x^*) = 0$ and $x^* \in Tx^*$, which concludes the proof. □

For single-valued mappings the following corollary holds.

Corollary 3.1. *Let $T : X \to X$ be a mapping and $f \in \mathcal{LS}(X)$ such that the following condition holds:*

(3.1.1) *for each $x \in X$, $f(Tx) + p(x, Tx) \leq f(x)$.*

Then, there exists $x^* \in X$ such that $f(x^*) < \infty$, $p(x^*, x^*) = 0$ and $x^* = Tx^*$.

An extension of the nonconvex minimization theorem according to Takahashi [24] can be stated as follows.

Theorem 3.3. *Let $f \in \mathcal{LS}(X)$ such that for any $u \in X$ which satisfies $\inf_{x\in X} f(x) < f(u) < \infty$, the following condition holds:*

(3.3.1) *there exists $v \in X \setminus \{u\}$ such that $f(v) + p(u, v) \leq f(u)$.*

Then there exists $y \in X$ such that $\inf_{x\in X} f(x) = f(y)$.

Proof. Suppose for every $y \in X$, $\inf_{x\in X} f(x) < f(y)$ and let $u \in \mathrm{dom}(f)$. From Theorem (3.1.1), $\preceq_f$ has a maximal element $x^* \in \mathrm{dom}(f)$ such that $u \preceq_f x^*$, and since $f(x^*) \leq f(u) < \infty$, (3.3.1) implies that there exists $v \in \mathrm{dom}(f) \setminus \{x^*\}$ such that $x^* \preceq_f v$. But, the maximality of x^* implies $v = x^*$, which is a contradiction. Therefore, there exists $y \in X$ such that $\inf_{x\in X} f(x) = f(y)$. This completes the proof. □

The following is an extension of Theorem 1.1 by Ekeland in [9].

Theorem 3.4. *Let $\epsilon > 0$, $\lambda > 0$ and $f \in \mathcal{LS}(X)$. Then, for every $u \in X$ satisfying $\inf_{x\in X} f(x) < f(u) < \inf_{x\in X} f(x)+\epsilon$, there exists $v \in X$ such that the following three conditions hold:*

(3.4.1) *$f(v) \leq f(u)$;*
(3.4.2) *$p(u, v) < \lambda$, whenever $u \neq v$; and*
(3.4.3) *for every $w \in X \setminus \{v\}$, $f(w) + (\epsilon/\lambda)p(v, w) > f(v)$.*

Proof. Let $u \in X$ satisfy $\inf_{x \in X} f(x) < f(u) < \inf_{x \in X} f(x) + \epsilon$ and $g = (\lambda/\epsilon)f$. Since $u \in \mathrm{dom}(f) = \mathrm{dom}(g)$ and $g \in \mathcal{LS}(X)$, (3.1.1) implies $\preceq_g$ has a maximal element $v \in \mathrm{dom}(f)$ such that $u \preceq_g v$. This fact implies that $u = v$ or $(\epsilon/\lambda)p(u, v) \leq f(u) - f(v) < \epsilon$, whenever $u \neq v$. Consequently, condition (3.4.2) holds and since f is non-increasing on $\mathrm{dom}(f)$, condition (3.4.1) so does. Finally, since v is a p-point for g, condition (3.4.3) follows and the proof is complete. □

4. Contractions

In this section, p stands for a w-distance on X.

Let $k \in [0, 1)$. A set-valued mapping $T : X \to 2^X$, with nonempty images, is said to satisfy the (p, k)-Banach orbital condition, if for any $u_0 \in X$ and $u_1 \in Tu_0$, there exists $u_2 \in Tu_1$ such that $p(u_1, u_2) \leq kp(u_0, u_1)$. The mapping T is said to be a (p, k)-contraction, if for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$, there exists $y_2 \in Tx_2$ such that $p(y_1, y_2) \leq kp(x_1, x_2)$.

Remark 4.1. *Each (p, k)-contraction satisfies the (p, k)-Banach orbital condition.*

Theorem 4.1. *Suppose $(X, \mathcal{U})$ is a complete Hausdorff uniform space, $k \in [0, 1)$ and $T : X \to 2^X$ is a set-valued mapping satisfying the (p, k)-Banach orbital condition. Then, there exists a symmetric w-distance q on X, such that*

(4.1.1) *T is a (q, k)-contraction; and*
(4.1.2) *there exists $x^* \in X$ such that $p(x^*, x^*) = q(x^*, x^*) = 0$ and $x^* \in Tx^*$.*

Proof. For a given $u_0 \in X$ fixed, there exists a sequence $\{u_n\}_{n \in \mathbb{N}}$ in X such that $u_{n+1} \in Tu_n$ and $p(u_n, u_{n+1}) \leq k^n p(u_0, u_1)$. Thus, for any $m, n \in \mathbb{N}$ with $m \geq n$,

$$p(u_n, u_m) \leq \frac{k^n}{1-k} p(u_0, u_1). \tag{4.2}$$

Let $\beta : X \to \mathbb{R}$ be the function defined by $\beta(x) = \lim_{n \to \infty} p(u_n, x)$ and $q : X \times X \to [0, \infty)$ such that $q(x, y) = \beta(x) + \beta(y)$. We need to prove that β is well-defined and q is a w-distance. From (4.2), for each $x \in X$ and $m, n \in \mathbb{N}$, we have

$$|p(u_n, x) - p(u_m, x)| \leq \frac{k^{\min\{m,n\}}}{1-k} p(u_0, u_1).$$

Hence, β is well-defined and for each $\alpha \in \mathbb{R}$,

$$\{x \in X : \beta(x) > \alpha\} = \bigcup_{n=0}^{\infty} \bigcup_{\gamma > \alpha} \bigcup_{m=n}^{\infty} \{x \in X : p(u_m, x) > \gamma\}.$$

Consequently, β is lower semicontinuous, which additionally implies that q satisfies condition (w3). Clearly, q is symmetric and enjoys condition (w1). Let $U \in \mathcal{U}$ and choose $\delta > 0$ such that $p(z, x) < \delta$ and $p(z, y) < \delta$ imply $(x, y) \in U$. Suppose $q(z, x) < \delta$ and $q(z, y) < \delta$. Accordingly, $\beta(x) < \delta$ and $\beta(y) < \delta$, and thus, there exists $n_0 \in \mathbb{N}$ such that $p(u_{n_0}, x) < \delta$ and $p(u_{n_0}, y) < \delta$. Hence $(x, y) \in U$, which proves that

q is a symmetric w-distance on X and it is easy to see that T is a (q,k)-contraction. This proves (4.1.1). Let $f : X \to (-\infty, \infty]$ such that $f(x) = (1+k)\beta(x)/(1-k)$. For each $x \in X$, we can choose any $y \in Tx$ and, since $\beta(y) \leq k\beta(x)$, we have $f(y) + q(x,y) \leq f(x)$. By Proposition 2.2, X is q-complete and Theorem 3.2 can be applied. Consequently, there exists $x^* \in X$ such that $f(x^*) < \infty$, $q(x^*, x^*) = 0$ and $x^* \in Tx^*$. Moreover, $p(x^*, x^*) \leq kp(x^*, x^*)$ and hence $p(x^*, x^*) = 0$, which concludes the proof. □

From Remark 4.1, we have the following corollary.

Corollary 4.1. *Suppose $(X, \mathcal{U})$ is a complete Hausdorff uniform space, $k \in [0,1)$ and $T : X \to 2^X$ is a (p,k)-contraction. Then, there exists $x^* \in X$ such that $x^* \in Tx^*$ and $p(x^*, x^*) = 0$.*

For single-valued contraction mappings, uniqueness of the fixed point is obtained as follows.

Corollary 4.2. *Suppose $(X, \mathcal{U})$ is a complete uniform space, $k \in [0,1)$ and $T : X \to X$ is a function satisfying $p(Tx, Ty) \leq kp(x,y)$, for all $x, y \in X$. Then there exists a unique $x^* \in X$ such that $x^* = Tx^*$. Moreover $p(x^*, x^*) = 0$.*

Proof. We only need to prove the uniqueness. If $y^* \in X$ is another point satisfying $y^* = Ty^*$, we have

$$p(x^*, y^*) = p(Tx^*, Ty^*) \leq kp(x^*, y^*)$$

and hence $p(x^*, y^*) = 0$. This fact along with $p(x^*, x^*) = 0$ and (w2) imply $x^* = y^*$, concluding the proof. □

In [25], Takahashi et al. defined the concept of Kannan mappings for w-distances in metric spaces. For $\alpha \in [0, 1/2)$ and a w-distance p on a uniform space X, within our context, a set-valued mapping $T : X \to 2^X$, with nonempty and closed images, is said to be a (p, α)-Kannan mapping, if for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$, there exists $y_2 \in Tx_2$ such that $p(y_1, y_2) \leq \alpha(p(x_1, y_1) + p(x_2, y_2))$. It is easy to see that each (p, α)-Kannan mapping satisfies the (p,k)-Banach orbital condition. Consequently, the following two corollaries hold.

Corollary 4.3. *Suppose $(X, \mathcal{U})$ is a complete Hausdorff uniform space, $\alpha \in [0, 1/2)$ and $T : X \to 2^X$ is a (p, α)-Kannan mapping. Then, there exists $x^* \in X$ such that $x^* \in Tx^*$ and $p(x^*, x^*) = 0$.*

Corollary 4.4. *Suppose $(X, \mathcal{U})$ is a complete Hausdorff uniform space, $\alpha \in [0, 1/2)$ and $T : X \to X$ is a single-valued (p, α)-Kannan mapping, i.e. $p(Tx, Ty) \leq \alpha(p(x, Tx) + p(y, Ty))$, for all $x, y \in X$. Then there exists a unique $x^* \in X$ such that $x^* = Tx^*$. Moreover $p(x^*, x^*) = 0$.*

The concept of weak contraction based in metrics was introduced by Berinde in [2] and it is naturally extended for w-distances. Let $\delta \in (0,1)$ and $L \geq 0$. We say $T : X \to 2^X$ is a (p, δ, L)-weak contraction, if for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$, there exists $y_2 \in Tx_2$ such that $p(y_1, y_2) \leq \delta p(x_1, x_2) + Lp(x_2, y_1)$. In order to this type of mappings satisfy a (p,k)-Banach orbital condition, for $k \in (0,1)$, we need to aggregate an additional property to p. On the other hand, a set-valued extension of the concept of p-contractively nonspreading mapping for w-distances, defined in [13] (see also [25]), is given as follows: let $\alpha \in [0,1)$ and $T : X \to 2^X$ be a set-valued mapping. We say T is (p,α)-contractively nonspreading mapping, if for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$, there exists $y_2 \in Tx_2$ such that $p(y_1, y_2) \le \alpha(p(x_1, y_2) + p(y_1, x_2))$. It is easy to see, whenever p is symmetric, that a (p,α)-contractively nonspreading mapping is a (p,δ,L)-weak contraction, where $\delta = \alpha/(1-\alpha)$ and $L = 2\alpha/(1-\alpha)$.

Proposition 4.1. *Let $(X,\mathcal{U})$ be a complete Hausdorff uniform space, $\delta \in [0,1)$ and $T : X \to 2^X$ be a (p,δ,L)-weak contraction. Suppose for each $x \in X$, $p(x,x) = 0$. Then, there exists a symmetric w-distance q on X, such that*

(4.1.1) *T is a (q,δ)-contraction; and*
(4.1.2) *there exists $x^* \in X$ such that $p(x^*,x^*) = q(x^*,x^*) = 0$ and $x^* \in Tx^*$.*

Proof. For each $u_0 \in X$ and $u_1 \in Tu_0$, there exists $u_2 \in Tu_1$ such that $p(u_1,u_2) \le \delta p(u_0,u_1) + Lp(u_1,u_1)$, i.e. $p(u_1,u_2) \le \delta p(u_0,u_1)$. Hence, T satisfies the (p,δ)-Banach orbital condition and the result follows from Theorem 4.1. □

Proposition 4.2. *Let $(X,\mathcal{U})$ be a complete Hausdorff uniform space, $\alpha \in [0,1/2)$ and $T : X \to 2^X$ be a (p,α)-contractively nonspreading mapping. Suppose for each $x \in X$, $p(x,x) = 0$. Then, there exists a symmetric w-distance q on X, such that*

(4.2.1) *T is a $(q,\alpha/(1-\alpha))$-contraction; and*
(4.2.2) *there exists $x^* \in X$ such that $p(x^*,x^*) = q(x^*,x^*) = 0$ and $x^* \in Tx^*$.*

Proof. Similar to the proof of Proposition 4.1. □

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