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CONVERGENCE OF MANN'S ITERATION FOR RELATIVELY NONEXPANSIVE MAPPINGS

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Abstract. We consider the Mann's iterative process to approximate the fixed points and best proximity points of a relatively non-expansive mapping $T : A \cup B \to A \cup B$, satisfying $\|Tx - Ty\| \leq \|x - y\|$ $\forall\ x \in A, y \in B$. These mappings need not be continuous.
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1. Introduction

It is well known that if T is a non expansive self map of a closed bounded convex subset of a uniformly convex Banach space and $T(A)$ is contained in a compact subset, then the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,\ \ \alpha_n \in (\epsilon, 1 - \epsilon),\ \ n = 1, 2, \dots$$

and $\epsilon > 0$ given, converges to a fixed point of T. In [1], the authors have obtained a convergence result based on the Krasnoselkii iteration

$$x_{n+1} = \frac{x_n + Tx_n}{2}$$

for relatively nonexpansive mappings $T : A \cup B \to A \cup B$, satisfying
i) $T(A) \subseteq A$ and $T(B) \subseteq B$ and
ii) $\|Tx - Ty\| \leq \|x - y\|\ \forall\ x \in A, y \in B$.
These results are interesting because such mappings need not be nonexpansive, in fact they need not be even continuous. In this paper we use the Mann's iterative process to obtain an extended version of Theorem 2.3 of [1], where the initial point belongs to A_0, which is contained in the boundary of the set A. We also discuss a stronger iteration which converges to a fixed point in a Hilbert space setting. Here the initial point is chosen arbitrarily. To prove our result, we assume the convergence of von Neumann sequences.

We state some of the results proved in that paper. Before that we give some basic notations. Define

$$P_A(x) = \{y \in A : \|x - y\| = d(x, A)\};$$
$$dist(A, B) = inf\{\|x - y\| : x \in A, y \in B\};$$
$$A_0 = \{x \in A : \|x - y'\| = dist\,(A, B) \text{ for some } y' \in B\}\,;$$
$$B_0 = \{y \in B : \|x' - y\| = dist\,(A, B) \text{ for some } x' \in A\}\,.$$

$P_A(x)$ is singleton when A is a closed convex subset of a strictly convex and reflexive space, and if A and B are closed, convex subsets of a reflexive space with one of them being bounded, then A_0 is nonempty.

Now let us see some basic concepts and known results which are related to our work.

Let A and B be nonempty subsets of a Banach Space X.

A mapping $T : A \cup B \to A \cup B$ is relatively non expansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x \in A,\ y \in B.$$

These type of mappings were studied in [1].

Example 1.1. Let $X = R$, $A = [-2, -1]$, $B = [1, 2]$.
Define T_1 and T_2 on A and B respectively by

$$T_1(x) = x + \frac{(1 - |2x + 3|)}{2} \text{ and } T_2(x) = x - \frac{(1 - |2x - 3|)}{2}.$$

Then both T_1 and T_2 are self maps on A and B. Now let $x \in A$ and $y \in B$,

$$\begin{aligned} |T_1(x) - T_2(y)| &= |x - y + 1/2(2 - |2x + 3| - |2y - 3|)| \\ &\leq |x - y|, \end{aligned}$$

since $x - y$ is ≤ 0 and both $|2x + 3|$, $|2y - 3|$ are ≤ 1. Here both T_1 and T_2 are not nonexpansive.

Theorem 1.2. [1] *Let A and B be nonempty closed convex bounded subsets of a uniformly convex Banach Space. Let $T : A \cup B \to A \cup B$ satisfy*

(i) $T\,(A) \subseteq B$ *and* $T\,(B) \subseteq A$; *and*

(ii) $\|Tx - Ty\| \leq \|x - y\|$ *for* $x \in A$, $y \in B$.

Then there exists $(x, y) \in A \times B$ such that $\|x - Tx\| = \|y - Ty\| = dist\,(A, B)$.

Such a point 'x' is called as a best proximity point.

Theorem 1.3. [1] *Let A and B be nonempty closed convex bounded subsets of a uniformly convex Banach Space. Suppose $T : A \cup B \to A \cup B$ satisfies*

(i) $T\,(A) \subseteq A$ *and* $T\,(B) \subseteq B$; *and*

(ii) $\|Tx - Ty\| \leq \|x - y\|$ *for* $x \in A$, $y \in B$.

Then there exist $x_0 \in A$ and $y_0 \in B$ such that

$$Tx_0 = x_0,\ Ty_0 = y_0,\ \textit{and}\ \|x_0 - y_0\| = dist\,(A, B)\,.$$

The following three results based on Mann's iteration process are well known.

Theorem 1.4. [6] *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space and suppose $T : K \to K$ is a non-expansive mapping. Let $x_0 \in K$ and define*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ \alpha_n \in (\epsilon, 1-\epsilon),\ n = 1, 2, \ldots$$

and $\epsilon > 0$ given. Then

$$\lim_n \|x_n - Tx_n\| = 0.$$

Moreover, if $T(K)$ lies in a compact set, $\{x_n\}$ converges to a fixed point of T.

Theorem 1.5. [1] *Let A and B be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $T : A \cup B \to A \cup B$ satisfies*

(i) *$T(A) \subseteq A$ and $T(B) \subseteq B$; and*

(ii) *$\|Tx - Ty\| \leq \|x - y\|$ for $x \in A$, $y \in B$.*

Let $x_0 \in A_0$. Define $x_{n+1} = \dfrac{x_n + Tx_n}{2}$, $n = 1, 2, \ldots$. Then

$$\lim_n \|x_n - Tx_n\| = 0.$$

Moreover, if $T(A)$ lies in a compact set, $\{x_n\}$ converges to a fixed point of T.

Proposition 1.6. [4] *If X is a uniformly convex space and $\alpha \in (0,1)$ and $\epsilon > 0$, then for any $d > 0$, if $x, y \in X$ are such that $\|x\| \leq d$, $\|y\| \leq d$, $\|x - y\| \geq \epsilon$, then there exists $\delta = \delta\left(\frac{\epsilon}{d}\right) > 0$ such that*

$$\|\alpha x + (1-\alpha)y\| \leq \left(1 - 2\delta\left(\frac{\epsilon}{d}\right) min(\alpha, 1-\alpha)\right) d.$$

Suppose X is a Hilbert Space and A is a closed convex subset of X. Then for any $x \in X$, $P_A(x)$ is the unique point of A which is nearest to x. It is well known that P_A is nonexpansive and characterized by the Kolmogorov's criterion:
$\langle x - P_A x, P_A x - z\rangle \geq 0$, for all $x \in X$ and $z \in A$.

Let A and B be two closed convex subsets of X. Suppose we define

$$P(x) = P_A(P_B(x)) \text{ for each } x \in X,$$

then the sequences $\{P^n(x)\} \in A$ and $\{P_B(P^n(x))\} \in B$. These sequences were first studied by von Neumann [7], who proved that both the sequences converges in norm whenever A and B are closed subspaces. The sequences $\{P^n(x)\}$ and $\{P_B(P^n(x))\}$ are called *von Neumann sequences* or *alternating projection algorithm* for two sets.

The norm convergence of $\{P^n(x)\}$ when A and B are arbitrary closed convex subsets was an open problem before it was answered in negative by Hundal in [5]. However, one can find some positive results when the sets A or B is boundedly compact. We summarize below some of the important facts from [3]:

(i) $\{P^n(x)\}$ converges weakly to some $y_0 \in A_0$ and $\{P_B(P^n(x))\}$ converges weakly to some $w_0 \in B_0$.

(ii) $P^n(x) - P_B(P^n(x)) \to y_0 - w_0$.

(iii) $P^n(x)$, $P_B(P^n(x))$ converges in norm whenever A or B is boundedly compact.

The most general result for the norm convergence was given by Bauschke and Borwein in [3].

Definition 1.7. [3] Let A and B be nonempty closed convex subsets of a Hilbert space X. We say that (A, B) is boundedly regular if for each bounded subset S of X and for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\max\{d(x, A), d(x, B - v)\} \leq \delta \Rightarrow d(x, B) \leq \epsilon \, \forall \, x \in X,$$

where $v = P_{\overline{B-A}}(0)$, the displacement vector from A to B. (v is the unique vector satisfying $\|v\| = dist(A, B)$).

Theorem 1.8. [3] *If (A, B) is boundedly regular, then the von Neumann sequences converges in norm.*

Theorem 1.9. [3] *If A or B is boundedly compact, then (A, B) is boundedly regular.*

Lemma 1.10. [2] *Let A be a nonempty closed and convex subset and B be nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$and$\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*
(i) $\|x_n - y_n\| \to dist(A, B)$, *and*
(ii) $\|z_n - y_n\| \to dist(A, B)$. *Then $\|x_n - z_n\|$ converges to zero.*

Proposition 1.11. [1] *Let A and B be two closed and convex subsets of a Hilbert space X. Then $P_B(A) \subseteq B$, $P_A(B) \subseteq A$, and $\|P_B x - P_A y\| \leq \|x - y\|$ for $x \in A$ and $y \in B$.*

Lemma 1.12. *Let A and B be two closed convex subsets of a Hilbert Space X. For each $x \in X$,*

$$\|P^{n+1}(x) - z\| \leq \|P^n(x) - z\|, \text{ for each } z \in A_0 \cup B_0.$$

Proof. The lemma follows from proposition [1.11] and from the fact that P_A is non-expansive. □

2. Main Results

Theorem 2.1. *Let A and B be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $T : A \cup B \to A \cup B$ satisfies*
(i) $T(A) \subseteq A$ *and* $T(B) \subseteq B$; *and*
(ii) $\|Tx - Ty\| \leq \|x - y\|$ *for* $x \in A$, $y \in B$.
Let $x_0 \in A$, and define

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \;\; \alpha_n \in (\epsilon, 1 - \epsilon),$$

where $\epsilon \in (0, 1/2]$ and $n = 1, 2, \ldots$. Suppose $d(x_n, A_0) \to 0$, then

$$\lim_n \|x_n - Tx_n\| = 0.$$

Moreover, if $T(A)$ lies in a compact set, $\{x_n\}$ converges to a fixed point of T.

Proof. If $dist\,(A,B) = 0$, then $A_0 = B_0 = A \cap B$ and the conclusion follows from Theorem 1.4 and the fact that $T : A \cap B \to A \cap B$ is nonexpansive. So let us assume that $dist\,(A,B) > 0$. By Theorem 1.3 there exists $y \in B_0$ such that $Ty = y$. Since $\{\|x_n - y\|\}$ is nonincreasing, there exists $d > 0$ such that $\lim_n \|x_n - y\| = d$.
Suppose there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an $\varepsilon > 0$ such that $\|x_{n_k} - Tx_{n_k}\| \geq \varepsilon > 0$ for all k.
Since the modulus of convexity δ of X is an increasing function (and continuous) it is possible to choose $\xi > 0$ so small such that

$$\left(1 - c\delta\left(\frac{\varepsilon}{d+\xi}\right)\right)(d+\xi) < d, \text{ where } c > 0.$$

Choose k, such that $\|x_{n_k} - y\| \leq d + \xi$. From proposition 1.1,

$$\begin{aligned}
\|y - x_{n_{k+1}}\| &= \|y - ((1-\alpha_n)x_{n_k} + \alpha_n Tx_{n_k})\| \\
&= \|(1-\alpha_n)y + \alpha_n y - ((1-\alpha_n)x_{n_k} + \alpha_n Tx_{n_k})\| \\
&= \|(1-\alpha_n)(y - x_{n_k}) + \alpha_n(y - Tx_{n_k})\| \\
&\leq \left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\{\alpha_n, 1-\alpha_n\}\right)(d+\xi).
\end{aligned}$$

Since we can find $l > 0$ such that $2 \cdot \min\{\alpha_n, 1-\alpha_n\} \geq l$,

$$\left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\{\alpha_n, 1-\alpha_n\}\right)(d+\xi) \leq \left(1 - l\delta\left(\frac{\epsilon}{d+\xi}\right)\right)(d+\xi).$$

By choosing $\xi > 0$ as small as we wish, we get

$$\left(1 - l\delta\left(\frac{\epsilon}{d+\xi}\right)\right)(d+\xi) < d,$$

a contradiction. This proves that

$$\lim_n \|x_n - Tx_n\| = \lim_n \|x_n - x_{n+1}\| = 0.$$

If $T\,(A)$ is compact then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to a point $z \in A$.
Since $\lim_n \|x_n - Tx_n\| = 0$, $\|Tx_n - y\| \to d$, for any $y \in B_0$.
As $d(x_n, A_0) \to 0$, $\exists\ a_n \in A_0$ such that $\|x_n - a_n\| \to 0$.

$$\therefore a_{n_k} \to z, \text{ which implies } z \in A_0.$$

Thus $\exists\ w \in B_0$ such that $\|z - w\| = d(A,B)$.
But $\|x_n - w\| \to \|z - w\| = d(A,B) = d$. $\therefore x_n \to z$.
Also, $\|Tx_n - w\| \to d(A,B)$ and $\|Tx_n - Tw\| \leq \|x_n - w\| \to d(A,B)$, giving $Tw = w$.
It follows that $Tz = z$. □

Example 2.2. Let $X = R^2$,

$$\begin{aligned}
A &= \{(x,y) : -2 \leq x \leq -1, -1 \leq y \leq 1\} \text{ and} \\
B &= \{(x,y) : 1 \leq x \leq 2, -1 \leq y \leq 1\}.
\end{aligned}$$

Define

$$T : A \to A \text{ by } T(x, y) = \left(\frac{x-2}{3}, -y\right) \text{ and}$$
$$T : B \to B \text{ by } T(x, y) = \left(\frac{x+2}{3}, -y\right).$$

Let $(x, y) \in$ A , $(x', y') \in$ B. Then,

$$\begin{aligned}
\|T(x,y) - T(x',y')\| &= \left\| \left(\frac{x-2}{3}, -y\right) - \left(\frac{x'+2}{3}, -y'\right) \right\| \\
&= \left\| \frac{x - x' - 4}{3}, y' - y \right\| \\
&= \sqrt{\left(\frac{x - x' - 4}{3}\right)^2 + (y' - y)^2} \\
&\le \sqrt{(x - x')^2 + (y - y')^2}, \text{ since } x' - x \ge 2.
\end{aligned}$$

$$\therefore \|T(x,y) - T(x',y')\| \le \sqrt{(x - x')^2 + (y - y')^2}.$$

Hence T is a relatively non-expansive mapping.
Let $(x, y) \in A$ and set $x_1 = (1 - \alpha_1)x + \alpha_1 Tx$.
We have

$$Tx = \frac{x-2}{3}.$$

$$\therefore x_1 = \frac{x(3 - 2\alpha_1) - 2\alpha_1}{3}$$

Now,

$$Tx_1 = \frac{x_1 - 2}{3} = \frac{x(3 - 2\alpha_1) - 2\alpha_1 - 6}{9}$$

We have, $x_2 = (1 - \alpha_2)x - 1 + \alpha_2 Tx_1$

$$\therefore x_2 = \frac{x(3 - 2\alpha_1)(3 - 2\alpha_2) - 2\alpha_1(3 - 2\alpha_1) - 6\alpha_2}{9}$$

In general,

$$\begin{aligned}
x_n \quad = \quad & \frac{1}{3^n}\Big\{x(3 - 2\alpha_1)...(3 - 2\alpha_n) - 2\alpha_1(3 - 2\alpha_2)...(3 - 2\alpha_n) - 2.3\alpha_2(3 - 2\alpha_3) \\
& ...(3 - 2\alpha_n) - \cdots\cdots - 2.3^{n-2}\alpha_{n-1}(3 - 2\alpha_n) - 2.3^{n-1}\alpha_n\Big\}
\end{aligned}$$

To see $x_n \to -1$, set

$$\begin{aligned}
\beta_n = {} & 2\alpha_1(3 - 2\alpha_2)...(3 - 2\alpha_n) + 2.3\alpha_2(3 - 2\alpha_3)...(3 - 2\alpha_n) + 2.3^2\alpha_3(3 - 2\alpha_4) \\
& ...(3 - 2\alpha_n) + ... + 2.3^{n-2}\alpha_{n-1}(3 - 2\alpha_n) + 2.3^{n-1}\alpha_n.
\end{aligned}$$

Then

$$x_n = \frac{x(3-2\alpha_1)(3-2\alpha_2)...(3-2\alpha_n)-\beta_n}{3^n}$$
$$= \frac{1}{3^n}(x(3-2\alpha_1)(3-2\alpha_2)...(3-2\alpha_n)) - \frac{\beta_n}{3^n}.$$

Since $\alpha_n \geq \epsilon$ for all n, $\dfrac{(3-2\alpha_1)(3-2\alpha_2)...(3-2\alpha_n)}{3^n} \to 0$.

Now, $3^n - 2.3^{n-1}\alpha_n = 3^{n-1}(3-2\alpha_n)$.

$$\therefore 3^n - 2.3^{n-1}\alpha_n - 2.3^{n-2}\alpha_{n-1}(3-2\alpha_n) = 3^{n-1}(3-2\alpha_n) - 2.3^{n-2}\alpha_{n-1}(3-2\alpha_n)$$

$$= 3^{n-2}(3-2\alpha_n)(3-2\alpha_{n-1})$$

Inductively.,

$$3^n - \beta_n = (3-2\alpha_n)(3-2\alpha_n-1)...(3-2\alpha_1).$$
$$\therefore \frac{3^n-\beta_n}{3^n} = \frac{(3-2\alpha_n)(3-2\alpha_n-1)...(3-2\alpha_1)}{3^n},$$

which converges to 0.

Thus $\dfrac{\beta_n}{3^n} \to$ to 1.

$$\therefore x_n = \frac{x(3-2\alpha_1)(3-2\alpha_2)...(3-2\alpha_n)}{3^n} - \frac{\beta_n}{3^n} \to -1.$$

Also, $y_n = y(1-2\alpha_1)(1-2\alpha_2)...(1-2\alpha_n) \to 0$.

Hence $(x_n, y_n) \to (-1, 0)$, which is a fixed point for T.

In a similar fashion we can show that if $(x', y') \in B$, then $(x'_n, y'_n) \to (1, 0)$, which is also a fixed point for T.

Corollary 2.3. *Let A and B be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $T : A \cup B \to A \cup B$ satisfies*

(i) *$T(A) \subseteq A$ and $T(B) \subseteq B$; and*

(ii) *$\|Tx - Ty\| \leq \|x - y\|$ for $x \in A$, $y \in B$.*

Let $x_0 \in A_0$, and define $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$, $\alpha_n \in (\epsilon, 1-\epsilon)$, $n = 1, 2, \ldots$ and $\epsilon > 0$ given. Then $\lim_n \|x_n - Tx_n\| = 0$. Moreover, if $T(A)$ lies in a compact set, $\{x_n\}$ converges to a fixed point of T.

The following modification of Example 1.1 shows it was necessary to choose the initial point x_0 in A_0. Let X=R, $A = [-2, 0]$ and $B = [1, 2]$.

Define $T : AUB \to AUB$ by

$$T(x) = x + \frac{(1 - |2x+3|)}{2}, x \in [-2, -1)$$
$$T(-1) = 0$$
$$T(x) = x, x \in (-1, 0]$$
$$T(x) = x - \frac{(1 - |2x-3|)}{2}, x \in [1, 2].$$

Then it is easy to see that $\|Tx - Ty\| \leq \|x - y\| \, \forall \, x \in A, \, y \in B$. Also T is not continuous, if $x_0 \in [-2, -1]$, then $x_n \to -1$, which is not a fixed point.
In the next result, we shall give a stronger algorithm in a Hilbert Space setting, where the initial point can be chosen arbitrarily.

Theorem 2.4. *Let A and B be two nonempty bounded closed convex subsets of a Hilbert space such that (A, B) is boundedly regular. Suppose $T : A \cup B \to A \cup B$ satisfies*

(i) $T(A) \subseteq A$ *and* $T(B) \subseteq B$; *and*

(ii) $\|Tx - Ty\| \leq \|x - y\|$ *for* $x \in A$, $y \in B$. *Let* $x_0 \in A$, *and define*

$$x_{n+1} = P^n \left((1 - \alpha_n) \, x_n + \alpha_n T x_n \right), \; n = 1, 2, \ldots, \; \alpha_n \in (\epsilon, 1 - \epsilon) \, .$$

Then $\lim_n \|x_n - Tx_n\| = 0$. *Moreover, if* $T(A)$ *lies in a compact set,* $\{x_n\}$ *converges to a fixed point of* T.

Proof. If $dist\,(A, B) = 0$, then $A_0 = B_0 = A \cap B$ and $T : A \cap B \to A \cap B$ is nonexpansive with $x_{n+1} = P^n \left((1 - \alpha_n) \, x_n + \alpha_n T x_n \right) = (1 - \alpha_n) \, x_n + \alpha_n T x_n$, the usual Mann's iteration. So we assume $dist\,(A, B) > 0$. By Theorem 1.3 there exists $y \in B_0$ such that $Ty = y$.

$$\begin{aligned}
\text{Now, } \|x_{n+1} - y\| &= \|P^n \left((1 - \alpha_n) \, x_n + \alpha_n T x_n \right) - y\| \\
&\leq \| \, (1 - \alpha_n) \, x_n + \alpha_n T x_n - y\| \\
&= \| \, (1 - \alpha_n) \, x_n + \alpha_n T x_n - \left((1 - \alpha_n) \, y + \alpha_n y \right) \| \\
&= \| \, (1 - \alpha_n) \, x_n + \alpha_n T x_n - \left((1 - \alpha_n) \, y + \alpha_n T y \right) \| \\
&= \| \, (1 - \alpha_n) \, (x_n - y) + \alpha_n \left(T x_n - T y \right) \| \\
&\leq (1 - \alpha_n) \, \|x_n - y\| + \alpha_n \|T x_n - T y\| \\
&\leq \|x_n - y\| \\
\text{Thus, } |x_{n+1} - y\| &\leq \|x_n - y\|
\end{aligned}$$

Hence $\{\|x_n - y\|\}$ is a non-increasing sequence and $\lim_n \|x_n - y\| = d > 0$.
As in the proof of Theorem 1.1, $\|x_n - Tx_n\|$ and $\|x_n - x_{n+1}\|$ converges to 0. If $T(A)$ is compact then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to a point $v_0 \in A$. Also $\{Tx_{n_k}\}$ converge to v_0. Now,

$$\|Tx_{n_k} - T(P_B(v_0))\| \leq \|x_{n_k} - P_B(v_0)\|$$

which implies

$$\|v_0 - T(P_B(v_0))\| \leq \|v_0 - P_B(v_0)\|.$$

Hence $T(P_B(v_0)) = P_B(v_0)$.
Also,

$$\|T(P(v_0)) - P_B(v_0)\| = \|T(P(v_0)) - T(P_B(v_0))\| \leq \|P(v_0) - P_B(v_0)\|.$$

So $T(P(v_0)) = P(v_0)$.
Now,

$$\|TP_B((P(v_0)) - P(v_0)\| = \|TP_B((P(v_0)) - T(P(v_0))\| \leq \|P_B(P(v_0)) - P(v_0)\|.$$

Thus $TP_B((P(v_0)) = P_B(P(v_0))$.
Indeed for any n, $T(P^n(v_0)) = P^n(v_0)$ and $T(P_B(P^n(v_0))) = P_B(P^n(v_0)$. By Theorem 1.8, for each $x \in A$ the sequence $\{P^n(x)\}$ converges to some $u(x) \in A_0$.

$$\begin{aligned} \text{Now, } \|u(v_0) - P_B(u(v_0))\| &\leq \lim_{n\to\infty} \|T(u(v_0)) - P_B(P^n(v_0))\| \\ &= \lim_{n\to\infty} \|T(u(v_0)) - T(P_B(P^n(v_0)))\| \\ &\leq \lim_{n\to\infty} \|u(v_0) - P_B(P^n(v_0))\| \\ &= \|u(v_0) - P_B(u(v_0))\|. \end{aligned}$$

So

$$\|T(u(v_0)) - P_B(u(v_0))\| = \|u(v_0) - P_B(u(v_0))\|.$$

Therefore $T(u(v_0)) = u(v_0)$ and similarly $T(P_B(u(v_0)) = P_B(u(v_0))$.
Define $g_n : A \to \mathbb{R}$ by $g_n(x) = \|P^n(x) - u(x)\|$.
Since $\|u(x)-u(y)\| = \lim_{n\to\infty} \|P^n(x)-P^n(y)\| \leq \|x-y\||$, u is continuous. Thus $g_n(x)$ is continuous and converges pointwise to zero. Since $u(x) \in A_0$, by lemma (1.12), $g_{n+1} \leq g_n$. Therefore g_n converges uniformly on the compact set

$$S = [\{(1-\alpha_{n_k})\, x_{n_k} + \alpha_{n_k} T x_{n_k}\} U\{v_0\}]\,.$$

$$\therefore \lim_{k\to\infty} \|P^{n_k}\left((1-\alpha_{n_k})\, x_{n_k} + \alpha_{n_k} T x_{n_k}\right) - u\left((1-\alpha_{n_k})\, x_{n_k} + T x_{n_k}\right)\| = 0.$$

Since $u\left((1-\alpha_n)\, x_{n_k} + \alpha_n T x_{n_k}\right) \to u(v_0)$, we have $x_{n_{k+1}} \to u(v_0)$.

$$\therefore \lim_n \|x_n - P_B(u(v_0))\| = \lim_k \|x_{n_{k+1}} - P_B(u(v_0))\| = d(A,B).$$

By Lemma 1.10 $x_n \to u(v_0)$, which implies $u(v_0) = v_0$.
Therefore $Tv_0 = Tu(v_0) = u(v_0) = v_0$.
This completes the proof. □

Suppose X is a Hilbert space and let T be as in Theorem 1.2. Consider

$$P_A T : A \to A \text{ and } P_B T : B \to B$$

From Proposition 1.11, $\|P_A T(x) - P_B T(y)\| \leq \|x-y\|$ for $x \in A$ and $y \in B$, by Theorem 2.1 and Theorem 2.4 we have the following two results on convergence of best proximity points.

Corollary 2.5. *Let A and B be nonempty, closed, bounded and convex subsets of a Hilbert space X. Let T be as in Theorem* 1.2. *If $T(A)$ is mapped into a compact subset of B, then for any $x_0 \in A_0$ the sequence defined by $x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n P_A\,(T x_n)$ converges to x in A_0 such that $\|x - Tx\| = d(A,B)$.*

Corollary 2.6. *Let A and B be nonempty, closed, bounded and convex subsets of a Hilbert space X. Let T be as in Theorem* 1.3. *If $T(A)$ is mapped into a compact subset of B, then for any $x_0 \in A$ the sequence defined by $x_{n+1} = P^n((1-\alpha_n)\, x_n + \alpha_n P_A\,(T x_n))$ converges to x in A_0 such that $\|x - Tx\| = d(A,B)$.*

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