Fixed Point Theory, 18(2017), No. 2, 503-522
DOI 10.24193/fpt-ro.2017.2.40
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

APPROXIMATING FIXED POINT SOLUTIONS OF VARIATIONAL INEQUALITIES USING EXPLICIT ITERATIONS FOR ASYMPTOTICALLY NONEXPANSIVE SEMIGROUP OF MAPPINGS IN BANACH SPACES

R.C. DIMRI AND PUSHPENDRA SEMWAL*

*Department of Mathematics, H.N.B. Garhwal University
Srinagar Garhwal, India-246174
E-mail: psrsdm@gmail.com

Abstract. Our purpose in this article is to introduce two new iterations and studying the strong convergence algorithms for finding approximating solutions of some variational inequalities on the set of common fixed point for a semigroup of asymptotically nonexpansive mappings. The results of this article improve and extend the recent work of Sunthrayuth and Kumam [21].
Key Words and Phrases: Common fixed point, variational inequality, semigroup of asymptotically nonexpansive mapping, strong convergence.
2010 Mathematics Subject Classification: 47H09, 47H10, 47H20, 47J20, 54H25.

1. Introduction

Construction of common fixed points of nonexpansive semigroups is an important subject in the theory of nonexpansive mappings and finds applications in a number of applied areas [2, 3, 17, 19, 27, 28], in particular, in image recovery and signal processing [8, 24, 25]. Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2 \text{ for all } x \in H. \tag{1.1}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \left\{ \frac{1}{2} \langle Ax, x \rangle - \langle x, u \rangle \right\}, \tag{1.2}$$

where C is the fixed point set of a nonexpansive mapping T on H, and u is a given point in H.

In 2003, Xu [26] proved that the sequence $\{x_n\}$ defined by the iterative method below with the initial guess $x_0 \in H$ choose arbitrarily

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u \ \forall n \geq 0, \tag{1.3}$$

converges strongly to the unique solution of the minimization problem (1.2), provided the sequence $\{\alpha_n\}$ satisfies certain conditions.Using the viscosity approximation method, Moudafi [15] introduced the following iterative process for nonexpansive mappings. Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, we define the sequence $\{x_n\}$ recursively by

$$x_{n+1} = \sigma_n f(x_n) + (1 - \sigma_n)Tx_n \ \forall n \geq 0, \tag{1.4}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved in [15, 25] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.4) strongly converges to a unique solution x^* of the variational inequality

$$\langle (f - I)x^*, x - x^* \rangle \leq 0 \ \ \forall x \in F(T). \tag{1.5}$$

In 2006, Marino and Xu [16] combined the iterative method (1.3) with the viscosity approximation method (1.4) considering the following general iterative process:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n \ \ \forall n \geq 0, \tag{1.6}$$

where $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. They proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a unique solution x^* of the variational inequality

$$\langle (\gamma f - I)x^*, x - x^* \rangle \leq 0 \ \ \forall x \in F(T), \tag{1.7}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \left\{ \frac{1}{2} \langle Ax, x \rangle - h(x) \right\},$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, Li et al.[13] considered the implicit and explicit viscosity iteration processes for a nonexpansive semigroup $S = \{T(t) : t \in R^+\}$ in a Hilbert space as follows:

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds \ \ \forall n \in N, \tag{1.8}$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds \ \ \forall n \in N, \tag{1.9}$$

where $\{\alpha_n\}$ and $\{t_n\}$ are two sequences satisfying certain conditions. They proved the sequence $\{x_n\}$ defined by (1.8) and (1.9) converges strongly to $x^* \in Fix(S)$, which solves the variational inequality (1.7). Recently Sunthrayuth and Kumam [20, 21] considered the same iteration processes (1.8) and (1.9) under the different conditions for $\{\alpha_n\}$ and $\{t_n\}$, proved the sequence $\{x_n\}$ defined by (1.8) and (1.9) converges strongly to $x^* \in Fix(S)$.

In this manuscript, motivated by the above results, we introduce two new iterative algorithms for finding a common fixed point of a semigroup of asymptotically nonexpansive mappings which is a unique solution of some variational inequality.We establish the strong convergence results in a uniformly convex Banach space.

2. Preliminaries

Throughout this manuscript, we denote N and R^+ the set of all positive integers and all positive real numbers respectively. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ weakly converges to x, as usual $x_n \to x$ will symbolize strong convergence. Let X be a real Banach space, X^* be its dual space. Let

$$U = \{x \in X : \|x\| = 1\}.$$

A Banach space X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists a $\delta > 0$ such that for each $x, y \in U$, $\|x - y\| \geq \epsilon$ implies

$$\frac{\|x + y\|}{2} \leq 1 - \delta.$$

We know that a uniformly convex Banach space is reflexive and strictly convex [22]. A Banach space is said to be smooth if the limit $\lim\limits_{t\to\infty} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. Let $J : X \to 2^{X^*}$ be normalized duality mappings by

$$J(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|\},$$

where X^* denotes the dual space of X, and $\langle\cdot,\cdot\rangle$ denotes the generalized duality paring. Let X be a real Banach space. A mapping $T : X \to X$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in X, \tag{2.1}$$

and T is asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers with $\lim\limits_{n} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \quad \forall n \geq 1 \text{ and } \forall x, y \in X. \tag{2.2}$$

We denote by $Fix(T)$ the set of fixed points of T. Recall that a self mapping $f : X \to X$ is L-Lipschitzian if there exists a constant $L \geq 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\| \text{ for all } x, y \in X.$$

Definition 2.1. A one parameter family $S = \{T(t) : t \in R^+\}$ of X into itself is said to be a strongly continuous semigroup of Lipschitzian mappings if the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in X$;

(ii) $T(s + t) = T(s)T(t)x$ for all $s, t \in R^+$;

(iii) for each $x \in X$ the mapping $T(\cdot)x$ from R^+ into X is continuous;

(iv) for each $t > 0$, there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that

$$\|T(t)x - T(t)y\| \leq L_t \|x - y\| \quad \forall x, y \in X.$$

A strongly continuous semigroup of Lipschitzian mappings S is called strongly continuous semigroup of nonexpansive mappings if $L_t = 1$ for all $t > 0$ and strongly continuous semigroup of asymptotically nonexpansive mappings if $\limsup_t L_t \leq 1$. Note that for the asymptotically nonexpansive semigroup S, we can always assume that the Lipschitzian constant $\{L_t\}_{t>0}$ is such that $L_t \geq 1$ for each $t > 0$, L_t is nonincreasing in t and $\lim_t L_t = 1$; otherwise, we replace L_t for each $t > 0$ with $\bar{L}_t = \max\{\sup_{s \geq t} L_s, 1\}$. S is said to have a fixed point if there exists $x_0 \in C$ such that $T(t)x_0 = x_0$ for all $t \geq 0$. We denote by $Fix(S)$ the set of fixed points of S, i.e.,

$$Fix(S) = \bigcap_{t \geq 0} F(T(t)) \quad [10-12].$$

A continuous operator of the semigroup $S = \{T(t) : t \in R^+\}$ is said to be uniformly asymptotically regular on X if for all $h \geq 0$ and any bounded subset C of X,

$$\lim_{t \to \infty} \sup_{x \in C} \|T(h)T(t)tx - T(t)x\| = 0$$

(see [1] for examples of uniformly asymptotically regular semigroups).

In [23], a Banach space X which admits a duality mapping J_ϕ with a gauge function ϕ, we say that an operator A is strongly positive if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J_\phi \rangle \geq \bar{\gamma} \|x\| \phi(\|x\|) \quad \forall x \in F(T) \tag{2.3}$$

and

$$\|aI - bI\| = \sup_{\|x\| \leq 1} |\langle (aI - bA)x, J_\phi \rangle|, \quad a \in [0, 1], b \in [-1, 1]. \tag{2.4}$$

As special cases of (2.3), we have the following results.

(1) If X is a smooth Banach space and $\phi(t) = t$ for all $t \in X$ [7], then the inequality (2.3) reduces to

$$\langle Ax, J(x) \rangle \geq \bar{\gamma} \|x\|^2 \quad \forall x \in F(T). \tag{2.5}$$

(2) If $X = H$ is a real Hilbert space, then the inequality (2.3) reduces to (1.1).

Lemma 2.2. [14] *Assume that a Banach space X has a weakly continuous duality mapping J_ϕ with a gauge ϕ, the following inequality/ identity holds:*

(i) For all $x, y \in X$, we have

$$\phi(\|x + y\|) \leq \phi(\|x\|) + \langle y, J_\phi(x + y) \rangle \quad \forall x \in F(T).$$

(ii) Assume that a sequence $\{x_n\}$ in X converges weakly to a point $x \in X$. Then:

$$\limsup_n \phi(\|x_n - y\|) = \limsup_n \phi(\|x_n - x\|) + \phi(\|y - x\|) \quad \forall x, y \in X.$$

Lemma 2.3. [23] *Assume that a Banach space X admits a duality mapping J_ϕ with a gauge ϕ. Let A be a strongly positive linear bounded operator on X with a coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \phi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \phi(1)(1 - \rho\bar{\gamma})$.*

Definition 2.4. [21] Let C be a closed convex subset of a real Banach space X. Let $S = \{T(t) : t \in R^+\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings from C into itself such that $Fix(S) \neq \phi$. Then S is said to be almost uniformly asymptotically regular on C, if for all $h \geq 0$,

$$\lim_t \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s)xds - T(h)\left(\frac{1}{t}\int_0^t T(s)xds\right) \right\| = 0. \tag{2.6}$$

Lemma 2.5. [29] *Let C be a closed convex subset of a uniformly convex Banach space X and $S = \{T(t) : t \in R^+\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings from C into itself with a sequence $\{L_t\} \subset [1, \infty)$ such that $Fix(S) \neq \phi$. Then for each $r > 0$ and $h \geq 0$,*

$$\lim_t \sup_{x \in C \cap B_r} \left\| \frac{1}{t} \int_0^t T(s)xds - T(h)\left(\frac{1}{t}\int_0^t T(s)xds\right) \right\| = 0. \tag{2.7}$$

Lemma 2.6. [22] *Let $\{x_n\}$ and $\{v_n\}$ be bounded sequences in Banach space X, let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$. Suppose*

$$x_{n+1} = (1 - \beta_n)v_n + \beta_n x_n$$

for all integers $n \geq 0$ and $\limsup_n(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then

$$\lim_n \|v_n - x_n\| = 0.$$

Lemma 2.7. [26] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \delta_n, \tag{2.8}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \sigma_n = \infty$;

(ii) $\limsup_n \frac{\delta_n}{\sigma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_n a_n = 0$.

3. Main results

Theorem 3.1. *Let X be a uniformly convex Banach space with a weakly continuous duality mapping J_ϕ with a gauge ϕ such that ϕ is invariant on $[0,1]$. Let*

$$S = \{T(t) : t \in R^+\}$$

be a strongly continuous semigroup of asymptotically nonexpansive mapping from X into itself with $\{L_t\} \subset [1, \infty)$ such that $Fix(S) \neq \phi$. Let $V : X \to X$ be a Lipschitzian *mapping with $L \geq 0$ and A be a strongly positive linear bounded operator with constant $\bar{\gamma} \in (0,1)$ such that $0 < \gamma < \frac{\phi(1)\bar{\gamma}}{L}$. For given $x_1 \in X$, let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = (1-\beta_n)x_n + \beta_n \left[\alpha_n \gamma V x_n + (I - \alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right)\right], \tag{3.1}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{t_n\}$ be a real divergent sequence with

(C_1) $\lim_n \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_n \beta_n = 1$,

(C_2) $\lim_n \dfrac{\frac{1}{t_n}\int_0^{t_n} L_s ds - 1}{\alpha_n} = 0.$

Then the sequence $\{x_n\}$ converges strongly to $x^* \in Fix(S)$ where x^* is the solution of the variational inequality

$$\langle \gamma V x^* - Ax^*, J_\phi(u - x^*)\rangle \leq 0 \quad \forall u \in Fix(S). \tag{3.2}$$

Proof. Using the condition (C_1), we may assume without loss of generality that

$$\alpha_n \leq \phi(1)\|A\|^{-1} \text{ for all } n \in N.$$

Using Lemma 2.3, we have

$$\|I - \rho A\| \leq \phi(1)(1 - \rho\bar{\gamma}).$$

First, we prove that $\{x_n\}$ is bounded. Take $p \in Fix(S)$ and $0 < \epsilon < \phi(1)\bar{\gamma} - \gamma L$. Since

$$\lim_{n\to\infty} \frac{\frac{1}{t_n}\int_0^{t_n} L_s ds - 1}{\alpha_n} = 0$$

we have

$$\frac{1}{t_n}\int_0^{t_n} L_s ds - 1 < \alpha_n \epsilon$$

for sufficiently large $n \geq 1$. Then from (3.1), we have

$$\|x_{n+1} - p\| = \left\|(1-\beta_n)x_n + \beta_n \left[\alpha_n \gamma V x_n + (I - \alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right) - p\right]\right\|$$

$$= \left\|(1-\beta_n)(x_n - p) + \beta_n \left[\alpha_n(\gamma V x_n - Ap) + (1 - \alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p\right)\right]\right\|$$

$$\leq (1-\beta_n)\|x_n - p\| + \beta_n \Big[\alpha_n \gamma L \|x_n - p\| + \phi(1)(1 - \alpha_n\bar{\gamma})$$

$$\times \left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)\|x_n - p\| + \alpha_n\|\gamma V p - Ap\|\Big]$$

$$= \left[1 + \beta_n\left(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1\right)\right.$$

$$-\alpha_n\beta_n\left(\bar{\gamma}\frac{1}{t_n}\int_0^{t_n}L_s ds-\gamma L\right)\Big]\|x_n-p\|+\alpha_n\beta_n\|\gamma Vp-Ap\|$$

$$\leq\left[1-\beta_n\left\{\alpha_n(\phi(1)\bar{\gamma}-\gamma L)-\left(\frac{1}{t_n}\int_0^{t_n}L_s ds-1\right)\right\}\right]\|x_n-p\|+\alpha_n\beta_n\|\gamma Vp-Ap\|$$

$$\leq[1-\alpha_n\beta_n(\phi(1)\bar{\gamma}-\gamma L-\epsilon)]\|x_n-p\|+\alpha_n\beta_n\|\gamma Vp-Ap\|$$

$$=[1-\alpha_n\beta_n(\phi(1)\bar{\gamma}-\gamma L-\epsilon)]\|x_n-p\|+\alpha_n\beta_n(\phi(1)\bar{\gamma}-\gamma L-\epsilon)\frac{\|\gamma Vp-Ap\|}{(\phi(1)\bar{\gamma}-\gamma L-\epsilon)}$$

$$\leq\max\left\{\|x_n-p\|,\frac{\|\gamma Vp-Ap\|}{(\phi(1)\bar{\gamma}-\gamma L-\epsilon)}\right\}.$$

By induction, we have

$$\|x_n-p\|\leq\max\left\{\|x_1-p\|,\frac{\|\gamma Vp-Ap\|}{(\phi(1)\bar{\gamma}-\gamma L-\epsilon)}\right\}.$$

Hence $\{x_n\}$ is bounded, so are $\{Vx_n\}$ and $\left\{A\left(\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\right)\right\}$.
Next, we shall show that $\|x_n-T(h)x_n\|\to 0$ as $n\to\infty$. From (3.1) we have

$$\|x_{n+1}-\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|\leq(1-\beta_n)\|x_n-\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|$$
$$+\alpha_n\beta_n\|\gamma Vx_n-A\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|.$$

From C_1 and C_2, we obtain

$$\lim_n\|x_{n+1}-\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|=0. \tag{3.3}$$

Now for all $h\geq 0$, we note that

$$\|x_{n+1}-T(h)x_{n+1}\|\leq\|x_{n+1}-\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|+\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds$$
$$-T(h)\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|+\|T(h)\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds-T(h)x_{n+1}\|$$
$$\leq\|x_{n+1}-\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|+\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds$$
$$-T(h)\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\|+L_h\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds-x_{n+1}\|.$$

From (3.3) and by Lemma 2.5, we get

$$\|x_{n+1}-T(h)x_{n+1}\|\to 0 \text{ as } n\to\infty \text{ for all } h\geq 0. \tag{3.4}$$

Hence $\lim_n\|x_n-T(h)x_n\|=0$ for all $h\geq 0$.

Next, we prove that $\limsup_n \langle \gamma V x^* - Ax^*, J_\phi(x_n - x^*)\rangle \leq 0$. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_n \langle \gamma V x^* - Ax^*, J_\phi(x_{n_j} - x^*)\rangle = \limsup_n \langle \gamma V x^* - Ax^*, J_\phi(x_{n_j} - x^*)\rangle.$$

By reflexivity of X and boundedness of $\{x_n\}$, there exists a weakly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup u \in X$ as $j \to \infty$. Since J_ϕ is weakly continuous, we have by Lemma 2.2 that

$$\limsup_n \phi(\|x_{n_j} - x\|) = \limsup_n \phi(\|x_{n_j} - u\|) + \phi(\|x - u\|), \quad \forall x \in X.$$

Let $G(x) = \limsup_j \phi(\|x_{n_j} - x\|)$ for all $x \in X$. Therefore

$$G(x) = G(u) + \phi(\|x - u\|) \text{ for all } x \in X.$$

Since ϕ is continuous and $\lim_h L_h = 1$, we have

$$\begin{aligned} G(\lim_h T(h)u) &= \lim_h G(T(h)u) \\ &= \lim_h \limsup_j \phi(\|x_{n_j} - T(h)u\|) \\ &= \lim_h \limsup_j \phi(\|T(h)x_{n_j} - T(h)u\|) \\ &\leq \lim_h \limsup_j \phi(L_h\|x_{n_j} - u\|) \\ &= \limsup_j \phi(\|x_{n_j} - u\|) \\ &= G(u). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} G(\lim_h T(h)u) &= \lim_h \limsup_j \phi(\|x_{n_j} - u\|) + \lim_h \phi(\|T(h)u - u\|) \\ &= \limsup_j \phi(\|x_{n_j} - u\|) + \phi(\lim_h \|T(h)u - u\|). \end{aligned} \tag{3.5}$$

Combining above, we obtain $\phi(\lim_h \|T(h)u - u\|) \leq 0$. The property of ϕ implies that $\lim_h T(h)u = u$. Since $T(t+h)x = T(t)T(h)x$ for all $x \in X$ and $t \geq 0$, we have

$$u = \lim_h T(h)u = \lim_h T(h+t)u = T(t)\lim_h T(h)u = T(t)u$$

for all $t \geq 0$. Hence $u \in Fix(S)$. Since J_ϕ is single-valued and weakly continuous, we obtain

$$\begin{aligned} \limsup_n \langle \gamma V x^* - Ax^*, J_\phi(x_n - x^*)\rangle &= \lim_n \langle \gamma V x^* - Ax^*, J_\phi(x_{n_j} - x^*)\rangle \\ &= \langle \gamma V x^* - Ax^*, J_\phi(u - x^*) \leq 0. \end{aligned}$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. Now from Lemma 2.2, we have

$$\phi(\|x_{n+1} - x^*\|) = \phi\Big(\Big\|(1-\beta_n)(x_n - x^*) + \beta_n\alpha_n\gamma(Vx_n - Vx^*)$$

$$+\beta_n(I-\alpha_n A)\Big(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*\Big) + \alpha_n\beta_n(\gamma Vx^* - Ax^*)\Big\|\Big)$$

$$\leq \phi\Big(\Big\|(1-\beta_n)(x_n - x^*) + \beta_n\alpha_n\gamma(Vx_n - Vx^*) + \beta_n(I-\alpha_n A)$$

$$\times\Big(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*\Big)\Big\|\Big) + \alpha_n\beta_n\langle\gamma Vx^* - Ax^*, J_\phi(x_{n+1}-x^*)\rangle$$

$$\leq \phi\Big(\Big\{1-\alpha_n\beta_n(\phi(1)\bar{\gamma}-\gamma L) + \beta_n\phi(1)(1-\alpha_n\bar{\gamma})\Big[\frac{1}{t_n}\int_0^{t_n} L_s ds - 1\Big](\|x_n - x^*\|)\Big\}\Big)$$

$$+\alpha_n\beta_n\langle\gamma Vx^* - Ax^*, J_\phi(x_{n+1}-x^*)\rangle$$

$$\leq (1-\alpha_n\beta_n(\phi(1)\bar{\gamma}-\gamma L))\phi(\|x_n - x^*\|) + \beta_n\phi(1)(1-\alpha_n\bar{\gamma})$$

$$\times\Big[\Big(\frac{1}{t_n}\int_0^{t-n} L_s ds\Big) - 1\Big]\phi(\|x_n - x^*\|) + \alpha_n\beta_n\langle\gamma Vx^* - Ax^*, J_\phi(x_{n+1}-x^*)\rangle$$

$$\leq (1-\alpha_n\beta_n(\phi(1)\bar{\gamma}-\gamma L))\phi(\|x_n - x^*\|) + \beta_n\phi(1)(1-\alpha_n\bar{\gamma})$$

$$\times\Big[\Big(\frac{1}{t_n}\int_0^{t-n} L_s ds\Big) - 1\Big]M + \alpha_n\beta_n\langle\gamma Vx^* - Ax^*, J_\phi(x_{n+1}-x^*)\rangle,$$

where $M = \sup_{n\geq 1}\phi(\|x_n - x^*\|)$. Put $\sigma_n = \alpha_n\beta_n(\phi(1)\bar{\gamma}-\gamma L)$ and

$$\delta_n = \beta_n\phi(1)(1-\alpha_n\bar{\gamma})\left[\left(\frac{1}{t_n}\int_0^{t-n} L_s ds\right) - 1\right]M + \alpha_n\beta_n\langle\gamma Vx^* - Ax^*, J_\phi(x_{n+1}-x^*)\rangle.$$

Thus, we have

$$\phi(\|x_{n+1}-x^*\|) \leq (1-\sigma_n)\phi(\|x_n - x*\|) + \delta_n.$$

It follows from condition C_1 and C_2, $\sum_{n=1}^{\infty}\sigma_n = \infty$ and

$$\limsup_n \frac{\delta_n}{\sigma_n} = \limsup_n \frac{1}{\phi(1)\bar{\gamma}-\gamma L}\Big[\frac{\phi(1)(1-\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1)M}{\alpha_n}$$
$$+ \langle\gamma Vx^* - Ax^*, J_\phi(x_{n+1}-x^*)\Big] \leq 0.$$

Hence by Lemma 2.3, we obtain $\phi(\|x_{n+1}-x^*\|) = 0$ as $n\to\infty$, and property of ϕ implies that $x_n \to x^*$ as $n\to\infty$. Finally, we prove the uniqueness of a solutions of the variational inequality (3.2). Suppose $x^*, x' \in Fix(S)$ are the solutions of (3.2), then we have

$$\langle\gamma Vx^* - Ax^*, J_\phi(x' - x^*)\rangle \leq 0, \tag{3.6}$$

and

$$\langle\gamma Vx' - Ax', J_\phi(x^* - x')\rangle \leq 0,. \tag{3.7}$$

Adding (3.6) and (3.7), we obtain

$$\begin{aligned}0 &\geq \langle(\gamma V x^* - Ax^*) - (\gamma V x' - Ax'), J_\phi(x' - x^*)\rangle \\ &= \langle A(x' - x^*), J_\phi(x' - x^*)\rangle - \gamma\langle V x' - V x^*, J_\phi(x' - x^*)\rangle \\ &\geq \bar{\gamma}\|x' - x^*\|\phi(\|x' - x^*\|) - \gamma L\phi(\|x' - x^*\|) \\ &\geq (\bar{\gamma} - \gamma L)\phi(\|x' - x^*\|) \\ &\geq (\phi(1)\bar{\gamma} - \gamma L)\phi(\|x' - x^*\|),\end{aligned}$$

a contradiction. Thus $x' = x^*$.

Theorem 3.2. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly Banach space X such that $C + C \subset C$. Let $S = \{T(t) : t \in R^+\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings from C into itself with a sequence $\{L_t\} \subset [1, \infty)$ such that $Fix(S) \neq \phi$. Let V be an L-Lipschitzian mapping from C into itself with a constant $L \geq 0$ and A be a strongly positive linear bounded operator with a constant $0 < \bar{\gamma} < 1$ such that $0 \leq \gamma L < \bar{\gamma}$. Let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n \gamma V x_n + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)y_n \tag{3.8}$$

$$y_n = \delta_n x_n + (1 - \delta_n)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds) \text{ for all } n \in N, \tag{3.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ and $\{t_n\}$ is a positive real divergent sequence such that $\lim_n \frac{t_n}{t_{n+1}} = 1$ satisfying the following conditions:

(C_1) $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = 0$ *and* $\lim_n \delta_n = 0$,

(C_2) $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$,

(C_3) $\lim_n \left(\frac{\frac{1}{t_n}\int_0^{t_n} L_s ds - 1}{\alpha_n}\right) = 0.$

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in Fix(S)$, where x^* is the unique solution of the variational inequality (3.2).

Proof. From the conditions (C_1) and (C_2), we may assume without loss of generality that

$$\alpha_n \leq (1 - \beta_n)\|A\|^{-1} \ \forall n \in N.$$

Since A is a strongly positive bounded linear operator on C, from (2.2) and (2.3), we have

$$\|A\| = sup\{|\langle Ax, J(x)\rangle| : x \in C, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1 - \beta_n)I - \alpha_n A)x, J(x)\rangle &= 1 - \beta_n - \alpha_n\langle Ax, J(x)\rangle \\ &\geq 1 - \beta_n - \alpha_n\|A\| \\ &\geq 0.\end{aligned}$$

This means that $((1-\beta_n)I-\alpha_n A)$ is positive. It follows that

$$\begin{aligned}\|((1-\beta_n)I-\alpha_n A)x\| &= \sup\{\langle((1-\beta_n)I-\alpha_n A)x, J(x)\rangle : x\in C, \|x\|=1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Ax, J(x)\rangle : x\in C, \|x\|=1\}\\ &\le 1-\beta_n-\alpha_n\bar{\gamma}.\end{aligned}$$

Firstly, we prove that $\{x_n\}$ defined by (3.8) and (3.9) is bounded. Take $q\in Fix(S)$ and $0<\epsilon<\bar{\gamma}-\gamma L$. Since

$$\lim_n\left(\frac{\frac{1}{t_n}\int_0^{t_n} L_s ds-1}{\alpha_n}\right)=0$$

implies that

$$(1-\beta_n-\alpha_n\bar{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s ds-1\right)<\epsilon\alpha_n$$

for sufficiently large $n\ge 1$, we have

$$\begin{aligned}\|y_n-q\| &= \left\|\delta_n x_n+(1-\delta_n)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right)-q\right\|\\ &\le \delta_n\|x_n-q\|+(1-\delta_n)\left\|\left(\frac{1}{t_n}\int_0^{t_n} L_s ds-q\right)\right\|\\ &\le \delta_n\|x_n-q\|+(1-\delta_n)\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)\|x_n-q\|\\ &= \|x_n-q\|+(1-\delta_n)\left(\frac{1}{t_n}\int_0^{t_n} L_s ds-1\right)\|x_n-q\|.\\ \|x_{n+1}-q\| &= \|\alpha_n\gamma Vx_n+\beta_n x_n+((1-\beta_n)I-\alpha_n A)y_n-q\|\\ &\le \alpha_n\|\gamma Vx_n-Aq\|+\beta_n\|x_n-q\|+(1-\beta_n-\alpha_n\bar{\gamma})\|y_n-q\|\\ &\le \alpha_n\gamma L\|x_n-q\|+\alpha_n\|\gamma Vq-Aq\|+\beta_n\|x_n-q\|\\ &\quad+(1-\beta_n-\alpha_n\bar{\gamma})\|y_n-q\|.\end{aligned}$$

Now, we have

$$\begin{aligned}\|x_{n+1}-q\| &= \|\alpha_n\gamma Vx_n+\beta_n x_n+((1-\beta_n)I-\alpha_n A)y_n-q\|\\ &\le \alpha_n\|\gamma Vx_n-Aq\|+\beta_n\|x_n-q\|+(1-\beta_n-\alpha_n\bar{\gamma})\|y_n-q\|\\ &\le \alpha_n\gamma L\|x_n-q\|+\alpha_n\|\gamma Vq-Aq\|+\beta_n\|x_n-q\|+(1-\beta_n-\alpha_n\bar{\gamma})\\ &\quad\times\left(\|x_n-q\|+(1-\delta_n)\left(\frac{1}{t_n}\int_0^{t_n} L_s ds-1\right)\|x_n-q\|\right)\\ &\le (1-\alpha_n(\bar{\gamma}-\gamma L-\epsilon))\|x_n-q\|-\delta_n\alpha_n\epsilon\|x_n-q\|+\alpha_n\|\gamma Vq-Aq\|\\ &\le (1-\alpha_n(\bar{\gamma}-\gamma L-\epsilon))\|x_n-q\|+\alpha_n\|\gamma Vq-Aq\|\\ &\le \max\left\{\|x_n-q\|,\frac{\|\gamma Vq-Aq\|}{\bar{\gamma}-\gamma L-\epsilon}\right\}\ \text{for all } n\ge 1.\end{aligned}$$

By induction, we have

$$\|x_{n+1} - q\| = \max\left\{\|x_1 - q\|, \frac{\|\gamma V q - Aq\|}{\bar{\gamma} - \gamma L - \epsilon}\right\}, \text{ for all } n \in N.$$

Hence the sequence $\{x_n\}$ is bounded, so are $\{Vx_n\}$ and $\left\{A\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right)\right\}$.
Next, we prove that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Now

$$\begin{aligned}
&\left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)x_{n+1}ds - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\| \\
&= \left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}[T(s)x_{n+1} - T(s)x_n]ds + \frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)x_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\| \\
&= \left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}[T(s)x_{n+1} - T(s)x_n]ds + \left(\frac{1}{t_n+1} - \frac{1}{t_n}\right)\int_0^{t_n} T(s)x_n ds\right. \\
&\qquad \left. + \frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)x_n ds\right\|.
\end{aligned}$$

For $q \in Fix(S)$, it follows that

$$\begin{aligned}
&\left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)x_{n+1}ds - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\| \\
&= \left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}[T(s)x_{n+1} - T(s)x_n]ds + \left(\frac{1}{t_n+1} - \frac{1}{t_n}\right)\int_0^{t_n}[T(s)x_n - q]ds\right. \\
&\qquad \left. + \frac{1}{t_{n+1}}\int_0^{t_{n+1}}[T(s)x_n - q]ds\right\| \\
&\le \left(\frac{1}{t_{n+1}}\int_0^{t_{n+1}} L_s ds\right)\|x_{n+1} - x_n\| + \left[\left|1 - \frac{t_n}{t_{n+1}}\right|\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)\right. \\
&\qquad \left. + \left|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} L_s ds - \left(\frac{t_n}{t_{n+1}}\right)\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)\right|\right] M, \qquad (3.10)
\end{aligned}$$

where $M > 0$ is a constant such that $M = \sup_{n\ge1}(\|x_n - q\|)$. Set

$$l_n = \frac{x_{n+1} - \beta_n x_n}{(1-\beta_n)} \ \forall n \in N.$$

Then

$$\begin{aligned}
l_{n+1} - l_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{(1-\beta_{n+1})} - \frac{x_{n+1} - \beta_n x_n}{(1-\beta_n)} \\
&= \frac{\alpha_n\gamma V x_{n+1} + ((1-\beta_{n+1})I - \alpha_{n+1}A)[\delta_{n+1}x_{n+1} + (1-\delta_{n+1})\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)x_{n+1}ds]}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma V x_n + ((1-\beta_n)I - \alpha_n A)[\delta_n x_n + (1-\delta_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds]}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{(1-\beta_{n+1})}\left\{\gamma V x_{n+1} - A\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)x_{n+1}ds\right\}
\end{aligned}$$

$$+\frac{\alpha_n}{(1-\beta_n)}\left\{A\frac{1}{t_n}\int_0^{t_n}T(s)x_nds-\gamma Vx_n\right\}$$
$$+\frac{\alpha_{n+1}\delta_{n+1}}{(1-\beta_{n+1})}\left\{A\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds-Ax_{n+1}\right\}$$
$$+\frac{\alpha_n\delta_n}{(1-\beta_n)}\left\{Ax_n-A\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\}$$
$$+\delta_{n+1}\left\{x_{n+1}-\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds\right\}+\delta_n\left\{\frac{1}{t_n}\int_0^{t_n}T(s)x_nds-x_n\right\}$$
$$+\left\{\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds-\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\}.$$
$$\|l_{n+1}-l_n\|=\frac{\alpha_{n+1}}{(1-\beta_{n+1})}\left\|\gamma Vx_{n+1}-A\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds\right\|$$
$$+\frac{\alpha_n}{(1-\beta_n)}\left\|A\frac{1}{t_n}\int_0^{t_n}T(s)x_nds-\gamma Vx_n\right\|$$
$$+\frac{\alpha_{n+1}\delta_{n+1}}{(1-\beta_{n+1})}\left\|A\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds-Ax_{n+1}\right\|$$
$$+\frac{\alpha_n\delta_n}{(1-\beta_n)}\left\|Ax_n-A\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\|$$
$$+\delta_{n+1}\left\|x_{n+1}-\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds\right\|+\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)x_nds-x_n\right\|$$
$$+\left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds-\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\|.$$

Using (3.10), we have

$$\|l_{n+1}-l_n\|=\frac{\alpha_{n+1}}{(1-\beta_{n+1})}\left\|\gamma Vx_{n+1}-A\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds\right\|$$
$$+\frac{\alpha_n}{(1-\beta_n)}\left\|A\frac{1}{t_n}\int_0^{t_n}T(s)x_nds-\gamma Vx_n\right\|$$
$$+\frac{\alpha_{n+1}\delta_{n+1}}{(1-\beta_{n+1})}\left\|A\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds-Ax_{n+1}\right\|$$
$$+\frac{\alpha_n\delta_n}{(1-\beta_n)}\left\|Ax_n-A\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\|$$
$$+\delta_{n+1}\left\|x_{n+1}-\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds\right\|+\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)x_nds-x_n\right\|$$
$$+\left(\frac{1}{t_{n+1}}\int_0^{t_{n+1}}L_sds\right)\|x_{n+1}-x_n\|$$
$$+\left[\left|1-\frac{t_n}{t_{n+1}}\right|\left(\frac{1}{t_n}\int_0^{t_n}L_sds\right)+\left|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}L_sds-\left(\frac{t_n}{t_{n+1}}\right)\left(\frac{1}{t_n}\int_0^{t_n}L_sds\right)\right|\right]M.$$

Therefore,

$$\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\| = \frac{\alpha_{n+1}}{(1-\beta_{n+1})}\left\|\gamma Vx_{n+1}-A\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds\right\|$$
$$+\frac{\alpha_n}{(1-\beta_n)}\left\|A\frac{1}{t_n}\int_0^{t_n}T(s)x_nds-\gamma Vx_n\right\|$$
$$+\frac{\alpha_{n+1}\delta_{n+1}}{(1-\beta_{n+1})}\left\|A\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds-Ax_{n+1}\right\|$$
$$+\frac{\alpha_n\delta_n}{(1-\beta_n)}\left\|Ax_n-A\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\|$$
$$+\delta_{n+1}\left\|x_{n+1}-\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)x_{n+1}ds\right\|+\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)x_nds-x_n\right\|$$
$$+\left[\left(\frac{1}{t_{n+1}}\int_0^{t_{n+1}}L_sds\right)-1\right]\|x_{n+1}-x_n\|$$
$$+\left[\left|1-\frac{t_n}{t_{n+1}}\right|\left(\frac{1}{t_n}\int_0^{t_n}L_sds\right)+\left|\frac{1}{t_{n+1}}\int_0^{t_n+1}L_sds-\left(\frac{t_n}{t_{n+1}}\right)\left(\frac{1}{t_n}\int_0^{t_n}L_sds\right)\right|\right]M.$$

By the conditions (C_1), (C_2) and (C_3), we have

$$\limsup_n(\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\|)\le 0.$$

From Lemma 2.6, we obtain

$$\lim_n(\|l_n-x_n\|)=0. \tag{3.11}$$

Since

$$x_{n+1}-x_n=(1-\beta_n)(l_n-x_n), \tag{3.12}$$

we have

$$\lim_n(\|x_{n+1}-x_n\|)=0. \tag{3.13}$$

Now

$$\left\|x_{n+1}-\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\|\le\alpha_n\left\|\gamma Vx_n-A\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\|$$
$$+\beta_n\left\|x_n-\frac{1}{t_n}T(s)x_nds\right\|$$
$$+\delta_n\left\|x_n-\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\|+\beta_n\delta_n\left\|\frac{1}{t_n}\int_0 t_nT(s)x_nds-x_n\right\|$$
$$+\alpha_n\delta_n\left\|A\frac{1}{t_n}T(s)x_nds-x_n\right\|$$
$$\le\frac{\alpha_n}{1-(\beta_n+\delta_n)}\left\|\gamma Vx_n-A\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\|$$
$$+\frac{\beta_n+\delta_n}{1-(\beta_n+\delta_n)}\|x_n-x_{n+1}\|+\frac{\beta_n\delta_n}{1-(\beta_n+\delta_n)}\left\|\frac{1}{t_n}\int_0 t_nT(s)x_nds-x_n\right\|$$

$$+\frac{\alpha_n\delta_n}{1-(\beta_n+\delta_n)}\left\|A\frac{1}{t_n}T(s)x_nds-x_n\right\|.$$

By the conditions (C_1) and (C_2) and (3.13), we get

$$\lim_n \|x_{n+1}-\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\|=0. \tag{3.14}$$

For all $h\geq 0$, we note that

$$\begin{aligned}\|x_{n+1}-T(h)x_{n+1}\| &\leq \left\|x_{n+1}-\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\| \\ &+\left\|\frac{1}{t_n}\int_0^{t_n}T(s)x_nds-T(h)\left(\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right)\right\| \\ &+\left\|T(h)\left(\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right)-T(h)x_{n+1}\right\| \\ &\leq \left\|x_{n+1}-\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right\| \\ &+\left\|\frac{1}{t_n}\int_0^{t_n}T(s)x_nds-T(h)\left(\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right)\right\| \\ &+L_h\left\|x_{n+1}-\left(\frac{1}{t_n}\int_0^{t_n}T(s)x_nds\right)\right\|.\end{aligned}$$

By Lemma 2.5 and (3.14), we obtain

$$\lim_n \|x_{n+1}-T(h)x_{n+1}\|=0, \forall\ h\geq 0. \tag{3.15}$$

Since

$$\|\frac{1}{t_m}\int_0^{t_m}T(s)x_{n+1}ds-x_{n+1}\|\leq\frac{1}{t_m}\int_0^{t_m}\|T(s)x_{n+1}-x_{n+1}\|ds.$$

Thus

$$\lim_n \|\frac{1}{t_m}\int_0^{t_m}T(s)x_{n+1}ds-x_{n+1}\|=0. \tag{3.16}$$

Next, we show that

$$\limsup_n\langle\gamma Vx^*-AX^*,J(x_{n+1}-x^*)\leq 0.$$

Let

$$z_m=\alpha_m\gamma Vz_m+\beta_m z_m+((1-\beta_m)I-\alpha_m A)\frac{1}{t_m}\int_0^{t_m}T(s)z_mds,$$

where $\{t_m\}$, $\{\alpha_m\}$ and $\{\beta_m\}$ satisfies the conditions (C_1) and (C_2). Then, we note that

$$\begin{aligned}z_m-x_{n+1}=\alpha_m\gamma Vz_m+\beta_m z_m+\frac{1}{t_m}\int_0^{t_m}T(s)z_mds-\beta_m\frac{1}{t_m}\int_0^{t_m}T(s)z_mds \\ -\alpha_m A\frac{1}{t_m}\int_0^{t_m}T(s)z_mds-x_{n+1}\end{aligned}$$

$$= \alpha_m(\gamma V z_m - A z_m) + \left(\frac{1}{t_m}\int_0^{t_m} T(s) z_m ds - x_{n+1}\right)$$
$$+\alpha_m \left(A z_m - A\left(\frac{1}{t_m}\int_0^{t_m} T(s) z_m ds\right)\right) + \beta_m \left(z_m - \frac{1}{t_m}\int_0^{t_m} T(s) z_m ds\right)$$
$$= \alpha_m(\gamma V z_m - A z_m) + \left(\frac{1}{t_m}\int_0^{t_m} T(s) z_m ds - \frac{1}{t_m}\int_0 t_m T(s) x_{n+1} ds\right)$$
$$+\left(\frac{1}{t_m}\int_0 t_m T(s) x_{n+1} ds - x_{n+1}\right) + \alpha_m^2 A\left(\gamma V z_m - A\left(\frac{1}{t_m}\int_0^{t_m} T(s) z_m ds\right)\right)$$
$$+\alpha_m \beta_m A\left(z_m - \frac{1}{t_m}\int_0^{t_m} T(s) z_m ds\right) + \beta_m\left(z_m - \frac{1}{t_m}\int_0^{t_m} T(s) z_m ds\right).$$

Therefore,

$$\|z_m - x_{n+1}\|^2 \le \alpha_m \langle \gamma V z_m - A z_m, J(z_m - x_{n+1})\rangle$$
$$+\left\|\frac{1}{t_m}\int_0^{t_m} T(s) x_{n+1} ds - x_{n+1}\right\| \|z_m - x_{n+1}\| + \left(\frac{1}{t_m}\int_0^{t_m} L_s ds\right)\|z_m - x_{n+1}\|^2$$
$$+\alpha_m^2 \left\|A\left(\gamma V z_m - \frac{1}{t_m}\int_0^{t_m} T(s) z_m ds\right)\right\| \|z_m - x_{n+1}\|$$
$$+\alpha_m \beta_m A \left\|z_m - \frac{1}{t_m}\int_0^{t_m} T(s) z_m ds\right\| \|z_m - x_{n+1}\|$$
$$+\beta_m \left\|z_m - \frac{1}{t_m}\int_0^{t_m} T(s) z_m ds\right\| \|z_m - x_{n+1}\|.$$

This implies that

$$\langle \gamma V z_m - A z_m, J(x_{n+1} - z_m)\rangle \le \left(\frac{(\frac{1}{t_m}\int_0^{t_m} L_s ds) - 1}{\alpha_m}\right)\|z_m - x_{n+1}\|^2$$
$$+\left(\frac{\|\frac{1}{t_m}\int_0^{t_m} T(s) x_{n+1} ds - x_{n+1}\|}{\alpha_m}\right)\|z_m - x_{n+1}\|$$
$$+\alpha_m \left\|A(\gamma V z_m - \frac{1}{t_m}\int_0^{t_m} T(s) z_m ds)\right\| \|z_m - x_{n+1}\|$$
$$+\beta_m A \left\|z_m - \frac{1}{t_m}\int_0^{t_m} T(s) z_m ds\|\|z_m - x_{n+1}\right\|$$
$$+\frac{\beta_m}{\alpha_m}\left\|z_m - \frac{1}{t_m}\int_0^{t_m} T(s) z_m ds\right\| \|z_m - x_{n+1}\|.$$

Now, taking the upper limit as $n \to \infty$, and then $m \to \infty$, we obtain

$$\limsup_n \langle \gamma V z_m - A z_m, J(x_{n+1} - z_m) \le 0. \tag{3.17}$$

On the other hand, we note that

$$\langle \gamma V x^* - AX^*, J(x_{n+1} - x^*) = \langle \gamma V x^* - A x^*, J(x_{n+1} - x^*) - J(x_{n+1} - z_m)\rangle$$
$$+\langle A z_m - A x^*, J(x_{n+1} - z_m)\rangle + \langle \gamma V x^* - \gamma V z_m, J(x_{n+1} - z_m)\rangle$$

$$+\langle \gamma V z_m - Az_m, J(x_{n+1} - z_m)\rangle.$$

Taking limit superior as $n \to \infty$, we have

$$\begin{aligned}\limsup_n \langle \gamma V x^* - AX^*, J(x_{n+1} - x^*)\rangle &\le \limsup_n \langle \gamma V x^* - Ax^*, J(x_{n+1} - x^*) \\ &-J(x_{n+1} - z_m)\rangle + \|A\|\|z_m - x^*\| \limsup_n \|x_{n+1} - z_m\| \\ &+\|\gamma V x^* - \gamma V z_m\| \limsup_n \|x_{n+1} - z_m\| \\ &+ \limsup_n \langle \gamma V z_m - Az_m, J(x_{n+1} - z_m)\rangle. \end{aligned} \tag{3.18}$$

Since X is uniformly smooth, the duality mapping J is norm-to-norm uniformly continuous on the bounded subset of C, then

$$\limsup_m \limsup_n \langle \gamma V x^* - AX^*, J(x_{n+1} - x^*) - J(x_{n+1} - z_m)\rangle = 0.$$

Therefore, from (3.18) we have

$$\begin{aligned}\limsup_n \langle \gamma V x^* - AX^*, J(x_{n+1} - x^*)\rangle &= \limsup \limsup_{m,n} \langle \gamma V x^* - AX^*, J(x_{n+1} - x^*)\rangle \\ &\le \limsup \limsup_{m,n} \langle \gamma V z_m - Az_m, J(x_{n+1} - z_m)\rangle \le 0.\end{aligned}$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. Now

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \alpha_n \langle \gamma V x_n - Aq, J(x_{n+1} - q)\rangle + \beta_n \langle x_n - q, J(x_{n+1} - q)\rangle \\ &+\langle ((1-\beta_n)I - \alpha_n A)y_n - q, J(x_{n+1} - q)\rangle \\ = \alpha_n \gamma L \langle x_n - q, J(x_{n+1} - q)\rangle &+ \alpha_n \langle \gamma V q - Aq, J(x_{n+1} - q)\rangle + \beta_n \langle x_n - q, J(x_{n+1} - q)\rangle \\ &+(1-\beta_n - \alpha_n\bar{\gamma})\delta\langle x_n - q, J(x_{n+1} - q)\rangle + (1-\beta_n - \alpha_n\bar{\gamma})(1-\delta_n) \\ &\times \left\langle \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - q, J(x_{n+1} - q)\right\rangle \\ &\le \alpha_n \gamma L \|x_n - q\|\|x_{n+1} - q\| + \beta_n \|x_n - q\|\|x_{n+1} - q\| \\ &+(1-\beta_n - \alpha_n\bar{\gamma})\delta_n \|x_n - q\|\|x_{n+1} - q\| \\ &+(1-\beta_n - \alpha_n\bar{\gamma})(1-\delta_n)\left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - q\|\|x_{n+1} - q \right\| \\ &+\alpha_n \langle \gamma V q - Aq, J(x_{n+1} - q)\rangle \\ &\le \alpha_n \gamma L \left(\frac{\|x_n - q\|^2 + \|x_{n+1} - q\|^2}{2}\right) + \beta_n \left(\frac{\|x_n - q\|^2 + \|x_{n+1} - q\|^2}{2}\right) \\ &+(1-\beta_n - \alpha_n\bar{\gamma})\delta_n \left(\frac{\|x_n - q\|^2 + \|x_{n+1} - q\|^2}{2}\right) \\ &+(1-\beta_n - \alpha_n\bar{\gamma})(1-\delta_n)\left(\frac{\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - q\|^2 + \|x_{n+1} - q\|^2}{2}\right) \\ &+\alpha_n \langle \gamma V q - Aq, J(x_{n+1} - q)\rangle \\ = \left[(1-\alpha_n(\bar{\gamma} - \gamma L)) + (1-\beta_n - \alpha_n\bar{\gamma})(1-\delta_n)\left\{\left(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1\right)^2\right\}\right] &\frac{\|x_n - q\|^2}{2}\end{aligned}$$

$$+(1-\alpha_n(\bar{\gamma}-\gamma L))\frac{\|x_{n+1}-q\|^2}{2}+\alpha_n\langle \gamma Vq - Aq, J(x_{n+1}-q)\rangle.$$

Therefore

$$\|x_{n+1}-q\|^2 \le \frac{(1-\alpha_n(\bar{\gamma}-\gamma L))}{(1+\alpha_n(\bar{\gamma}-\gamma L))}\|x_n-q\|^2$$

$$+\frac{(1-\beta_n-\alpha_n\bar{\gamma})(1-\delta_n)}{(1+\alpha_n(\bar{\gamma}-\gamma L))}\left\{\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)^2-1\right\}\|x_n-q\|^2$$

$$+\frac{2\alpha_n}{(1+\alpha_n(\bar{\gamma}-\gamma L))}\langle \gamma Vq - Aq, J(x_{n+1}-q)\rangle$$

$$\le \frac{(1-\alpha_n(\bar{\gamma}-\gamma L))}{(1+\alpha_n(\bar{\gamma}-\gamma L))}\|x_n-q\|^2+\frac{(1-\beta_n-\alpha_n\bar{\gamma})(1-\delta_n)}{(1+\alpha_n(\bar{\gamma}-\gamma L))}\left\{\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)^2-1\right\}M$$

$$+\frac{2\alpha_n}{(1+\alpha_n(\bar{\gamma}-\gamma L))}\langle \gamma Vq - Aq, J(x_{n+1}-q)\rangle,$$

where $M > 0$ is a constant such that

$$M=\sup_{n\ge 1}\{\|x_n-q\|^2\}.$$

Put

$$\sigma_n=\frac{2(\bar{\gamma}-\gamma L)\alpha_n}{1+(\bar{\gamma}-\gamma L)\alpha_n}$$

and

$$\lambda_n=\frac{(1-\beta_n-\alpha_n\bar{\gamma})(1-\delta_n)}{(1+\alpha_n(\bar{\gamma}-\gamma L))}\left\{\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)^2-1\right\}M$$

$$+\frac{2\alpha_n}{(1+\alpha_n(\bar{\gamma}-\gamma L))}\langle \gamma Vq - Aq, J(x_{n+1}-q)\rangle.$$

Thus

$$\|x_{n+1}-q\|^2 \le (1-\sigma_n)\|x_n-q\|^2+\lambda_n. \tag{3.19}$$

Using conditions (C_1), (C_2) and (C_3), we have

$$\sum_{n=1}^{\infty}\sigma_n=\infty$$

and

$$\limsup_n \frac{\lambda_n}{\sigma_n}=\limsup_n\left[\frac{(1-\beta_n-\alpha_n\bar{\gamma})(1-\delta_n)}{2(\bar{\gamma}-\gamma L)\alpha_n}\left\{\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)^2-1\right\}\right]$$

$$+\frac{1}{(\bar{\gamma}-\gamma L)}\langle \gamma Vq - Aq, J(x_{n+1}-q)\rangle \le 0.$$

From Lemma 2.7, we obtain that $x_n \to\ 0$ as $n\to\infty$.

References

[1] I.A. Aleyner, Y.Censor, *Best approximation to common fixed points of a semigroup of nonexpansive operators*, J. Nonlinear Convex Anal., **6**(2005), 137-151.

[2] W.L. Bynum, *Normal structure coefficients for Banach spaces*, Pacific J. Math., **86**(2001), 427-436.

[3] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image recostruction*, Inverse Probl., **20**(2004), 103-120.

[4] F.E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach Spaces*, Math. Z., **100**(1967), 201-225.

[5] P. Cholamjiak, S. Suantai, *Viscosity approximation methods for a nonexpansive semigroup in Banach spaces with gauge functions*, J. Global Optim., **54**(2012), 185-197.

[6] R. Chen, Y. Song, *Convergence to common fixed point of nonexpansive semigroup*, J. Comput. Appl. Math., **200**(2007), 500-575.

[7] G. Cai, C.S. Hu, *Strong convergence theorems of a general iterative process for a finite family of λ-strict pseudo-contraction in q-uniformly smooth Banach spaces*, Comput. Math. Appl., **59**(2010), 49-160.

[8] F. Deutsch, I. Yamada, *Minimizing certain convex functions over the intersection of the fixed points sets of nonexpansive mappings*, Numer. Funct. Anal. Optim., **19**(1998), 33-56.

[9] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpensive mappings*, Proc. Amer. Math.Soc., **35**(1972), 171-174.

[10] A.T.-M. Lau, *Amenability and fixed point property for semigroup of nenexpansive mapping*, In: Thera, M.A., Baillon, J.B. (eds), Fixed Point Theory and Application, Pitman Res. Notes Math., Set, Zongman, Harlow, **252**(1991), 303-313.

[11] A.T.-M. Lau, *Invanant means and fixed point properties of semigroup of nonexpansive mappings*, Taiwanese J. Math., **12**(2008), 1525-1542.

[12] A.T.-M. Lau, H. Miyake, W. Takahashi, *Approximation of fixed points for amenable semigroups of nonexpansive mappings in Banach spaces*, Nonlinear Anal., **67**(2007), 1211-1225.

[13] S. Li, L. Li, Y. Su, *General iterative methods for one-parameter nonexpansive semigroup in Hilbert space*, Nonlinear Anal., **70**(2009), 3065-3071.

[14] T.C. Lim, H.K. Xu, *Fixed point theorems for asymptotically nonexpansive mappings*, Nonlinear Anal., **22**(1994), 1345-1355.

[15] A. Moudafi, *Viscosity approximation methods for fixed point problems*, J. Math. Anal. Appl., **241**(2000), 46-55.

[16] G. Marino, H.K. Xu, *A general iterative for nonexpansive mapping in Hilbert spaces*, J. Math. Anal. Appl., **318**(2006), 43-52.

[17] C.I. Podilchuk, R.J. Mammone, *Image recovery by convex projections using a least squares constraint*, J. Opt. Soc. Am., **A7**(2005), 517-521.

[18] S. Reich, D. Shoiklet, *Nonlinear Semigroups, Fixed Points and Geometry of Domains in Banach Spaces*, Imperial Collage Press, London, 2005.

[19] M.I. Sezan, H. Stark, *Applications of convex projection theory to image recovery in tomography and related areas*, In: Stark, H. (ed.), Image Recovery Theory and Applications, Academic Press, Orlando, (1987), 415-462.

[20] P. Sunthrayuth, P. Kumam, *Approximating solutions of variational inequalities on the sets of common fixed points for a semigroup of asymptotically nonexpansive mappings in Banach spaces*, Math. Sci., **6**(20)(2012).

[21] P. Sunthrayuth, P. Kumam, *Fixed point solutions of variational inequalities for a semigroup of asymptotically nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., **177**(2012).

[22] T. Suzuki, *Strong convergence of Krasnoselski and Mann's type sequence for one-parameter nonexpansive semigroup without Bochner integrals*, J. Math. Anal. Appl., **305**(2005), 227-239.

[23] W. Takahashi, *Nonlinear Functional Analysis*, Yokahama Publishers Yokahama , 2000.

[24] R. Wangkeeree, N. Petrot, R. Wangkeeree, *The general iterative methods of nonexpansive mappings in Banach spaces*, J. Giobal Optim., **51**(2011), 27-46.

[25] H.K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl., **116**(2003), 659-678.

[26] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298**(2004), 279-291.

[27] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2002), no. 1, 240-256.

[28] D. Youla, *Mathematical theory of image restoration by the method of convex projections*, In: Stark, H. (ed.), Image Recovery Theory and Applications, Academic Press, Orlando, 1987, 29-77.

[29] D. Youla, *On deterministic convergence of iterations of relaxed projection operators*, J. Vis. Commun. Image Represent., **1**(1990), 12-20.

[30] H. Zegeye, N. Shahzad, O.A. Daman, *Strong convergence theorems for a semigroup of asymptotically nonexpansive mappings*, Math. Comput. Model., **54**(2011), 2077-2086.

[31] H. Zegeye, N. Shahzad, *Convergence theorems for strongly continuous semi-groups of asymptotically nonexpansive mappings*, Nonlinear Anal., **71**(2009), 2308-2315.

[32] H. Zegeye, N. Shahzad, *Strong convergence theorems for continuous semigroups of asymptotically nonexpansive mappings*, Numer. Funct, Anal. Optim., **30**(2009), no. 7, 833-848.

Received: March 12, 2015; Accepted: May 20, 2015.