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FIXED POINT THEOREMS FOR COMPACT POTENTIAL OPERATORS IN HILBERT SPACES

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Abstract. The aim of this paper is to present new fixed point theorems for compact potential operators in Hilbert spaces. A variational approach is used and applications to boundary value problems illustrate the existence results.
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1. Introduction

Given a Banach space X, an operator $\Phi : X \longrightarrow X'$ is called a potential operator (or gradient operator) on a subset $\Omega \subset X$, if there exists a Gâteaux differentiable functional $\varphi : \Omega \longrightarrow \mathbb{R}$ such that $\nabla\varphi(x) = \Phi(x)$, for every $x \in \Omega$ (see [6]). In other words

$$\lim_{\lambda\to 0} \frac{\varphi(x+\lambda y) - \varphi(x)}{\lambda} = < \Phi(x), y >_{X',X},$$

for all $x, y \in X$. Here $< \cdot,\cdot >_{X',X}$ refers to the duality pairing between X and its topological dual X'. For a given potential, we always assume that $\varphi(0) = 0$. Potential operators arise in many steady-state phenomena in physical problems stem

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from quantum mechanics such that the potential of the Hamiltonian operator in the Schrödinger equation. In [1, 9], the Leray-Schauder degree of the gradient of a coercive functional φ, $\nabla\varphi = I - \Phi$ with Φ compact, on a large ball of a Hilbert space is proved to be equal to one; as a consequence the potential operator has at least one zero, i.e., Φ has a fixed point in that ball. Recall that a mapping is said to be compact if it maps bounded sets into relatively compact sets. In [5], the authors have considered nonlinear mappings $\phi \in C^1(H, \mathbb{R})$ defined on a Hilbert space H ordered by a cone P and such that ϕ satisfies the (PS) condition (see Definition 1.1) and $\phi' = I - A$. Then when A satisfies some growth conditions, A is proved to have a fixed point. A combination of topological and variational methods are used and an application to a second-order dynamic equation is given. Regarding potential operator equations, a mathematical theory is developed in [4].

In this work, we present new fixed point theorems for compact potential operators in Hilbert spaces, including operators with sub-linear like growth. The proofs are based on a variational approach. Then the main existence theorem is applied to Dirichlet boundary value problems associated to ordinary and fractional differential equations with an illustrative example of application.

First, recall one concept from critical point theory.

Definition 1.1. [4, 6, 7] Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$. If any sequence $(u_n)_n \subset X$ for which $(\varphi(u_n))_n$ is bounded in $\mathbb{R}$ and $\varphi'(u_n) \longrightarrow 0$ as $n \to +\infty$ in X' possesses a convergent subsequence, then we say that φ satisfies the Palais-Smale condition, (PS) condition for short.

A fundamental result in minimization of functionals is the following

Lemma 1.2. [7] *Let H be a Hilbert space and $\varphi \in C^1(H, \mathbb{R})$. Suppose that the functional φ is bounded from below and verifies the Palais-Smale condition at level c with $c = \inf\limits_{u \in H} \varphi(u)$. Then there exists a critical point for φ at level c.*

In fact, φ achieves a minimum (see [4, Corollary 1.1.1]) and it is easy to see that every local point of minimum u_0 of a Gâteaux differentiable functional φ is a critical point, i.e., $\varphi'(u_0) = 0$. Finally, an important auxiliary result in the sequel is:

Theorem 1.3. [2] *Let X and Y be two Banach spaces, Ω an open subset of X, and $\varphi : \Omega \longrightarrow Y$ a mapping of class C^1. Given $x, y \in \Omega$, if $x + ty \in \Omega$ for all $t \in [0, 1]$, then*

$$\varphi(x+y) = \varphi(x) + \int_0^1 < D\varphi(x+ty), y > dt.$$

Indeed, this result makes connection between the potential operator Φ and the Gâteaux differentiable functional φ for it can be checked that

$$\varphi(x) = \int_0^1 < \Phi(sx), x > ds. \tag{1.1}$$

2. Existence results

Our first result in this paper is:

Theorem 2.1. *Let H be a Hilbert space and $A : H \longrightarrow H$ a compact potential operator such that there exist $v^* \in H$ and a bounded linear operator B on H with $\|B\| < 1$ such that:*

$$\Big(A(su),\, u\Big) \leq \Big(B(su),\, u\Big) + (v^*,\, u),\ \forall\, s \in [0,1],\ \forall\, u \in H. \tag{2.1}$$

Then, the operator A has a fixed point in H.

Proof. Since A is a potential operator, there exists a Gâteaux differentiable functional $T : H \longrightarrow \mathbb{R}$ such that $T' = A$. By (1.1), A can be represented in the form

$$T(u) = \int_0^1 \Big(A(su), u\Big)\, ds.$$

Consider the functional $\varphi : H \longrightarrow \mathbb{R}$ defined by $\varphi = K - T$, where $Ku = \frac{1}{2}\|u\|^2$. Then clearly $K' = I$, $\varphi \in C^1(H, \mathbb{R})$, and $\varphi' = I - A$.
Step 1. φ is bounded from below. By Assumption (2.1), we have the estimates:

$$\begin{aligned}
\varphi(u) \;&\geq\; \frac{1}{2}\|u\|^2 - \int_0^1 \Big[\Big(B(su),\, u\Big) + (v^*, u)\Big]\, ds \\
&\geq\; \frac{1}{2}\|u\|^2 - \int_0^1 s\|u\|^2\|B\|\, ds - \int_0^1 (v^*, u)\, ds \\
&\geq\; \frac{1}{2}\|u\|^2 - \frac{1}{2}\|u\|^2\|B\| - \|v^*\|\|u\| \\
&\geq\; \frac{1}{2}\Big(1 - \|B\|\Big)\|u\|^2 - \|v^*\|\|u\| \geq -c,
\end{aligned}$$

for all $c > \frac{\|\nu^*\|}{2(1-\|B\|)}$.
Step 2. φ verifies the Palais-Smale condition. Let $(u_n)_n$ be a sequence in H such that $\lim\limits_{n\to+\infty} \varphi'(u_n) = 0$ and $(\varphi(u_n))_n$ is bounded. i.e., there exists some positive constant C such that $|\varphi(u_n)| \leq C$, for all positive integers n. In view of hypothesis (2.1), we have

$$\begin{aligned}
C \geq \varphi(u_n) &\geq \frac{1}{2}\|u_n\|^2 - \int_0^1 \Big[\Big(B(su_n),\, u_n\Big) + (v^*, u_n)\Big]\, ds \\
&\geq \frac{1}{2}\|u_n\|^2 - \int_0^1 s\|u_n\|^2\|B\|ds - \int_0^1 (v^*, u_n)\, ds \\
&\geq \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u_n\|^2\|B\| - \|v^*\|\|u_n\| \\
&\geq \frac{1}{2}\Big(1 - \|B\|\Big)\|u_n\|^2 - \|v^*\|\|u_n\|,
\end{aligned}$$

which implies that $(u_n)_n$ is bounded in H. We note that $\varphi'(u_n) = u_n - A(u_n)$ with $\lim\limits_{n\to+\infty} \varphi'(u_n) = 0$. Since the sequence $(u_n)_n$ is bounded and the operator A is compact, the sequence $(A(u_n))_n$ is relatively compact; as a consequence there exists a subsequence $(u_{n_k})_k \subset (u_n)_n$ such that $A(u_{n_k}) \longrightarrow w$; hence $u_{n_k} \longrightarrow w$ in H, as $k \to +\infty$. Indeed,

$$\|u_{n_k} - w\| \leq \|u_{n_k} - A(u_{n_k})\| + \|A(u_{n_k}) - w\| \longrightarrow 0.$$

Thus, the (PS) condition is satisfied.
Finally, by Lemma 1.2, we conclude that φ has a critical point which is a fixed point for the operator A. $\square$

The following result deals with the sub-linear like growth case. We will denote by $\|u\|_1 = \int_0^1 |u(s)|ds$ the standard norm of the Lebesgue space of measurable functions such that the map $s \mapsto |u(s)|$ is Lebesgue integrable on $(0,1)$.

Theorem 2.2. *Let H be a Hilbert space and $A : H \longrightarrow H$ a compact potential operator. Assume that there exist two mappings:*

$$\psi_1 : [0,1] \longrightarrow \mathbb{R}^+$$
$$\psi_2 : [0,+\infty) \longrightarrow \mathbb{R}^+$$

such that

$$\|A(su)\| \leq \psi_1(s)\psi_2(\|su\|),\ \forall\, s \in [0,1] \ \ \text{and}\ \forall\, u \in H \tag{2.2}$$

with $s \mapsto \frac{\psi_1(s)}{s} \in L^1$ and

$$x\psi_2(x)\|s\psi_1(s)\|_1 \leq \frac{x^2}{2} + M, \quad \forall\, x \geq 0,$$

for some constant M. Then, the operator A has a fixed point in H.

Proof. Since A is a potential operator, there exists a Gâteaux differentiable functional $T : H \longrightarrow \mathbb{R}$ such that $T' = A$. By (1.1), A can be represented in the form:

$$T(u) = \int_0^1 \Big(A(su), u\Big)\, ds.$$

Define the operator $\varphi = K - T$, where $Ku = \frac{1}{2}\|u\|^2$. Then $\varphi \in C^1(H,\mathbb{R})$ and $\varphi' = I - A$. As in the proof of Theorem 2.1, we can check that φ satisfies the Palais-Smale condition. Moreover φ is bounded from below. Indeed, using hypothesis (2.2), we derive the estimates:

$$\begin{aligned}\int_0^1 (A(su),u)ds &\leq \int_0^1 \|A(su)\|.\|u\|ds\\ &\leq \int_0^1 \|u\|\psi_1(s)\psi_2(s\|u\|)ds\\ &\leq \int_0^1 \frac{\psi_1(s)}{s\|s\psi_1(s)\|_1}\left(\frac{s^2\|u\|^2}{2}+M\right)ds\\ &\leq \frac{\|u\|^2}{2}+\frac{M}{\|s\psi_1(s)\|_1}\int_0^1 \frac{\psi_1(s)}{s}ds.\end{aligned}$$

Hence

$$\varphi(u) \geq -\frac{M\left\|\frac{\psi_1(s)}{s}\right\|_1}{\|s\psi_1(s)\|_1}.$$

Lemma 1.2 then guarantees that φ has a critical point which is a fixed point for the operator A. □

Remark 2.3. Notice that by (2.2), the operator A satisfies:

$$\|A(u)\| \leq \frac{\psi_1(1)}{\|s\psi_1(s)\|_1}\left(\frac{\|u\|}{2}+\frac{M}{\|u\|}\right), \quad \forall\, u \in H_0^1(0,1)\setminus\{0\}.$$

However if, for instance $\dfrac{\psi_1(1)}{\|s\psi_1(s)\|_1} = 2$, then the Schauder fixed point theorem does not apply.

3. Applications

Consider the Dirichlet boundary value problem

$$\begin{cases} -u''(t) = f(t,u(t)), & t \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

Lemma 3.1. *If u is a solution of the integral equation*

$$u(t) = \int_0^1 G(t,s)f(s,u(s))\,ds,$$

where

$$G(t,s) = \begin{cases} t(1-s), & t \leq s, \\ s(1-t), & s \leq t, \end{cases} \tag{3.2}$$

then u is a solution of problem (3.1).

Let $H_0^1 = H_0^1(0,1)$ be the standard Sobolev space endowed with the norm

$$\|u\|_{H_0^1} = \left(\int_0^1 u'^2(t)dt\right)^{\frac{1}{2}}$$

and A the operator defined on H_0^1 by

$$Au(t) = \int_0^1 G(t,s)f(s,u(s))\,ds.$$

Then, A satisfies the Dirichlet bvp

$$\begin{cases} -(Au)''(t) = f(t,u(t)), & t \in [0,1], \\ (Au)(0) = (Au)(1) = 0. \end{cases} \tag{3.3}$$

Define the functional

$$\varphi : H_0^1 \longrightarrow \mathbb{R}$$

by $\varphi(u) = Ku - \int_0^1 F(t,u(t))dt$, where $Ku = \frac{1}{2}\|u\|^2$ and $F(t,u) = \int_0^u f(t,s)ds$. Then $\varphi' = I - A$. Indeed, from (3.3), we have, using integrations by parts, for all $u, v \in H_0^1$

$$\begin{aligned} (\varphi'(u), v) &= \int_0^1 u'(t)v'(t)\,dt - \int_0^1 f(t,u(t))v(t)\,dt \\ &= \int_0^1 u'(t)v'(t)\,dt + \int_0^1 (Au)''(t)v(t)\,dt \\ &= \int_0^1 \left(u'(t)v'(t) - (Au)'(t)v'(t)\right)\,dt \\ &= (u,v) - (Au,v) = (u - Au, v) = ((I-A)u, v). \end{aligned}$$

Definition 3.2. We say that $u \in H_0^1$ is a weak solution of (3.1) if

$$\int_0^1 \left[u'(t)v'(t) - f(t,u(t))v(t)\right]\,dt = 0, \quad \text{for all } \ v \in H_0^1.$$

To prove that problem (3.1) has a weak solution, we first study the compactness of A.

Lemma 3.3. *The operator $A : H_0^1 \longrightarrow H_0^1$ is compact.*

Proof. Let $(u_n)_n$ be a bounded sequence in the reflexive space H_0^1. Then there exists $u \in H_0^1$ such that $u_{n_k} \rightharpoonup u$ in H_0^1. We prove that $Au_{n_k} \longrightarrow Au$ in H_0^1. Making use of the Lebesgue's dominated convergence theorem, we obtain the following estimates:

$$\begin{aligned}\|Au_{n_k}-Au\|_{H_0^1} &= \sup_{\|v\|_{H_0^1}\le 1} |\langle Au_{n_k}-Au, v\rangle| \\ &= \sup_{\|v\|_{H_0^1}\le 1} \left|(Au_{n_k}-Au, v)_{H_0^1}\right| \\ &= \sup_{\|v\|_{H_0^1}\le 1} \left|\int_0^1 (Au_{n_k}-Au)'(t)v'(t)dt\right| \\ &= \sup_{\|v\|_{H_0^1}\le 1} \left|\int_0^1 \left(-(Au_{n_k})''(t)+(Au)''(t)\right)v(t)\,dt\right| \\ &= \sup_{\|v\|_{H_0^1}\le 1} \left|\int_0^1 \left(f(t,u_{n_k}(t))-f(t,u(t))\right)v(t)\,dt\right|.\end{aligned}$$

Hence

$$\begin{aligned}\|Au_{n_k}-Au\|_{H_0^1} &\le \sup_{\|v\|_{H_0^1}\le 1} \left(\int_0^1 \left(f(t,u_{n_k}(t))-f(t,u(t))\right)^2 dt\right)^{\frac12} \left(\int_0^1 v^2(t)\,dt\right)^{\frac12} \\ &\le \sup_{\|v\|_{H_0^1}\le 1} \left(\int_0^1 \left(f(t,u_{n_k}(t))-f(t,u(t))\right)^2 dt\right)^{\frac12} \|v\|_{L^2} \\ &\le \sup_{\|v\|_{H_0^1}\le 1} \left(\int_0^1 \left(f(t,u_{n_k}(t))-f(t,u(t))\right)^2 dt\right)^{\frac12} \frac{1}{\sqrt{\lambda_1}}\|v\|_{H_0^1} \\ &\le \frac{1}{\sqrt{\lambda_1}} \left(\int_0^1 \left(f(t,u_{n_k}(t))-f(t,u(t))\right)^2 dt\right)^{\frac12} \longrightarrow 0,\end{aligned}$$

as $k \longrightarrow +\infty$. Here $\lambda_1 = \pi^2$ is the first eigenvalue of the linear Dirichlet problem

$$\begin{cases} -u''(t) = \lambda u(t), & t\in[0,1], \\ u(0)=u(1) = 0. \end{cases}$$

We have also used the Poincaré's inequality (see, e.g., [3])

$$\|u\|_{L^2} \le \frac{1}{\sqrt{\lambda_1}}\|u'\|_{L^2}, \;\; \forall\, u \in H_0^1(0,1). \qquad \square$$

Theorem 3.4. *Assume that there exist functions $a, b \in L^1([0,1])$ with*

$$\|a\|_\infty = \sup_{0\le t\le 1} |a(t)| < \pi^2$$

such that

$$\operatorname{sgn}(u)\,(f(t,u)-a(t)u-b(t)) \le 0, \;\; \textit{for all } t\in[0,1] \textit{ and all } u\in\mathbb{R}. \tag{3.4}$$

Then problem (3.1) has at least one solution $u \in C^2[0,1]$.

Proof. Let

$$Bu(t) = \int_0^1 G(t,s)a(s)u(s)\,ds.$$

We check that the operator A verifies the conditions of Theorem 2.1.
Step 1. The operator A satisfies hypothesis (2.1). Let

$$v^*(t) = \int_0^1 G(t,s)b(s)\,ds.$$

For $u, v \in H_0^1$, we have

$$\begin{aligned}\Big(Bv - Av + v^*, u\Big) &= \int_0^1 (Bv - Av + v^*)'(t)u'(t)\,dt \\ &= \int_0^1 \left(-(Bv)''(t) + (Av)''(t) - (v^*)''(t)\right) u(t)\,dt \\ &= \int_0^1 \left(a(t)v(t) - f(t,v(t)) + b(t)\right) u(t)\,dt.\end{aligned}$$

Taking $u = \frac{v}{s}$ and using hypothesis (3.4), we deduce that $(Bv - Av + v^*, v) \geq 0$.
Step 2. $\|B\| < 1$. For all $u \in H_0^1$, we have

$$\begin{aligned}\|Bu\|_{H_0^1} = \sup_{\|v\|_{H_0^1}\leq 1} |\langle Bu, v\rangle| &= \sup_{\|v\|_{H_0^1}\leq 1} \left|(Bu,v)_{H_0^1}\right| \\ &= \sup_{\|v\|_{H_0^1}\leq 1} \left|\int_0^1 (Bu)'(t)v'(t)\,dt\right| \\ &= \sup_{\|v\|_{H_0^1}\leq 1} \left|\int_0^1 -(Bu)''v(t)\,dt\right| \\ &= \sup_{\|v\|_{H_0^1}\leq 1} \left|\int_0^1 a(t)u(t)v(t)\,dt\right|.\end{aligned}$$

Hence

$$\begin{aligned}\|Bu\|_{H_0^1} &\leq \|a\|_\infty \sup_{\|v\|_{H_0^1}\leq 1} \int_0^1 |u(t)v(t)|\,dt \\ &\leq \|a\|_\infty \sup_{\|v\|_{H_0^1}\leq 1} \|u\|_{L^2}\|v\|_{L^2} \\ &\leq \|a\|_\infty \|u\|_{L^2} \sup_{\|v\|_{H_0^1}\leq 1} \|v\|_{L^2} \\ &\leq \|a\|_\infty \frac{1}{\sqrt{\lambda_1}}\|u\|_{H_0^1} \sup_{\|v\|_{H_0^1}\leq 1} \frac{1}{\sqrt{\lambda_1}}\|v\|_{H_0^1} \\ &\leq \frac{1}{\lambda_1}\|a\|_\infty \|u\|_{H_0^1}.\end{aligned}$$

Since B is a linear operator, i.e., $\|Bu\|_{H_0^1} \leq \|B\|\|u\|_{H_0^1}$, we get

$$\|B\| \leq \frac{\|a\|_\infty}{\lambda_1} = \frac{\|a\|_\infty}{\pi^2} < 1.$$

By Theorem 2.1, we conclude that the operator A has a fixed point u, weak solution of problem (3.1). Finally, since $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $u \in C^2[0,1]$. □

In the same way, we can study the following fractional boundary value problem:

$$\begin{cases} D^\alpha_{1^-}(D^\alpha_{0^+}u(t)) = f(t, u(t)), \quad t \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{3.5}$$

where $0 < \alpha < 1$ and $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies assumption (3.4) with $\|a\|_\infty < \Gamma^2(\alpha + 1)$. Then

Theorem 3.5. *Problem (3.5) has a solution.*

The proof follows the same line as in Theorem 3.4 and is omitted. Definitions and main properties of fractional operators may be found in [8].

Example 3.6. Consider the boundary value problem for a fractional operator:

$$\begin{cases} D^{\frac{1}{2}}_{1^-}(D^{\frac{1}{2}}_{0^+}u(t)) = f(t, u(t)), \quad t \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{3.6}$$

where $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$f(t,u) = \begin{cases} \frac{1}{2}t^{n_1} \sin u + t^{n_2}, & \text{if} \quad u \geq 0,\ 0 < t < 1 \\ \frac{1}{2}t^{n_1} u^2 + t^{n_2}, & \text{if} \quad u \leq 0,\ 0 < t < 1 \end{cases} \tag{3.7}$$

and $n_i,\ i = 1, 2$ are positive integers. Here $a(t) = \frac{1}{2}t^{n_1}$ and $b(t) = t^{n_2}$. Since (3.4) is clearly satisfied and

$$\|a\|_\infty = \left\|\frac{1}{2}t^{n_1}\right\|_\infty = \frac{1}{2} < \Gamma^2\left(\frac{1}{2} + 1\right) = \Gamma^2\left(\frac{3}{2}\right) = \frac{\pi}{4},$$

then problem has at least one nontrivial weak solution.

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