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A FIXED POINT THEOREM FOR CARISTI-TYPE CYCLIC MAPPINGS

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Abstract. We discuss two results for Caristi-type cyclic mappings due to Du and Karapinar [3]. We show that they can be deduced from our best proximity point theorem. Our result can be regarded as a generalized result of a fixed point theorem proved by Bollenbacher and Hicks [1] in the setting of cyclic mappings.
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1. Introduction and preliminaries

Let X be a nonempty set and $T : X \to X$ be a mapping. By a fixed point of T, we understand a point $x \in X$ such that

$$x = Tx.$$

The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$. In 1976, Caristi [2] proved the following fixed point theorem in a metric space which is an extension of the well-known Banach fixed point theorem.

Theorem C. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that*

$$d(x, Tx) + f(Tx) \leq f(x) \quad \forall x \in X,$$

where $f : X \to (-\infty, \infty]$ is a proper, bounded below and lower semicontinuous function. Then there exists $u \in \mathrm{Fix}(T)$ such that $f(u) < \infty$.

There are many results related to Theorem C. One of them we concern is the following result proved by Du and Karapinar.

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Theorem DK1. ([3, Theorem 3.4]) *Let M be a nonempty subset of a metric space (X,d). Let $f : M \to (-\infty, \infty]$ be a proper and bounded below function and $\varphi : \mathbb{R} \to (0, \infty)$ be a nondecreasing function. Suppose that $T : M \to M$ is a mapping of Caristi type dominated by φ and f, that is,*

$$d(x, Tx) \leq \varphi(f(x))(f(x) - f(Tx)) \quad \forall x \in M. \tag{DK1}$$

Assume that M is closed and X is complete, and one of the following conditions is satisfied:

(D1) *T is continuous;*
(D2) *$G(T) := \{(x, Tx) : x \in M\}$ is closed in $M \times M$;*
(D3) *The function $x \mapsto g(x) := d(x, Tx)$ is lower semicontinuous.*

Then, for any $u \in M$ with $f(u) < \infty$, the sequence $\{T^n u\}$ converges to a fixed point of T.

It is clear that if $\varphi(t) = 1$ for all $t \in \mathbb{R}$, then the mapping of Caristi type in (DK1) of Theorem DK1 becomes the mapping in Theorem C. As mentioned by Du and Karapinar [3], Theorem DK1 is different from Theorem C because it does not require the lower semicontinuity of the dominated function f. Moreover, Theorem DK1 is applied to conclude an interesting fixed point theorem for $\mathcal{MT}$-contractions due to Mizoguchi–Takahashi [7].

Remark 1.1. Let us discuss the statements of Theorem DK1.

(1) It suffices to consider (M, d) as a complete metric space.
(2) The quantity $\varphi(f(x))$ in the condition (DK1) is not defined unless $f(x) < \infty$ because ∞ does not belong to the domain of φ. The term $f(x) - f(Tx)$ in the condition (DK1) is not determined if $f(x) = f(Tx) = \infty$. To illustrate this, let $X = [-1, 1]$ be equipped with the usual metric and $T : X \to X$ be defined by $Tx = \frac{x}{2}$ for all $x \in X$. Let $f : X \to [0, \infty]$ be defined by $f(x) = 2d(x, Tx)$ if $x \in [0, 1]$ and $f(x) = \infty$ if $x \in [-1, 0)$ and $\varphi(t) = 1$ for all $t \in [0, \infty]$. However, it does not effect the proof given there. So we assume in the ststement of Theorem DK1 that f is finite everywhere.

In the paper of Du and Karapinar [3], they also discuss the situation that T does not have a fixed point. Let us recall the setting for this problem: Suppose that A and B are nonempty subsets of a metric space (X, d). Now we are interested in the *cyclic* mapping $T : A \cup B \to A \cup B$, that is, T satisfies

$$T(A) \subset B \quad \text{and} \quad T(B) \subset A.$$

By a *best proximity point* of T, we understand a point $x \in A \cup B$ such that

$$d(x, Tx) = D(A, B) := \inf\{d(a, b) : a \in A, b \in B\}.$$

The set of all best proximity points of T is denoted by $\mathrm{BP}(T)$. If $A = B = X$, then $D(A, B) = 0$ and hence $\mathrm{BP}(T) = \mathrm{Fix}(T)$. In the other word, the problem of finding a best proximity point includes that of finding a fixed point as a special case.

The following result is analogous to Theorem DK1 in this situation.

Theorem DK2. ([3, Theorem 2.2]) *Let A and B be nonempty subsets of a metric space (X, d). Let $f : A \cup B \to \mathbb{R}$ be a proper and bounded below function and*

$\varphi : \mathbb{R} \to (0, \infty)$ be a nondecreasing function. Suppose that $T : A \cup B \to A \cup B$ is a cyclic mapping of Caristi type dominated by φ and f, that is, it is cyclic and satisfies

$$d(x, Tx) - D(A, B) \le \varphi(f(x))(f(x) - f(Tx)) \quad \forall x \in A \cup B. \tag{DK2}$$

Suppose that one of the following conditions is satisfied:

(H1) *T is continuous on $A \cup B$;*

(H2) *$d(Tx, Ty) \le d(x, y)$ for all $(x, y) \in A \times B$;*

(D3) *The function $x \mapsto g(x) := d(x, Tx)$ is lower semicontinuous.*

Let $x_0 \in A$. Then the following statements hold true.

(a) *If $\{T^{2n}x_0\}$ has a convergent subsequence in A, then there exists $\widehat{x} \in A$ such that $d(\widehat{x}, T\widehat{x}) = D(A, B)$.*

(b) *If $\{T^{2n+1}x_0\}$ has a convergent subsequence in B, then there exists $\widehat{x} \in B$ such that $d(\widehat{x}, T\widehat{x}) = D(A, B)$.*

2. Main results

First, we start with a result of Eisenfeld and Lakshmikantham ([4]) in the setting of cyclic mappings.

Theorem 2.1. *Let (X, d) be a metric space and A, B be two nonempty subsets of X. Let $T : A \cup B \to A \cup B$ be a cyclic mapping. Then the following statements are equivalent.*

(i) *There exists a function $f : A \cup B \to [0, \infty)$ such that*

$$d(x, Tx) - D(A, B) \le f(x) - f(Tx) \quad \forall x \in A \cup B.$$

(ii) $\sum_{n=0}^{\infty} (d(T^n x, T^{n+1} x) - D(A, B)) < \infty$ *for all $x \in A \cup B$.*

Proof. (i) $\Rightarrow$ (ii) Assume that (i) holds. Let $x \in A \cup B$. Since $T^{n+1}x = T(T^n x)$, we have

$$d(T^n x, T^{n+1} x) - D(A, B) \le f(T^n x) - f(T^{n+1} x) \quad \forall n \ge 0.$$

Hence $f(T^{n+1}x) \le f(T^n x)$ for all $n \ge 0$. Then $\lim_{n\to\infty} f(T^n x) = \alpha$ for some $\alpha \ge 0$. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty}(d(T^nx, T^{n+1}x) - D(A,B)) \\
&= \lim_{k\to\infty}\sum_{n=0}^{k}(d(T^nx, T^{n+1}x) - D(A,B)) \\
&\le \lim_{k\to\infty}\sum_{n=0}^{k}(f(T^nx) - f(T^{n+1}x)) \\
&= \lim_{k\to\infty}(f(x) - f(T^{n+1}x)) \\
&= f(x) - \lim_{k\to\infty} f(T^{k+1}x) \\
&= f(x) - \alpha < \infty.
\end{aligned}$$

(ii) $\Rightarrow$ (i) Assume that (ii) holds. Define a function $f : A \cup B \to [0, \infty)$ by

$$f(x) = \sum_{n=0}^{\infty}(d(T^nx, T^{n+1}x) - D(A,B)) \quad \forall x \in A \cup B.$$

Note that, for each $k \in \mathbb{N}$, we have

$$\begin{aligned}
&d(x, Tx) - D(A,B) \\
&= \sum_{n=0}^{k+1}(d(T^nx, T^{n+1}x) - D(A,B)) - \sum_{n=0}^{k}(d(T^{n+1}x, T^{n+2}x) - D(A,B)).
\end{aligned}$$

Moreover,

$$\lim_{k\to\infty}\sum_{n=0}^{k+1}(d(T^nx, T^{n+1}x) - D(A,B)) = f(x)$$

and

$$\lim_{k\to\infty}\sum_{n=0}^{k}(d(T^{n+1}x, T^{n+2}x) - D(A,B)) = f(Tx).$$

Hence

$$d(x, Tx) - D(A,B) = f(x) - f(Tx).$$

This completes the proof. □

Setting $A = B = X$ in Theorem 2.1 gives the following corollary which is a result in [4] (see also [1]).

Corollary 2.2. *Let (X, d) be a metric space and let $T : X \to X$ be any mapping. Then the following statements are equivalent.*

(i) *There exists a function $f : X \to [0, \infty)$ such that*

$$d(x, Tx) \le f(x) - f(Tx) \quad \forall x \in X.$$

(ii) $\sum_{n=0}^{\infty} d(T^nx, T^{n+1}x) < \infty$ *for all $x \in X$.*

Let X be a set and $T : X \to X$. Let $x_0 \in X$. By $O(x_0, \infty)$, we denote the set

$$O(x_0, \infty) = \{x_0, Tx_0, T^2x_0, \dots\}.$$

Definition 2.3. Let (X, d) be a metric space, $T : X \to X$ and $x_0 \in X$. A function $g : X \to [0, \infty)$ is said to be T-*orbitally lower semicontinuous at* x_0 if $\{x_n\}$ is a sequence in $O(x_0, \infty)$ and $\lim_{n\to\infty} x_n = x^* \in X$ implies $g(x^*) \le \liminf_{n\to\infty} g(x_n)$.

Lemma 2.4. *Let (X, d) be a metric space. Suppose that $T : X \to X$ and $x_0 \in X$. If $\{y_n\}$ is a sequence in $O(x_0, \infty)$ such that $\lim_{n\to\infty} y_n = y \in X$, then one of the following statements holds.*

(a) *There exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ is a subsequence of $\{T^n x_0\}$. In particular, there is a strictly increasing sequence $\{p_k\}$ of natural numbers such that $y_{n_k} = T^{p_k} x_0$ for all $k \in \mathbb{N}$.*

(b) *There exists $N \in \mathbb{N}$ such that $y_n = y$ for all $n \ge N$.*

Proof. Assume that $\{y_n\}$ is a sequence in $O(x_0, \infty)$ and $\lim_{n\to\infty} y_n = y \in X$. For each $n \in \mathbb{N}$, let $m(n)$ be the smallest number k such that $T^k x_0 = y_n$. We consider the set

$$\mathbb{K} = \{m(n) : n \in \mathbb{N}\}.$$

Case 1. $\mathbb{K}$ is an infinite set. So, there exists a strictly increasing sequence $\{n_k\}$ on $\mathbb{N}$ such that $m(n_k) < m(n_{k+1})$ for all $k \in \mathbb{N}$. Hence $\{y_{n_k}\}$ is a subsequence of $\{T^n x_0\}$.

Case 2. $\mathbb{K}$ is a finite set. Since $\{y_n\}$ is a sequence in a finite set $\{T^j x_0 : j \in \mathbb{K}\}$ and y_n converges to y, there exist $k \in \mathbb{K}$ and $N \in \mathbb{N}$ such that $y_n = T^k x_0$ for all $n \ge N$. Hence $y = T^k x_0$ and the conclusion follows. □

For a sequence $\{z_n\}$ in a metric space (X, d), we define

$$\omega(\{z_n\}) = \left\{ z \in X : z = \lim_{k\to\infty} z_{n_k} \text{ for some subsequence } \{z_{n_k}\} \text{ of } \{z_n\} \right\}.$$

Theorem 2.5. *Let A and B be nonempty subsets of a metric space (X, d). Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping and $f : A \cup B \to [0, \infty]$. Suppose that there exists an $x_0 \in A \cup B$ such that $f(x_0) < \infty$ and*

$$d(y, Ty) - D(A, B) \le f(y) - f(Ty) \quad \forall y \in O(x_0, \infty).$$

Then the following statements hold.

(a) $\sum_{n=0}^{\infty} (d(T^n x_0, T^{n+1} x_0) - D(A, B)) < \infty.$

(b) *If $\omega(\{T^n x_0\}) = \varnothing$, then $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0.*

(c) *Assume that $\omega(\{T^n x_0\}) \ne \varnothing$. Then the following statements are equivalent.*
   (i) $\omega(\{T^n x_0\}) \subset \mathrm{BP}(T)$.
   (ii) *$\omega(\{T^n x_0\}) \subset A \cup B$ and $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0.*

Proof. (a) We can follow the proof of Theorem 2.1

(b) Suppose that $\omega(\{T^n x_0\}) = \varnothing$. Let $\{y_n\}$ be a sequence in $O(x_0, \infty)$ such that

$$\lim_{n\to\infty} y_n = y$$

for some $y \in X$. It follows from Lemma 2.4 that there exists $N \in \mathbb{N}$ such that $g(y_n) = g(y)$ for all $n \geq N$, that is, g is T-orbitally lower semicontinuous at x_0.

(c) Suppose that $\omega(\{T^n x_0\}) \neq \varnothing$.

(i) $\Rightarrow$ (ii) Assume that $\omega(\{T^n x_0\}) \subset \mathrm{BP}(T)$. It is clear that $\omega(\{T^n x_0\}) \subset A \cup B$. Let $\{y_n\}$ be a sequence in $O(x_0, \infty)$ such that $\lim_{n\to\infty} y_n = y$ for some $y \in X$.
By Lemma 2.4, we consider the following two cases.
Case 1. There exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ is a subsequence of $\{T^n x_0\}$. Then

$$y \in \omega(\{T^n x_0\}) \subset \mathrm{BP}(T).$$

Hence

$$g(y) = d(y, Ty) = D(A, B) \leq \liminf_{n\to\infty} d(y_n, Ty_n) = \liminf_{n\to\infty} g(y_n).$$

Case 2. There exists $N \in \mathbb{N}$ such that $y_n = y$ for all $n \geq N$. Hence $g(y_n) = g(y)$ for all $n \geq N$.

Therefore, g is T-orbitally lower semicontinuous at x_0.

(ii) $\Rightarrow$ (i) Suppose that $\omega(\{T^n x_0\}) \subset A \cup B$ and g is T-orbitally lower semicontinuous at x_0. Let $y \in \omega(\{T^n x_0\})$. So $y \in A \cup B$. We show that $d(y, Ty) = D(A, B)$. Since $y \in \omega(\{T^n x_0\})$ there exists a subsequence $\{T^{n_k} x_0\}$ of $\{T^n x_0\}$ such that

$$\lim_{k\to\infty} T^{n_k} x_0 = y.$$

It follows from the T-obitally lower semicontinuity of g at x_0 that

$$\begin{aligned} d(y, Ty) &= g(y) \\ &\leq \liminf_{k\to\infty} g(T^{n_k} x_0) \\ &= \liminf_{k\to\infty} d(T^{n_k} x_0, T(T^{n_k} x_0)) \\ &= \liminf_{k\to\infty} d(T^{n_k} x_0, T^{n_k+1} x_0) \\ &= \lim_{n\to\infty} d(T^n x_0, T^{n+1} x_0) = D(A, B). \end{aligned}$$

Hence $y \in \mathrm{BP}(T)$. Therefore, $\omega(\{T^n x_0\}) \subset \mathrm{BP}(T)$. □

Remark 2.6. In the setting of Theorem 2.5, if $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0, then

(a) $\omega(\{T^n x_0\}) \cap (A \cup B) \subset \mathrm{BP}(T)$,

(b) $(\omega(\{T^{2n+1} x_0\}) \cap A) \cup (\omega(\{T^{2n} x_0\}) \cap B) \subset \mathrm{Fix}(T)$ provided that $x_0 \in A$.

Proof. (a) It follows directly from Theorem 2.5(c).

(b) Assume that $x_0 \in A$. Then $\{T^{2n} x_0\}$ is a sequence in A and $\{T^{2n+1} x_0\}$ is a sequence in B. If

$$(\omega(\{T^{2n+1} x_0\}) \cap A) \cup (\omega(\{T^{2n} x_0\}) \cap B) = \varnothing,$$

then we are done. Suppose that $y \in \omega(\{T^{2n+1} x_0\}) \cap A$. Then there exists a subsequence $\{T^{2n_k+1} x_0\}$ of $\{T^{2n+1} x_0\}$ such that $\lim_{k\to\infty} T^{2n_k+1} x_0 = y \in A$. Then

$$D(A, B) \leq \lim_{k\to\infty} d(y, T^{2n_k+1} x_0) = 0.$$

It follows from (a) that $y \in \mathrm{BP}(T) = \mathrm{Fix}(T)$, that is, $(\omega(\{T^{2n+1}x_0\}) \cap A) \subset \mathrm{Fix}(T)$. Similarly, we have $(\omega(\{T^{2n}x_0\}) \cap B) \subset \mathrm{Fix}(T)$. This completes the proof. □

We deduce the following result due to Bollenbacher and Hicks (see [1, Theorem 3]).

Corollary 2.7. *Let (X, d) be a metric space and let $x_0 \in X$. Suppose that $f : X \to [0, \infty)$ is any function and $T : X \to X$ is a mapping such that*

$$d(y, Ty) + f(Ty) \leq f(y) \quad \forall y \in O(x_0, \infty).$$

Suppose that every Cauchy sequence in $O(x_0, \infty)$ converges to an element in X. Then the following statements are true.

(a) *There exists an element $\widehat{x} \in X$ such that $\widehat{x} = \lim_n T^n x_0$.*

(b) *$\widehat{x} \in \mathrm{Fix}(T)$ if an only if $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0.*

Proof. Setting $A = B = X$ in Theorem 2.5 gives $D(A, B) = 0$ and $\mathrm{BP}(T) = \mathrm{Fix}(T)$. It follows from (a) of Theorem 2.5 that $\sum_{n=0}^{\infty} d(T^n x_0, T^{n+1}x_0) < \infty$, that is, $\{T^n x_0\}$ is a Cauchy sequence in $O(x_0, \infty)$. Hence $\lim_{n\to\infty} T^n x_0 = \widehat{x}$ for some $\widehat{x} \in X$ and hence $\omega(\{T^n x_0\}) = \{\widehat{x}\}$. Moreover, by (c) of Theorem 2.5, we have $\widehat{x} \in \mathrm{Fix}(T)$ if an only if $g(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at x_0. □

Before we show that Theorem DK2 (and hence Theorem DK1) follows from our Theorem 2.5, we observe the following facts.

Lemma 2.8. *Let (X, d) be a metric space, $T : X \to X$ and $x_0 \in X$. Define*

$$g(x) = d(x, Tx) \text{ for all } x \in X.$$

Then the following statements hold.

(a) *If T is continuous, then g is lower semicontinuous and*

$$G(T) := \{(x, Tx) : x \in X\}$$

is closed in $X \times X$.

(b) *If g is lower semicontinuous, then g is T-orbitally lower semicontinuous at x_0.*

Proof. The proof is straightforward, so it is omitted. □

Lemma 2.9. *Let (X, d) be a metric space and $x_0 \in X$. Suppose that $T : X \to X$ is a mapping such that $\lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0) = 0$. If $G(T) := \{(x, Tx) : x \in X\}$ is closed in $X \times X$, then $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0.*

Proof. Let $\{y_n\}$ be a sequence in $O(x_0, \infty)$ such that $\lim_{n\to\infty} y_n = y$ for some $y \in X$. By Lemma 2.4, we consider the following two cases.

Case 1. There exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ is a subsequence of $\{T^n x_0\}$. Note that $\lim_{k\to\infty} d(y_{n_k}, Ty_{n_k}) = 0$. Since $\lim_{k\to\infty} d(y_{n_k}, y) = 0$, we have $\lim_{k\to\infty} d(Ty_{n_k}, y) = 0$. Since $G(T)$ is closed, we have $(y, y) \in G(T)$, that is, $y = Ty$. Then

$$g(y) = d(y, Ty) = 0 \leq \liminf_{n\to\infty} d(y_n, Ty_n) = \liminf_{n\to\infty} g(y_n).$$

Case 2. There exists $N \in \mathbb{N}$ such that $y_n = y$ for all $n \geq N$. Hence

$$g(y) = d(y, Ty) = \liminf_{n\to\infty} d(y_n, Ty_n) = \liminf_{n\to\infty} g(y_n).$$

Therefore, $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0. □

Lemma 2.10. *Let A and B be nonempty subsets of a metric space (X, d) and $x_0 \in A \cup B$. Suppose that $T : A \cup B \to A \cup B$ is a cyclic mapping such that*

$$\omega(\{T^n x_0\}) \subset A \cup B \text{ and } \lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0) = D(A, B).$$

If $d(Tx, Ty) \leq d(x, y)$ for all $x \in A$ and $y \in B$, then $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0.

Proof. Let $\{y_n\}$ be a sequence in $O(x_0, \infty)$ such that $\lim_{n\to\infty} y_n = y$ for some $y \in X$. It is obvious that

$$\liminf_{n\to\infty} g(y_n) = \liminf_{n\to\infty} d(y_n, Ty_n) \geq D(A, B).$$

By Lemma 2.4, we consider the following two cases.

Case 1. There exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ is a subsequence of $\{T^n x_0\}$. We may assume without loss of generality that $y_{n_k} \in A$ for all $k \in \mathbb{N}$. We also assume that there is a strictly increasing sequence $\{p_k\}$ of natural numbers such that $y_{n_k} = T^{p_k} x_0$ for all $k \in \mathbb{N}$. Since $\omega(\{T^n x_0\}) \subset A \cup B$ and $\lim_{k\to\infty} y_{n_k} = y$, we have $y \in A \cup B$. Then we consider the following two subcases.

Subcase 1.1. $y \in A$. Then

$$\begin{aligned}
D(A, B) \leq d(y, Ty) &\leq \liminf_{k\to\infty}(d(y, T^{p_k}x_0) + d(T^{p_k}x_0, Ty)) \\
&= \lim_{k\to\infty} d(y, T^{p_k}x_0) + \liminf_{k\to\infty} d(T^{p_k}x_0, Ty) \\
&= \liminf_{k\to\infty} d(T(T^{p_k-1}x_0), Ty) \\
&\leq \liminf_{k\to\infty} d(T^{p_k-1}x_0, y) \\
&\leq \liminf_{k\to\infty}(d(T^{p_k-1}x_0, T^{p_k}x_0) + d(T^{p_k}x_0, y)) \\
&= \lim_{k\to\infty} d(T^{p_k-1}x_0, T^{p_k}x_0) + \lim_{k\to\infty} d(T^{p_k}x_0, y) \\
&= \lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0) \\
&= D(A, B).
\end{aligned}$$

Thus $g(y) = d(y, Ty) = D(A, B) \leq \liminf_{n\to\infty} g(y_n)$.

Subcase 1.2. $y \in B$. Since $y_{n_k} \in A$ for all $k \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = y \in B$, we have $D(A, B) = 0$. Since $\lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0) = D(A, B)$, we have

$$\lim_{k\to\infty} d(T^{p_k}x_0, T^{p_k+1}x_0) = D(A, B) = 0.$$

Hence

$$\lim_{k\to\infty} Ty_{n_k} = \lim_{k\to\infty} T^{p_k+1}x_0 = y.$$

Then

$$\begin{aligned} d(y, Ty) &\le \liminf_{k\to\infty}(d(y, Ty_{n_k}) + d(Ty_{n_k}, Ty)) \\ &\le \liminf_{k\to\infty}(d(y, Ty_{n_k}) + d(y_{n_k}, y)) \\ &\le \liminf_{k\to\infty} d(y, Ty_{n_k}) + \lim_{k\to\infty} d(y_{n_k}, y) \\ &= 0. \end{aligned}$$

Thus $g(y) = d(y, Ty) = 0 \le \liminf_{n\to\infty} g(y_n)$.

Case 2. There exists $N \in \mathbb{N}$ such that $y_n = y$ for all $n \ge N$. Hence

$$g(y) = d(y, Ty) = \liminf_{n\to\infty} d(y_n, Ty_n) = \liminf_{n\to\infty} g(y_n).$$

Therefore, $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0. □

We show that Theorem DK2 follows from our result.

Theorem DK2 (revisited). *Let A and B be nonempty subsets of a metric space (X, d). Let $f : A \cup B \to \mathbb{R}$ be a bounded below function and $\varphi : \mathbb{R} \to (0, \infty)$ be a nondecreasing function. Suppose that $T : A \cup B \to A \cup B$ is a cyclic mapping of Caristi type dominated by φ and f, that is, it is cyclic and satisfies*

$$d(x, Tx) - D(A, B) \le \varphi(f(x))(f(x) - f(Tx)) \quad \forall x \in A \cup B. \tag{DK2}$$

Suppose that one of the following conditions is satisfied:

- (H1) *T is continuous on $A \cup B$;*
- (H2) *$d(Tx, Ty) \le d(x, y)$ for all $(x, y) \in A \times B$;*
- (D3) *The function $x \mapsto g(x) := d(x, Tx)$ is lower semicontinuous.*

Let $x_0 \in A$. Then the following statements hold true.

- (a) *If $\{T^{2n}x_0\}$ has a convergent subsequence in A, then there exists $\widehat{x} \in A$ such that $d(\widehat{x}, T\widehat{x}) = D(A, B)$.*
- (b) *If $\{T^{2n+1}x_0\}$ has a convergent subsequence in B, then there exists $\widehat{x} \in B$ such that $d(\widehat{x}, T\widehat{x}) = D(A, B)$.*

Proof of Theorem DK2 (revisited) via Theorem 2.5 and Remark 2.6. Let $x_0 \in A$. Then $T^{2n}x_0 \in A$ for all $n \ge 0$ and $T^{2n+1}x_0 \in B$ for all $n \ge 0$. Let $\alpha = \varphi(f(x_0))$. Siunce f is bounded below, there exists a real number m such that $f(x) \ge m$ for all $x \in A \cup B$. Define a function $\widehat{f} : A \cup B \to [0, \infty)$ by

$$\widehat{f}(x) = \alpha f(x) - \alpha m \text{ for all } x \in A \cup B.$$

Then, for each $n \in \mathbb{N} \cup \{0\}$, we have

$$0 \le d(T^n x_0, T^{n+1}x_0) - D(A, B) \le \varphi(f(T^n x_0))(f(T^n x_0) - f(T^{n+1}x_0)).$$

In particular, since $\varphi(f(T^n x_0)) > 0$, we have $f(T^{n+1}x_0) \le f(T^n x_0)$ for all $n \ge 0$. Consequently, since φ is nondecreasing, we have $\varphi(f(T^n x_0)) \le \alpha$ for all $n \ge 0$. We now conclude that

$$d(y, Ty) - D(A, B) \le \widehat{f}(y) - \widehat{f}(Ty) \quad \forall y \in O(x_0, \infty).$$

Assume that (H1) or (H2) or (D3) holds. Then by Lemma 2.8 and Lemma 2.10, we have $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0.

We assume that $\{T^{2n}x_0\}$ has a convergent subsequence in A. Then there are an element $\widehat{x} \in A$ and a subsequence $\{T^{2n_k}x_0\}$ of $\{T^{2n}x_0\}$ such that $\lim_{k\to\infty} T^{2n_k}x_0 = \widehat{x}$. Hence $\widehat{x} \in \omega(\{T^n x_0\}) \cap A$. It follows then that $\widehat{x} \in \mathrm{BP}(T)$.

For the case that $\{T^{2n+1}x_0\}$ has a convergent subsequence in B, we can prove similarly. □

Finally, we present a nonself version of a Banach type fixed point theorem of Hicks and Rhoades [5] with an assumption of Kada, Suzuki and Takahashi [6].

Theorem 2.11. *Let A and B be nonempty subsets of a metric space (X, d). Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping. Suppose that there exist a constant $k \in (0, 1)$ and an element $x_0 \in A \cup B$ such that*

$$d(Ty, T^2y) \le kd(y, Ty) + (1 - k)D(A, B) \quad \forall y \in O(x_0, \infty).$$

Suppose that the following conditions hold:

(C1) $\varnothing \neq \omega(\{T^n x_0\}) \subset A \cup B$.

(C2) *If $d(z, Tz) > D(A, B)$, then*

$$\inf\{d(y, z) + d(y, Ty) : y \in O(x_0, \infty)\} > D(A, B).$$

Then $\omega(\{T^n x_0\}) \subset \mathrm{BP}(T)$.

Lemma 2.12. *Let A and B be nonempty subsets of a metric space (X, d) and $x_0 \in A \cup B$. Suppose that $T : A \cup B \to A \cup B$ is a cyclic mapping such that*

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0) = D(A, B).$$

Assume that the following condition holds.

(C1) $\omega(\{T^n x_0\}) \subset A \cup B$.

(C2) *If $d(z, Tz) > D(A, B)$, then*

$$\inf\{d(y, z) + d(y, Ty) : y \in O(x_0, \infty)\} > D(A, B).$$

Then $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0.

Proof. Let $\{y_n\}$ be a sequence in $O(x_0, \infty)$ such that $\lim_{n\to\infty} y_n = y$ for some $y \in X$. Then $y \in A \cup B$. By Lemma 2.4, we consider the following two cases.

Case 1. There exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ is a subsequence of $\{T^n x_0\}$. Then

$$\lim_{k\to\infty} d(y_{n_k}, y) + \lim_{k\to\infty} d(y_{n_k}, Ty_{n_k}) = D(A, B).$$

Hence

$$\inf\{d(x, y) + d(x, Tx) : x \in O(x_0, \infty)\} = D(A, B).$$

Then, by (C2), we have $d(y, Ty) = D(A, B)$. Hence

$$g(y) = D(A, B) \le \liminf_{n\to\infty} d(y_n, Ty_n) = \liminf_{n\to\infty} g(y_n).$$

Case 2. There exists $N \in \mathbb{N}$ such that $y_n = y$ for all $n \ge N$. Hence

$$g(y) = d(y, Ty) = \liminf_{n\to\infty} d(y_n, Ty_n) = \liminf_{n\to\infty} g(y_n).$$

Therefore, $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0. □

Proof of Theorem 2.11. Define a function $f : A \cup B \to [0, \infty)$ by

$$f(x) = \frac{1}{1-k} d(x, Tx) \text{ for all } x \in A \cup B.$$

Let $y \in O(x_0, \infty)$. Then we have

$$d(Ty, T^2y) \leq kd(y, Ty) + (1-k)D(A, B).$$

It follows that

$$d(Ty, T^2y) + (1-k)d(y, Ty) \leq d(y, Ty) + (1-k)D(A, B).$$

Hence

$$d(y, Ty) - D(A, B) \leq \frac{1}{1-k} d(y, Ty) - \frac{1}{1-k} d(Ty, T^2y) = f(y) - f(Ty).$$

Then, by Lemma 2.12, $g(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at x_0. Hence, by Theorem 2.5, we have $\omega(\{T^n x_0\}) \subset \mathrm{BP}(T)$. □

Set $A = B = X$ in Theorem 2.11 gives the following one which is a result of Kada, Suzuki and Takahashi (see [6]) in the setting of a metric space.

Corollary 2.13. *Let* (X, d) *be a complete metric space and* $k \in (0, 1)$. *Assume that* $T : X \to X$ *is a mapping such that*

$$d(Tx, T^2x) \leq kd(x, Tx) \quad \forall x \in X.$$

Suppose that $\inf\{d(x, z) + d(x, Tx) : x \in X\} > 0$ *for all* $z \neq Tz$. *Then, for every* $x \in X$, *the sequence* $\{T^n x\}$ *converges to a fixed point of* T.

Finally, we illustrate our Theorem 2.5 by the following two examples.

Example 2.14. Let $A = [-2, -1]$, $B = [1, 2]$ and $X = A \cup B$ be equiped with the usual metric. Note that $D(A, B) = 2$. Let $T : A \cup B \to A \cup B$ be a cyclic mapping defined by $Tx = \frac{-x+1}{2}$ if $x \in A$ and $Tx = \frac{-x-1}{2}$ if $x \in B$. Let $f : A \cup B \to [0, \infty)$ be defined by $f(x) = -4x$ for all $x \in A$ and $f(x) = 4x$ for all $x \in B$. Then

$$d(x, Tx) - D(A, B) \leq f(x) - f(Tx) \quad \forall x \in A \cup B.$$

It is easy to see that $\omega(\{T^n x_0\}) = \mathrm{BP}(T) = \{-1, 1\}$ for all $x_0 \in X$.

Example 2.15. We modify the sets A and B in the preceding example, that is, $A = [-2, -1)$ and $B = (1, 2]$. Let X, T, f be the same as in Example 2.14. Now, $\mathrm{BP}(T) = \varnothing = \omega(\{T^n x_0\})$ for all $x_0 \in X$.

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