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ON PROPERTIES OF CONTRACTIONS AND NONEXPANSIVE MAPPINGS ON SPHERICAL CAPS IN HILBERT SPACES

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Abstract. Let H be an at least two-dimensional real Hilbert space with the unit sphere S_H. For $\alpha \in [-1,1]$ and $n \in S_H$ we define an (α, n)-spherical cap by $S_{\alpha,n} = \{x \in S_H : \langle x, n \rangle \geq \alpha\}$. We show that the distance between the set of contractions $T : S_{\alpha,n} \to S_{\alpha,n}$ and the identity mapping is positive iff $\alpha < 0$. We also study the fixed point property and the minimal displacement problem in this setting for nonexpansive mappings.
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1. Introduction and preliminaries

Let (M, ρ) be a metric space. By $B_M(x, r)$ we denote the closed ball with center at $x \in M$ and radius r. A mapping $T : M \to M$ is said to be k-lipschitzian if there exists a constant $k \geq 0$ such that $\rho(Tx, Ty) \leq k\rho(x, y)$ for all $x, y \in M$. If $k = 1$, then the map T is called nonexpansive. A contraction is a k-lipschitzian map with $k < 1$. The space M has the fixed point property for nonexpansive mappings (FPP for short) if any nonexpansive mapping $T : M \to M$ has a fixed point. The quantity

$$d_T = \inf_{x \in M} \rho(x, Tx)$$

(see [7]) is called the minimal displacement of a mapping $T : M \to M$. The space M has the almost fixed point property (AFPP for short) if $d_T = 0$ for all nonexpansive mappings $T : M \to M$. If $d_T > 0$, then we say that the map T has a positive minimal displacement.

Let C be the set of all contractions from M into itself. Put

$$I_M = \inf_{T \in C} \left\{ \sup_{x \in M} \rho(x, Tx) \right\}.$$

This coefficient describes the degree of accuracy with which the identity can be estimated by contractions. We shall refer to I_M as the distance between the identity and contractions.

Throughout the paper, H denotes a real Hilbert space of dimension at least 2 with inner product $\langle \cdot, \cdot \rangle$. By B_H and S_H we denote the closed unit ball and the unit sphere of H, respectively. The angle between $x, y \in H \setminus \{0\}$ is defined by

$$\angle (x, y) = \arccos \frac{\langle x, y \rangle}{\|x\| \, \|y\|}.$$

Given $n \in S_H$ ("the north pole"), and $\alpha, \beta \in [-1, 1]$, we define a spherical segment by

$$S_{\alpha,\beta,n} = \{x \in S_H : \alpha \le \langle x, n \rangle \le \beta\}$$

and an (α, n)-spherical cap $S_{\alpha,n}$ by

$$S_{\alpha,n} = S_{\alpha,1,n}.$$

In the same way we define spherical rings and caps for any $z \in S_H$. The $(0, z)$-spherical cap is called a hemisphere.

The Banach contraction principle [1] states that if $T : M \to M$ is a contraction and M is complete, then T has a unique fixed point. Using the notion of I_M we get the following theorem.

Theorem 1.1. *If (M, ρ) is a complete metric space such that $I_M = 0$, then M has the almost fixed point property.*

Proof. Let $\varepsilon > 0$ and $T : M \to M$ be a nonexpansive mapping. In view of the definition of I_M, there exists a contraction $T_\varepsilon : M \to M$ such that $\rho(x, T_\varepsilon x) \le \varepsilon$ for all $x \in M$. The composition $T_\varepsilon \circ T$ is a contraction on M. Therefore, by the Banach contraction principle, it has a fixed point x_0, and consequently $\rho(Tx_0, x_0) = \rho(Tx_0, T_\varepsilon(Tx_0)) \le \varepsilon$. As $\varepsilon > 0$ is arbitrary, this proves the result.

It is easy to see that Theorem 1.1 applies to any nonempty closed convex and bounded set in a Banach space. However, some non-convex sets also satisfy assumptions of Theorem 1.1.

Claim 1.2. *If $w \in S_H$ and $\varepsilon \in (0, 1]$, then the mapping $T_\varepsilon : S_{0,w} \to S_{0,w}$ defined by*

$$T_\varepsilon x = \frac{(1 - \varepsilon) x + \varepsilon w}{\|(1 - \varepsilon) x + \varepsilon w\|}$$

is a contraction, which satisfies the inequality

$$\sup \{\|x - T_\varepsilon x\| : x \in S_{0,w}\} \le \sqrt{2\varepsilon}. \tag{1.1}$$

Proof. For arbitrary $u, v \in H \setminus \{0\}$ and $z \in S_{0,w}$, we have

$$\begin{aligned} \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| &= \left(2 - 2\left\langle \frac{u}{\|u\|}, \frac{v}{\|v\|} \right\rangle\right)^{\frac{1}{2}} \le \left(\frac{\|u\|^2 - 2\langle u, v \rangle + \|v\|^2}{\|u\| \, \|v\|}\right)^{\frac{1}{2}} \\ &\le \frac{\|u - v\|}{\min\{\|u\|, \|v\|\}} \end{aligned}$$

and $\|(1-\varepsilon)z+\varepsilon w\|^2 = (1-\varepsilon)^2 + 2(1-\varepsilon)\varepsilon\langle z,w\rangle + \varepsilon^2 \geq (1-\varepsilon)^2+\varepsilon^2$. From this we obtain

$$\|T_\varepsilon x - T_\varepsilon y\| \leq \frac{(1-\varepsilon)\|x-y\|}{\min\{\|(1-\varepsilon)x+\varepsilon w\|, \|(1-\varepsilon)y+\varepsilon w\|\}} \leq \frac{1-\varepsilon}{\sqrt{(1-\varepsilon)^2+\varepsilon^2}}\|x-y\|,$$

where $x, y \in S_{0,w}$. T_ε is a contraction because $\frac{1-\varepsilon}{\sqrt{(1-\varepsilon)^2+\varepsilon^2}} < 1$.

The inequality (1.1) follows from the estimate

$$\|x-T_\varepsilon x\|^2 = 2-2\langle x, T_\varepsilon x\rangle \leq 2-2\langle x,(1-\varepsilon)x+\varepsilon w\rangle = 2-2(1-\varepsilon)-2\varepsilon\langle x,w\rangle \leq 2\varepsilon,$$

where $x \in S_{0,w}$.

Claim 1.2 shows that the condition $I_M = 0$ is satisfied for hemispheres of inner product spaces. Hence, by Theorem 1.1, we obtain:

Corollary 1.3. *Hemispheres in Hilbert spaces have the almost fixed point property.*

In the sequel, the following lemma will play an important role.

Lemma 1.4. *Let $T : D \to S_H$, where $D \subset S_H$, be a nonexpansive mapping. Then:*

(1) *If for some $x \in D$ we have $-x \in D$ and $T(-x) \neq -Tx$, then $T(D) \subset S_{0,w}$, where $w = \frac{Tx+T(-x)}{\|Tx+T(-x)\|}$.*

(2) *Let*

$$E = \{x \in D : -x \in D \wedge T(-x) = -Tx\}.$$

For any $y \in D$, the mapping $T|_{E\cup\{y\}}$ is an isometry.

Proof. Choose $x \in D$ satisfying conditions $-x \in D$ and $T(-x) \neq -Tx$. Next, choose an arbitrary $y \in D$. Since T is nonexpansive, $\langle Tx, Ty\rangle \geq \langle x,y\rangle$. Similarly, $\langle T(-x), Ty\rangle \geq \langle -x, y\rangle$. Adding the last two inequalities, we find that

$$\langle Tx+T(-x), Ty\rangle = \langle Tx,Ty\rangle + \langle T(-x),Ty\rangle \geq \langle x,y\rangle + \langle -x,y\rangle = 0.$$

This proves the first statement.

If $E = \emptyset$, then the second statement is trivial. Assume that $E \neq \emptyset$ and $y \in D$. Consider two distinct elements x, z such that $x \in E$ and $z \in E \cup \{y\}$. Since T is nonexpansive,

$$\langle Tx, Tz\rangle \geq \langle x,z\rangle \tag{1.2}$$

and $\langle T(-x), Tz\rangle \geq \langle -x, z\rangle$. Applying $T(-x) = -Tx$, we get

$$\langle Tx, Tz\rangle \leq \langle x,z\rangle. \tag{1.3}$$

Combining (1.2) and (1.3), we obtain $\langle Tx,Tz\rangle = \langle x,z\rangle$. Therefore, $T|_{E\cup\{y\}}$ is an isometry.

Corollary 1.5. *Let $\alpha \in (-1,0)$, $n \in S_H$ and $M = S_{\alpha,n}$. If $T : M \to M$ is a nonexpansive mapping, then there exists $w \in S_H$ such that $T(M) \subset S_{0,w}$, or T is an isometry.*

Proof. Assume that there is no $w \in S_H$ such that $T(M) \subset S_{0,w}$. Then by Lemma 1.4, for any $y \in M$ the mapping $T|_{N\cup\{y\}}$, where $N = S_{\alpha,|\alpha|,n}$, is an isometry. We shall prove that T is an isometry.

Clearly, it is sufficient to consider the case u, $v \in S_{|\alpha|,n}$, $u \neq v$. Choose $x \in N \cap \operatorname{span}\{u,v\}$ such that $\|x-u\| \leq \|x-v\|$. Observe that the conditions $\|Tz_1 - Tz_2\| = \|z_1 - z_2\|$ and $\angle(Tz_1, Tz_2) = \angle(z_1, z_2)$ are equivalent for any z_1, $z_2 \in M$. Since $T|_{N\cup\{u\}}$ and $T|_{N\cup\{v\}}$ are isometries, $\angle(Tx,Tu) = \angle(x,u)$ and $\angle(Tv, T(-x)) = \angle(v,-x)$. It is easy to see that $\angle(x,u) + \angle(u,v) + \angle(v,-x) = \pi$. Applying the spherical triangle inequality, we get

$$\begin{aligned}\pi &= \angle(Tx, T(-x)) \leq \angle(Tx,Tu) + \angle(Tu,Tv) + \angle(Tv,T(-x)) \\ &\leq \angle(x,u) + \angle(u,v) + \angle(v,-x) = \pi,\end{aligned}$$

which shows that $\angle(Tu,Tv) = \angle(u,v)$. Therefore, T is an isometry.

2. The property E and the nonexpansive extension property

Given two metric spaces (M,ρ) and (M',ρ'), a mapping T from M into M' is called nonexpansive if

$$\rho'(Tx,Ty) \leq \rho(x,y)$$

for all $x, y \in M$. We say that the pair $(M,\ M')$ has the nonexpansive extension property if for each subset $D \subset M$ and each nonexpansive mapping $T : D \to M'$, there is a nonexpansive mapping $T' : M \to M'$ which extends T, that is, $T'|_D = T$.

In [12], F. A. Valentine showed that the pair $(M,\ M')$ has the nonexpansive extension property if it has the following property.

Property E. *We say that the pair $(M,\ M')$ has the property E if for any at least two-element set I, and for any $x_i \in M$, $x_i' \in M'$, $r_i > 0$, $i \in I$ such that*

$$\rho'\left(x'_i, x'_j\right) \leq \rho(x_i, x_j), \quad i,j \in I, \tag{2.1}$$

the condition

$$\bigcap_{i\in I} B_M(x_i, r_i) \neq \emptyset \tag{2.2}$$

implies

$$\bigcap_{i\in I} B_{M'}(x'_i, r_i) \neq \emptyset. \tag{2.3}$$

Applying this property, Valentine proved the nonexpansive extension property if each M and M' is:

(1) a Hilbert space,
(2) a sphere of a n-dimensional Euclidean space.

The following theorem from Valentine's paper will be helpful later on.

Theorem 2.1. *If H is finite-dimensional, then the pair of spaces $M = M' = S_H$ has the property E.*

The next corollary follows straightforwardly from the above theorem.

Corollary 2.2. *Let H be finite-dimensional, $D \subset S_H$ and let $T : D \to S_H$ be a nonexpansive mapping. Then there exists a nonexpansive mapping $T' : S_H \to S_H$ such that $T'|_D = T$.*

The below lemma shows that the pair consisting of a spherical cap and a hemisphere of a Hilbert space also has the nonexpansive extension property.

Lemma 2.3. *Let $w \in S_H$. The property E holds for the pair of spaces $M = S_H$ and $M' = S_{0,w}$.*

Proof. Consider a set I with at least two elements. Assume that $x_i \in M$, $x_i' \in M'$, $r_i > 0$, $i \in I$ satisfy conditions (2.1) and (2.2). Consider a finite at least two-element subset J of I. Define sets $N = S_Y$, $N' = S_{Y'}$, where Y, Y' are subspaces of H such that $\{x_i : i \in J\} \subset Y$, $\{x_i' : i \in J\} \subset Y'$ and $\dim Y = \dim Y' = \mathrm{card}\,(J)$. Choose an arbitrary $x \in \bigcap_{i \in I} B_M\,(x_i, r_i)$. It is easy to see that there is $m \in N$ such that $\langle x_i, m\rangle \geq 0$ for all $i \in J$. Let $\hat{x}$ be the orthogonal projection of x onto Y. Put $\check{x} = \hat{x} + \alpha m$, where $\alpha \geq 0$ is chosen so that $\check{x} \in N$. It is easy to see that $\check{x} \in \bigcap_{i \in J} B_N\,(x_i, r_i)$, and so $\bigcap_{i \in J} B_N\,(x_i, r_i) \neq \emptyset$. By Theorem 2.1, we have

$$\bigcap_{i \in J} B_{N'}\,(x_i', r_i) \neq \emptyset. \tag{2.4}$$

Let $P : H \to \{w\}^{\perp}$ be the orthogonal projection. Sets $P\,(B_{M'}\,(x_i', r_i))$ are closed and convex. In view of (2.4) and $P\,(B_{M'}\,(x_i', r_i)) = P\,(B_{S_H}\,(x_i', r_i)) \supset P\,(B_{N'}\,(x_i', r_i))$, these sets have the finite intersection property. The ball $B_{\{w\}^{\perp}}$ is weakly compact, hence

$$Z := \bigcap_{i \in I} P\,(B_{M'}\,(x_i', r_i)) \neq \emptyset.$$

Since $\widetilde{P} = P|_{M'}$ is a bijection from M' onto $B_{\{w\}^{\perp}}$, so $\widetilde{P}^{-1}\,(Z) \subset B_{M'}\,(x_i', r_i)$ for all $i \in I$. This completes the proof.

3. The distance between the identity and contractions

By Claim 1.2, $I_M = 0$ if M is a hemisphere of a Hilbert space. Using the same argument one can easily show that $I_M = 0$ if M is a (α, n)-spherical cap and $\alpha \in (0, 1]$. In this section we apply the result from the last section to obtain the distance between the identity and contractions for (α, n)-spherical caps in case of $\alpha \in [-1, 0)$.

Theorem 3.1. *If $\alpha \in [-1, 0)$, $n \in S_H$ and $M = S_{\alpha,n}$, then $I_M = 2\,|\alpha|$.*

Proof. Let us consider two cases:

i) H is a finite-dimensional Hilbert space. Let $T : M \to M$ be a contraction. By Corollary 2.2, there exists a nonexpansive mapping $T' : S_H \to S_H$ such that $T'|_M = T$. In view of Lemma 1.4, $T'\,(M) \subset S_{0,w}$ for some $w \in S_H$. Given $\varepsilon \in (0, 1)$, let $T_\varepsilon : S_{0,w} \to S_{0,w}$ be the mapping defined in Claim 1.2. Put $\tilde{T} = T_\varepsilon \circ T'$. The mapping $-\tilde{T}$ is a contraction, so it has a fixed point x_0, hence $\tilde{T}x_0 = -x_0$.

Suppose that $x_0 \in M$. Applying Claim 1.2, we obtain

$$2 = \left\| x_0 - \tilde{T}x_0 \right\| \leq \|x_0 - T'x_0\| + \|T'x_0 - T_\varepsilon\,(T'x_0)\| \leq \|x_0 - Tx_0\| + \sqrt{2\varepsilon}.$$

Hence

$$\|x_0 - Tx_0\| \geq 2 - \sqrt{2\varepsilon} \geq 2\,|\alpha| - \sqrt{2\varepsilon}. \tag{3.1}$$

Assume now that $x_0 \notin M$. Consider a two-dimensional subspace W of H such that $x_0, n \in W$. Let $u, v \in S_{\alpha,\alpha,n} \cap W$, where $u \neq v$, and $\angle(x_0, u) \leq \angle(x_0, v)$. Since $\angle(x_0, u) + \angle\left(u, \tilde{T}u\right) + \angle\left(\tilde{T}u, -x_0\right) \geq \pi$, so

$$\begin{aligned}\angle\left(u, \tilde{T}u\right) &\geq \pi - \angle(x_0, u) - \angle\left(\tilde{T}u, -x_0\right) = \pi - \angle(x_0, u) - \angle\left(\tilde{T}u, \tilde{T}x_0\right) \\ &\geq \pi - 2\angle(x_0, u) \geq \pi - \angle(x_0, u) - \angle(x_0, v) = 2\angle(u, n) - \pi.\end{aligned}$$

Hence

$$\begin{aligned}\left\|u - \tilde{T}u\right\|^2 &= 2 - 2\left\langle u, \tilde{T}u\right\rangle = 2 - 2\cos\angle\left(u, \tilde{T}u\right) \geq 2 - 2\cos\left(2\angle(u, n) - \pi\right) \\ &= 2 + 2\cos\left(2\angle(u, n)\right) = 4\left(\cos\angle(u, n)\right)^2 = 4\langle u, n\rangle^2 = 4\alpha^2.\end{aligned}$$

From this we obtain $\|u - Tu\| \geq 2|\alpha| - \sqrt{2\varepsilon}$. Since $\varepsilon \in (0, 1)$ is arbitrary, this and (3.1) show that $\sup_{x \in M} \|x - Tx\| \geq 2|\alpha|$, and hence $I_M \geq 2|\alpha|$.

Observe that for the mapping $T : M \to M$,

$$Tx = \begin{cases} x, & x \in S_{0,n}, \\ x - 2\langle x, n\rangle n, & x \in M \setminus S_{0,n}, \end{cases}$$

we have $\sup_{x \in M} \|x - Tx\| = 2|\alpha|$, therefore $I_M = 2|\alpha|$.

ii) H is an infinite-dimensional Hilbert space. We consider a contraction $T : M \to M$. In view of Lemma 1.4, $T(M) \subset S_{0,w}$ for some $w \in S_H$. By Lemma 2.3 and Valentine theorem from the previous section, there is an extension T' of T onto S_H such that $T'|_M = T$ and T' is a nonexpansive mapping. Now, it is enough to repeat the reasoning from the proof of the previous case.

4. Fpp, afpp and minimal displacement

Let T be a nonexpansive mapping on an (α, n)-spherical cap. The problem of the existence of fixed points of such a mapping is trivial for $\alpha = -1$ and $\alpha = 1$. Namely, if $\alpha = 1$, then $S_{1,n} = \{n\}$. If $\alpha = -1$, then $S_{-1,n} = S_H$ and the map $Tx = -x$ is fixed point free and moreover $d_T = 2$. Next, observe that if $\alpha \in [0, 1)$, then, in view of Corollary 1.3, we have $d_T = 0$. More generally, it is easy to get the following lemma.

Lemma 4.1. *If $\alpha \in [-1, 1]$ and $T : S_{\alpha,n} \to S_{\alpha,n}$ is a nonexpansive mapping such that $T(S_{\alpha,n})$ is contained in a hemisphere $\widetilde{S}$ which is contained in $S_{\alpha,n}$, then $d_T = 0$.*

Proof. Define the mapping $\widetilde{T} : \widetilde{S} \to \widetilde{S}$ by the formula $\widetilde{T}x = Tx$. Observe that

$$d_T \leq d_{\widetilde{T}} = \inf_{x \in \widetilde{S}} \|x - Tx\| = 0,$$

which finishes the proof.

The situation is also completely clear if H is a finite-dimensional Hilbert space.

Theorem 4.2. *Let $\alpha \in (-1, 1)$ and H be a finite-dimensional Hilbert space. Then every continuous mapping $T : S_{\alpha,n} \to S_{\alpha,n}$ has a fixed point.*

Proof. Observe that $S_{\alpha,n}$ is homeomorphic with B_Y, where $Y = \{n\}^{\perp}$. According to Brouwer's fixed point theorem [5], every continuous self-map of B_Y has a fixed point, which finishes the proof.

If H is an infinite-dimensional Hilbert space, then the situation for nonexpansive mappings $T : S_{\alpha,n} \to S_{\alpha,n}$ is more complicated. We present an example showing that d_T can be positive. But before, recall that in all infinite-dimensional Banach spaces X there exist Lipschitz retractions of B_X onto S_X. This fact was proved by Nowak [11] for some spaces and extended to all spaces by Benyamini and Sternfeld [3]. Let $k_0(X)$ denote the so-called retraction constant, being the infimum of the set of all positive k for which there exists a Lipschitz retraction $R : B_X \to S_X$ with the constant k. It is known that $k_0(H) \geq 4.55$ [8]. On the other hand, there are known some constructions of such retractions [10], [2], [4] and the best known estimation from above is $k_0(H) < 28.99$ [2].

Example 4.3. *Let $k > k_0(H)$, $\alpha \in \left(-1, \frac{-k}{\sqrt{1-k^2}}\right)$ and $n \in S_H$. Define the mapping $T : S_{\alpha,n} \to S_{\alpha,n}$ by the formula*

$$Tx = \begin{cases} -(T_3 \circ T_2 \circ T_1)(x) & \text{if } x \in S_{0,n}, \\ -T_3(x) & \text{if } x \in S_{\alpha,n} \setminus S_{0,n}, \end{cases}$$

where T_1 is the orthogonal projection on the subspace $Y = \{n\}^{\perp}$, T_2 is a Lipschitz retraction of the ball B_Y onto the sphere S_Y with the Lipschitz constant k, and $T_3 : S_{\alpha,0,n} \to S_{\alpha,\alpha,n}$ is the closest point projection. We shall show that T is a nonexpansive mapping. Obviously, if $x, y \in S_{0,n}$ or $x, y \in S_{\alpha,0,n}$, then $\|Tx - Ty\| \leq \|x - y\|$. Assume now that $x \in S_{0,n}$ and $y \in S_{\alpha,n} \setminus S_{0,n}$. Choose $z \in S_Y$ such that $\angle(x, z) + \angle(z, y) = \angle(x, y)$. Then $\angle(Tx, Tz) \leq \angle(x, z)$ and $\angle(Tz, Ty) \leq \angle(z, y)$, and therefore

$$\angle(Tx, Ty) \leq \angle(Tx, Tz) + \angle(Tz, Ty) \leq \angle(x, z) + \angle(z, y) = \angle(x, y),$$

so T is a nonexpansive mapping. The mapping T has the minimal displacement equal to $\frac{2}{k}$.

Now we present some lemmas and theorems, which guarantee AFPP in an infinite-dimensional Hilbert space H for $\alpha \in (-1, 0)$.

Lemma 4.4. *Let $\alpha \in (-1, 0)$, $n \in S_H$, $M = S_{\alpha,n}$ and let $T : M \to M$ be an isometry. Then the mapping $\tilde{T} : H \to H$ defined by the formula*

$$\tilde{T}(\beta x) = \begin{cases} \beta Tx, & x \in M, \\ -\beta T(-x), & x \in S_H \setminus M, \end{cases}$$

where $\beta \in [0, \infty)$, is an isometry such that $\tilde{T}|_M = T$.

Proof. First, we shall prove that $\tilde{T}|_{S_H}$ is an isometry. Let $u, v \in S_H$. The case $u, v \in M$ is obvious. If $u, v \notin M$, then $\left\|\tilde{T}u - \tilde{T}v\right\| = \|T(-u) - T(-v)\| = \|u - v\|$. Assume now that $u \notin M$, $v \in M$. Then

$$\angle\left(\tilde{T}u, \tilde{T}v\right) = \angle(-T(-u), Tv) = \pi - \angle(T(-u), Tv) = \pi - \angle(-u, v) = \angle(u, v).$$

Therefore, $\tilde{T}|_{S_H}$ is an isometry. Any two elements of H can be expressed in the forms βx, γy, where β, $\gamma \geq 0$ and x, $y \in S_H$. Then

$$\left\|\tilde{T}(\beta x) - \tilde{T}(\gamma y)\right\|^2 = \beta^2 - 2\beta\gamma \left\langle \tilde{T}x, \tilde{T}y \right\rangle + \gamma^2 = \beta^2 - 2\beta\gamma \langle x, y\rangle + \gamma^2 = \|\beta x - \gamma y\|^2 .$$

Thus $\tilde{T}$ is an isometry.

Theorem 4.5. (Baker) [6, Theorem 1.3.8] *If T is an isometry from a real normed space X into a strictly convex, real normed space Y such that $T(0) = 0$, then T is a linear mapping.*

Theorem 4.6. *Let $\alpha \in (-1, 0)$, $n \in S_H$ and $M = S_{\alpha,n}$. If $T : M \to M$ is an isometry, then $d_T = 0$.*

Proof. Let $\tilde{T}$ be the isometry defined in Lemma 4.4. By Theorem 4.5, $\tilde{T}$ is a linear mapping. Clearly, $\tilde{T}(H)$ is a closed subspace of H. Assume that $\tilde{T}(H) = H$. Then $T(S_{\alpha,\alpha,n}) \subset S_{\alpha,\alpha,Tn}$. For an arbitrary $y \in S_{\alpha,\alpha,Tn}$ there is $x \in S_H$ such that $y = \tilde{T}x$. Since $\left\langle \tilde{T}x, \tilde{T}n \right\rangle = \langle x, n\rangle$, we have $x \in S_{\alpha,\alpha,n}$. It follows that $T(S_{\alpha,\alpha,n}) = S_{\alpha,\alpha,Tn}$. But $S_{\alpha,\alpha,Tn} \subset M$, so $Tn = n$, and therefore $d_T = 0$.

Now consider the case where $\tilde{T}(H) \neq H$. Choose an arbitrary $w \in \left(\tilde{T}(H)\right)^{\perp} \cap S_{0,n}$. Then $T(S_{0,w} \cap M) \subset S_{0,w} \cap M$. Let $\varepsilon \in (0, 1)$ and T_ε be the mapping defined in Claim 1.2. We have $T_\varepsilon(S_{0,w} \cap M) \subset S_{0,w} \cap M$, since

$$\langle T_\varepsilon x, n\rangle = \left\langle \frac{(1-\varepsilon)x + \varepsilon w}{\|(1-\varepsilon)x + \varepsilon w\|}, n \right\rangle \geq \frac{1-\varepsilon}{\|(1-\varepsilon)x + \varepsilon w\|} \langle x, n\rangle \geq \alpha$$

for any $x \in S_{0,w} \cap M$. Applying the reasoning similar to that in the proof of Theorem 1.1 for the mapping $T_\varepsilon \circ T$ from $S_{0,w} \cap M$ into itself, we obtain $d_T = 0$.

Note that, by Lemma 4.1, the above theorem is also true for $\alpha \in [0, 1]$.

Lemma 4.7. *Given $\alpha \in (-1, 0)$, let $T : S_{\alpha,n} \to S_{\alpha,n}$ be a nonexpansive mapping for which $T(S_{\alpha,n})$ is contained in a hemisphere $\widetilde{S}$ such that $\widetilde{S} \not\subset S_{\alpha,n}$, and let $T' : S_H \to \widetilde{S}$ be a nonexpansive extension of T. If:*

(1) *$T'(-n) \neq -n$, or*
(2) *$T'(-n) = -n$ and $T|_{S_{\alpha,\alpha,n}}$ is not an isometry onto $S_{\alpha,\alpha,n}$,*

then $d_T = 0$.

Proof. (1) Assume that $d_T > 0$. Let $r = \min\left\{\frac{1}{3}\|n + T'(-n)\|, d_T\right\}$. Given $\varepsilon \in (0, r)$, observe that, if $x \in B_{S_H}(-n, r)$ or $x \in S_{\alpha,n}$, then $\|x_\varepsilon - T'x_\varepsilon\| > \varepsilon$, and therefore, by Lemma 4.1, there exists $x_\varepsilon \in \widetilde{S} \setminus (S_{\alpha,n} \cup B_{S_H}(-n, r))$ such that $\|x - T'x\| \leq \varepsilon$. It is easy to see that for any such x_ε there exists a unique $y_\varepsilon \in S_{\alpha,\alpha,n}$ such that $\|x_\varepsilon - y_\varepsilon\| = \operatorname{dist}(x_\varepsilon, S_{\alpha,\alpha,n})$. From the triangle inequality, we have $Ty_\varepsilon \in B_H(x_\varepsilon, \|x_\varepsilon - y_\varepsilon\| + \varepsilon) \cap S_{\alpha,n}$. Using the cosine rule, one can prove that

$$\operatorname{diam}(B_H(x_\varepsilon, \|x_\varepsilon - y_\varepsilon\| + \varepsilon) \cap S_{\alpha,n}) \leq 2\sqrt{\frac{\pi^2\varepsilon^2}{r^2(4-r^2)} + 3\varepsilon}.$$

The above diameter tends to 0 as $\varepsilon \to 0$, which contradicts with $d_T > 0$.

(2) Since the mapping T' is nonexpansive, we have that $T(S_{\alpha,\alpha,n}) \subset S_{\alpha,\alpha,n}$. We can treat the affine hull of $S_{\alpha,\alpha,n}$ as a Hilbert space G so that $S_{\alpha,\alpha,n}$ is the unit sphere, i.e. $S_G = S_{\alpha,\alpha,n}$. Put $T_1 = T|_{S_G}$. Applying Lemma 1.4 to T_1, we see that $T_1(S_G)$ is contained in a hemisphere $\widetilde{S}_G$ of G, or T_1 is an isometry. In the first case, by Lemma 4.1, we get $d_{T_1} = 0$. In the second case, since T_1 is not onto S_G, so, applying the reasoning from the proof of Theorem 4.6, we obtain $d_{T_1} = 0$. In view of $d_T \le d_{T_1}$, we have $d_T = 0$, as claimed.

From Lemmas 4.1 and 4.7, we have that if $\alpha \in (-1, 0)$ and $T : S_{\alpha,n} \to S_{\alpha,n}$ is a nonexpansive mapping but not an isometry, then d_T can be positive only if $\widetilde{S} \not\subset S_{\alpha,n}$ and $T|_{S_{\alpha,\alpha,n}}$ is an isometry onto $S_{\alpha,\alpha,n}$. We shall use this observation in the proof of the following theorem.

Theorem 4.8. *Let $\alpha \in \left(-\frac{1}{2}, 0\right)$ and $T : S_{\alpha,n} \to S_{\alpha,n}$ be a nonexpansive mapping, then $d_T = 0$.*

Proof. Suppose that $d_T > 0$. Theorem 4.6 implies that T is not an isometry. From Lemmas 1.4 and 4.1, we have that $T(S_{\alpha,n})$ is contained in a hemisphere $\widetilde{S}$ such that $\widetilde{S} \not\subset S_{\alpha,n}$. By Lemma 4.7, $T'(-n) = -n$, where $T' : S_H \to \widetilde{S}$ is a nonexpansive extension of T, and $T|_{S_{\alpha,\alpha,n}}$ is an isometry onto $S_{\alpha,\alpha,n}$. Observe that $S_{\alpha,\alpha,n} \subset \widetilde{S}$, hence $n \notin \widetilde{S}$. Let $y \in S_{\alpha,\alpha,n}$ be such that $Ty = -(P \circ T)n$, where P is the closest point projection onto $S_{\alpha,\alpha,n}$. Since P is nonexpansive on $\widetilde{S} \cap S_{\alpha,n}$,

$$\|Tn - Ty\| \ge \|(P \circ T)n - (P \circ T)y\| = 2\|Ty\| = 2\sqrt{1-\alpha^2}$$

and $\|n - y\| = \sqrt{2 - 2\alpha}$.

It is easy to check that if $\alpha \in \left(-\frac{1}{2}, 0\right)$, then $\|Tn - Ty\| > \|n - y\|$, which is a contradiction.

It is an open problem to find the supremum of α such that the minimal displacement is positive.

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