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NONLINEAR BOUNDARY VALUE PROBLEM FOR IMPLICIT DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER IN BANACH SPACES

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Abstract. In this paper, we establish sufficient conditions for the existence of solutions for a class of boundary value problem for implicit fractional differential equations with Caputo fractional derivative. We apply the technique of measure of noncompactness and the fixed point theorems of Darbo and Mönch. As an application, two examples are included to show the applicability of our results.
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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). In recent years, fractional differential equations arise naturally in various fields such as engineering, electrochemistry, viscoelasticity, rheology, fractals, image and signal processing, modeling and control theory, biophysics, bioengineering and biomedical applications, . . . ; Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes; see the monographs [10, 22, 29, 30, 32], and references therein.

Recently, fractional differential equations have been studied by Abbas *et al.* [1, 2, 3, 4], Baleanu *et al.* [9, 11], Diethelm [19], Kilbas and Marzan [23], Srivastava *et al.*

[24], Lakshmikantham *et al.* [26], Samko *et al.* [30], and Zhou [35]. More recently, some authors have considered some classes of boundary value problems for fractional differential equations. In [14], the authors studied the problem involving Caputo's derivative:

$$^cD^{\alpha}u(t) = f(t, u(t), {}^cD^{\alpha-1}u(t)), \text{ for each, } t \in J := [0, \infty), \ 1 < \alpha \leq 2,$$

$$u(0) = u_0, u \text{ is bounded on } J,$$

where $^cD^{\alpha}$ is the Caputo fractional derivative, $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a continuous function. In [31], by means of Schauder fixed-point theorem, Su and Liu studied the nonlinear fractional boundary value problem involving Caputo's derivative:

$$^cD^{\alpha}u(t) = f(t, u(t), {}^cD^{\beta}u(t)), \text{ for each, } t \in (0, 1), \ 1 < \alpha \leq 2, \ 0 < \beta \leq 1,$$

$$u(0) = u^{'}(1) = 0, \ or \ u^{'}(1) = u(1) = 0, or \ u(0) = u(1) = 0,$$

where $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Many techniques have been developed for studying the existence and uniqueness of solutions of initial and boundary value problems for fractional differential equations. Several authors tried to develop a technique that depends on the Darbo or the Mönch fixed point theorems with the Hausdorff or Kuratowski measure of noncompactness. The notion of the measure of noncompactness was defined in many ways. In 1930, Kuratowski [25] defined the measure of noncompactness, $\alpha(A)$, of a bounded subset A of a metric space (X, d), and in 1955, Darbo [17] introduced a new type of fixed point theorem for set contractions.

Motivated by all the works above, the purpose of this paper, is to establish existence and uniqueness results to the following implicit fractional-order differential equations:

$$^cD^{\nu}y(t) = f(t, y(t), {}^cD^{\nu}y(t)), \text{ for each, } t \in J := [0, T], \ T > 0, \ 0 < \nu \leq 1, \quad (1.1)$$

$$ay(0) + by(T) = c, \quad (1.2)$$

where $^cD^{\nu}$ is the Caputo fractional derivative, $(E, \|.\|)$ is a real Banach space, $f : J \times E \times E \to E$ is given function and a, b are real with $a + b \neq 0$ and $c \in E$, and

$$^cD^{\nu}y(t) = f(t, y(t), {}^cD^{\nu}y(t)), \quad \text{for every } t \in J := [0, T], T > 0, \quad 0 < \nu \leq 1 \quad (1.3)$$

$$y(0) + g(y) = y_0, \quad (1.4)$$

where $g : C([0, T], E) \longrightarrow E$ a continuous function and $y_0 \in E$. This type of non-local Cauchy problem was introduced by Byszewski [15, 16] (see also [18]). He observed that the non-local condition is more appropriate than the local condition (initial) to describe correctly some physics phenomenons, he proved the existence and the uniqueness of weak solutions and also classical solutions for this type of problems. We take an example of non-local conditions as follows:

$$g(y) = \sum_{i=1}^{p} c_i y(t_i)$$

where $c_i, i = 1, ..., p$ are constants and $0 < t_1 < ... < t_p \leq T$.

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some preliminary concepts about fractional calculus and the Kuratowski's measure of noncompactness and auxiliary results. In Section 3, two results are provided; the first one is based on Darbo's fixed point theorem combined with the measure of noncompactness, and the second on Mönch's fixed point theorem. At last, two examples are given to demonstrate the application of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $(E; \|.\|)$ be a Banach space. We denote by $C(J, E)$ the space of E-valued continuous functions on J with the usual supremum norm

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\}$$

for any $y \in C(J, E)$.

A measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable.

Let $L^1(J, E)$ denote the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T \|y(t)\| \, dt.$$

For properties of the Bochner integrable functions, see [34].

Definition 2.1. ([24, 29]). The fractional (arbitrary) order integral of the function $h \in L^1([0,T], E)$ of order $\nu \in \mathbb{R}_+$ is defined by

$$I^\nu h(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s) ds,$$

where Γ is the Euler gamma function defined by

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt, \ \nu > 0.$$

Definition 2.2. ([23]). For a function h given on the interval $[0,T]$, the Caputo fractional-order derivative of order ν of h, is defined by

$$({}^cD^\nu h)(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} h^{(n)}(s) ds,$$

where $n = [\nu] + 1$.

Lemma 2.3. ([27]). *Let $\nu \geq 0$ and $n = [\nu] + 1$. Then*

$$I^\nu({}^cDf(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

Lemma 2.4. ([29]). *Let $\nu > 0$, so the homogenous differential equation of fractional order ${}^cD^\nu h(t) = 0$, has a solution $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$, where c_i, $i = 1, \ldots, n$ are constants and $n = [\nu] + 1$.*

Remark 2.5. ([27]) The Caputo derivative of a constant is equal to zero.

Moreover, for a given set V of functions $v : J \to E$ let us denote by

$$V(t) = \{v(t), v \in V\},\ t \in J$$

and

$$V(J) = \{v(t) : v \in V,\ t \in J\}.$$

Next we give the definition of the measure of noncompactness and some auxiliary result; for more details see [6, 8, 12, 13] and the references therein.

Definition 2.6. ([12]). Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty)$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } \mathrm{diam}(B_i) \leq \epsilon\};\ \text{here } B \in \Omega_E,$$

where $\mathrm{diam}(B_i) = \sup\{\|x - y\| : x, y \in B_i\}$.

The Kuratowski measure of noncompactness satisfies the following properties.

Lemma 2.7. ([6, 12, 13]). *Let A and B be bounded sets.*

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is relatively compact), where $\overline{B}$ denotes the closure of B.*
(b) *nonsingularity: α is equal to zero on every one element-set.*
(c) $\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$*, where $convB$ is the convex hull of B.*
(d) *monotonicity: $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.*
(e) *algebraic semi-additivity: $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, where*

$$A + B = \{x + y : x \in A, \quad y \in B\}.$$

(f) *semi-homogencity: $\alpha(\lambda B) = |\lambda|\alpha(B)$; $\lambda \in \mathbb{R}$, where $\lambda(B) = \{\lambda x : x \in B\}$.*
(g) *semi-additivity: $\alpha(A \bigcup B) = \max\{\alpha(A), \alpha(B)\}$.*
(h) *invariance under translations: $\alpha(B + x_0) = \alpha(B)$ for any $x_0 \in E$.*

For our purpose we will need the following fixed point theorem.

Theorem 2.8. ([20]) (Darbo's Fixed Point Theorem). *Let X be a Banach space, and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $N : C \to C$ is such that for all closed subsets D of C,*

$$\alpha(N(D)) \leq k\alpha(D), \tag{2.1}$$

where $0 \leq k < 1$. Then N has a fixed point in C.

Remark 2.9. Mappings satisfying the Darbo-condition (2.1) have subsequently been called k-*set contractions.*

Theorem 2.10. ([5, 28]) (Mönch's Fixed Point Theorem). *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.11. ([21]). *If $V \subset C(J, E)$ is a bounded and equicontinuous set, then*

(i) *the function $t \to \alpha(V(t))$ is continuous on J, and*

$$\alpha_c(V) = \sup_{0 \leq t \leq T} \alpha(V(t)).$$

(ii) $\alpha\left(\int_0^T x(s)ds : x \in V\right) \leq \int_0^T \alpha(V(s))ds,$

where $V(s) = \{x(s) : x \in V\},\ s \in J.$

We state the following generalization of Gronwall's lemma for singular kernels.

Lemma 2.12. ([33]). *Let* $v : [0,T] \to [0,+\infty)$ *be a real function and* $w(\cdot)$ *is a nonnegative, locally integrable function on* $[0,T]$ *and there are constants* $a > 0$ *and* $0 < \nu < 1$ *such that*

$$v(t) \le w(t) + a\int_0^t (t-s)^{-\nu} v(s)ds,$$

Then, there exists a constant $K = K(\nu)$ *such that*

$$v(t) \le w(t) + Ka\int_0^t (t-s)^{\nu} w(s)ds,$$

for every $t \in [0,T]$.

3. Existence of Solutions

Let us defining what we mean by a solution of problem (1.1)-(1.2).

Definition 3.1. A function $y \in C^1(J,E)$ is said to be a solution of the problem (1.1)–(1.2) if y satisfied equation (1.1) and conditions (1.2).

For the existence of solutions for the problem $(1.1)-(1.2)$, we need the following auxiliary lemma:

Lemma 3.2. ([10]). *Let* $0 < \nu \le 1$ *and* $h : [0,T] \longrightarrow E$ *be a continuous function. Then the linear problem*

$$^cD^{\nu} y(t) = h(t),\ t \in J$$

$$ay(0) + by(T) = c$$

has a unique solution which is given by:

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} h(s)ds \\ &- \frac{1}{a+b}\left[\frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1} h(s)ds - c\right]. \end{aligned}$$

Lemma 3.3. *Let* $f(t,u,v) : J \times E \times E \to E$ *be a continuous function, then the problem* $(1.1)-(1.2)$ *is equivalent to the problem:*

$$y(t) = \tilde{A} + I^{\nu} g(t) \tag{3.1}$$

where $g \in C(J,E)$ *satisfies the functional equation*

$$g(t) = f(t, \tilde{A} + I^{\nu} g(t), g(t))$$

and

$$\tilde{A} = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1} g(s)ds\right].$$

Proof. Let y be solution of (3.1). We shall show that y is solution of $(1.1)-(1.2)$. We have

$$y(t) = \tilde{A} + I^{\nu} g(t).$$

So, $y(0)=\tilde{A}$ and $y(T)=\tilde{A}+\dfrac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}g(s)ds$.

$$\begin{aligned}
ay(0)+by(T) &= \frac{-ab}{(a+b)\Gamma(\nu)}\int_0^T (T-s)^{\alpha-1}g(s)ds \\
&\quad +\frac{ac}{a+b}-\frac{b^2}{(a+b)\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}g(s)ds \\
&\quad +\frac{bc}{a+b}+\frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}g(s)ds. \\
&= c.
\end{aligned}$$

Then

$$ay(0)+by(T)=c.$$

On the other hand, we have

$$\begin{aligned}
{}^cD^\nu y(t) &= {}^cD^\nu(\tilde{A}+I^\nu g(t))=g(t) \\
&= f(t,y(t),{}^cD^\nu y(t)).
\end{aligned}$$

Thus, y is solution of problem (1.1) – (1.2).

First we list the following hypotheses:

($H1$) The function $f: J\times E\times E\to E$ is continuous.

($H2$) There exist constants $K>0$ and $0<L<1$ such that

$$\|f(t,u,v)-f(t,\bar{u},\bar{v})\|\le K\|u-\bar{u}\|+L\|v-\bar{v}\|$$

for any $u,v,\bar{u},\bar{v}\in E$ and $t\in J$.

We are now in a position to state and prove our existence result for the problem (1.1) – (1.2) based on the concept of measure of noncompactness and *Darbo's* fixed point theorem.

Remark 3.4. ([7]) Condition ($H2$) is equivalent to the inequality

$$\alpha\Big(f(t,B_1,B_2)\Big)\le K\alpha(B_1)+L\alpha(B_2),$$

for any bounded sets $B_1,B_2\subseteq E$ and for each $t\in J$.

Theorem 3.5. *Assume* ($H1$) *and* ($H2$) *hold. If*

$$\frac{(\|b\|+\|a+b\|)T^\nu K}{|a+b|\Gamma(\nu+1)(1-L)}<1 \tag{3.2}$$

and

$$\frac{KT^\nu}{(1-L)\Gamma(\nu+1)}<1, \tag{3.3}$$

then the problem (1.1) – (1.2) *has at least one solution on* J.

Proof. Transform the problem (1.1) – (1.2) into a fixed point problem. Define the operator $N: C(J,E)\to C(J,E)$ by

$$N(y)(t)=\tilde{A}+I^\nu g(t), \tag{3.4}$$

where $g\in C(J,E)$ satisfies the functional equation

$$g(t)=f(t,y(t),g(t)),$$

and

$$\tilde{A} = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1} g(s)ds\right].$$

Clearly, the fixed points of operator N are solutions of problem $(1.1)-(1.2)$. We shall show that N satisfies the assumption of *Darbo's* fixed point Theorem. The proof will be given in several claims.

Claim 1. *N is continuous.*

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J,E)$. Then for each $t \in J$

$$\begin{aligned} \|N(u_n)(t) - N(u)(t)\| &\leq \frac{\|b\|}{\|a+b\|\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}\|g_n(s)-g(s)\|ds \\ &+ \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\|g_n(s)-g(s)\|ds, \end{aligned} \tag{3.5}$$

where $g_n, g \in C(J,E)$ are such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

By $(H2)$ we have, for each $t \in J$,

$$\begin{aligned} \|g_n(t) - g(t)\| &= \|f(t,u_n(t),g_n(t)) - f(t,u(t),g(t))\| \\ &\leq K\|u_n(t)-u(t)\| + L\|g_n(t)-g(t)\|. \end{aligned}$$

Then

$$\|g_n(t) - g(t)\| \leq \frac{K}{1-L}\|u_n(t)-u(t)\|.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$.

Let $\eta > 0$ be such that, for each $t \in J$, we have $\|g_n(t)\| \leq \eta$ and $\|g(t)\| \leq \eta$.

Then we have,

$$\begin{aligned} (t-s)^{\nu-1}\|g_n(s)-g(s)\| &\leq (t-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|] \\ &\leq 2\eta(t-s)^{\nu-1}. \end{aligned}$$

For each $t \in J$, the function $s \to 2\eta(t-s)^{\nu-1}$ is integrable on $[0,t]$, then the Lebesgue Dominated Convergence Theorem and (3.5) imply

$$\|N(u_n)(t) - N(u)(t)\| \to 0 \ as \ n \to \infty.$$

Then

$$\|N(u_n) - N(u)\|_\infty \to 0 \ as \ n \to \infty.$$

Consequently, N is continuous.

Let the constant R be such that

$$R \geq \frac{\|c\|\Gamma(\nu+1)(1-L) + (\|b\| + \|a+b\|)T^\nu f^*}{\|a+b\|\Gamma(\nu+1)(1-L) - (\|b\| + \|a+b\|)T^\nu K}, \tag{3.6}$$

where $f^* = \sup_{t\in J}\|f(t,0,0)\|$.

Define

$$D_R = \{u \in C(J,E) : \|u\|_\infty \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $C(J,E)$.
Claim 2. $N(D_R) \subset D_R$. Let $u \in D_R$ we show that $Nu \in D_R$. We have, for each $t \in J$

$$\begin{aligned} \|Nu(t)\| &\leq \frac{\|c\|}{\|a+b\|} + \frac{|b|}{\|a+b\|\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} \|g(s)\| ds \\ &+ \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \|g(s)\| ds. \end{aligned} \tag{3.7}$$

By $(H2)$ we have for each $t \in J$,

$$\begin{aligned} \|g(t)\| &= \|f(t,u(t),g(t)) - f(t,0,0) + f(t,0,0)\| \\ &\leq \|f(t,u(t),g(t)) - f(t,0,0)\| + \|f(t,0,0)\| \\ &\leq K\|u(t)\| + L\|g(t)\| + f^* \\ &\leq KR + L\|g(t)\| + f^*. \end{aligned}$$

Then

$$\|g(t)\| \leq \frac{f^* + KR}{1-L} := M$$

Thus, (3.6) and (3.7) imply that

$$\begin{aligned} \|Nu(t)\| &\leq \frac{\|c\|}{\|a+b\|} + \left[\frac{\|b\|}{\|a+b\|} + 1\right] \frac{T^\nu}{\Gamma(\nu+1)} \left(\frac{f^* + KR}{1-L}\right) \\ &\leq \frac{\|c\|}{\|a+b\|} + \frac{(\|b\| + \|a+b\|)T^\nu f^*}{|a+b|\Gamma(\nu+1)(1-L)} \\ &+ \frac{(\|b\| + \|a+b\|)T^\nu KR}{\|a+b\|\Gamma(\nu+1)(1-L)} \\ &\leq R. \end{aligned}$$

Consequently,

$$N(D_R) \subset D_R.$$

Claim 3. *$N(D_R)$ is bounded and equicontinuous.* By Claim 2 we have $N(D_R) = \{N(u) : u \in D_R\} \subset D_R$. Thus, for each $u \in D_R$ we have $\|N(u)\|_\infty \leq R$ which means that $N(D_R)$ is bounded. Let $t_1, t_2 \in J,\ t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned} \|N(u)(t_2) - N(u)(t_1)\| &= \Big\| \frac{1}{\Gamma(\nu)} \int_0^{t_1} [(t_2-s)^{\nu-1} - (t_1-s)^{\nu-1}] g(s) ds \\ &\quad + \frac{1}{\Gamma(\nu)} \int_{t1}^{t_2} (t_2-s)^{\nu-1} g(s) ds \Big\| \\ &\leq \frac{M}{\Gamma(\nu+1)} (t_2^\nu - t_1^\nu + 2(t_2-t_1)^\nu). \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Claim 4. *The operator $N : D_R \to D_R$ is a strict set contraction.* Let $V \subset D_R$ and $t \in J$, then we have,

$$\begin{aligned}\alpha(N(V)(t)) &= \alpha((Ny)(t), y \in V)\\ &\le \frac{1}{\Gamma(\nu)}\Big\{\int_0^t (t-s)^{\nu-1}\alpha(g(s))ds, y \in V\Big\}.\end{aligned}$$

Then Remark 3.4 and Lemma 2.7 imply that, for each $s \in J$,

$$\begin{aligned}\alpha(\{g(s), y \in V\}) &= \alpha(\{f(s, y(s), g(s)), y \in V\})\\ &\le K\alpha(\{y(s), y \in V\}) + L\alpha(\{g(s), y \in V\}).\end{aligned}$$

Thus

$$\alpha\left(\{g(s), y \in V\}\right) \le \frac{K}{1-L}\alpha\{y(s), y \in V\}.$$

Then

$$\begin{aligned}\alpha(N(V)(t)) &\le \frac{K}{(1-L)\Gamma(\nu)}\left\{\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))\}ds, y \in V\right\}\\ &\le \frac{K\alpha_c(V)}{(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}ds\\ &\le \frac{KT^\nu}{(1-L)\Gamma(\nu+1)}\alpha_c(V).\end{aligned}$$

Therefore

$$\alpha_c(NV) \le \frac{KT^\nu}{(1-L)\Gamma(\nu+1)}\alpha_c(V).$$

So, by (3.3), the operator N is a set contraction. As a consequence of Theorem 2.8, we deduce that N has a fixed point which is solution to the problem (1.1) − (1.2). This completes the proof.

Our next existence result for the problem (1.1) − (1.2) is based on concept of measure of noncompactness and *Mönch's* fixed point theorem.

Theorem 3.6. *Assume* $(H1)-(H2)$ *and* (3.2) *hold. Then the BVP* (1.1) − (1.2) *has at least one solution.*

Proof. Consider the operator N defined in (3.4). We shall show that N satisfies the assumption of *Mönch's* fixed point theorem. We know that $N : D_R \to D_R$ is bounded and continuous, we need to prove that the implication

$$V = \overline{conv}N(V) \quad \text{or} \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D_R. Now let V be a subset of D_R such that $V \subset \overline{conv}(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function

$t \to v(t) = \alpha(V(t))$ is continuous on J. By Remark 3.4, Lemma 2.11 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned} v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\ &\leq \alpha(N(V)(t)) \\ &\leq \alpha\{(Ny)(t), y \in V\} \\ &\leq \frac{K}{(1-L)\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \{\alpha(y(s))ds, y \in V\} \\ &\leq \frac{K}{(1-L)\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} v(s)ds. \end{aligned}$$

Lemma 2.12 implies that $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_R. Applying now Theorem 2.10 we conclude that N has a fixed point $y \in D_R$. Hence N has a fixed point which is solution to the problem (1.1)-(1.2). This completes the proof.

Remark 3.7. Our results for the boundary value problem (1.1)-(1.2) are appropriate for the following problems:

- Initial value problem: $a = 1, b = 0, c = 0$.
- Terminal value problem: $a = 0, b = 1, c$ arbitrary.
- Anti-periodic problem: $a = 1$, $b = 1$, $c = 0$.

However, they are not for the periodic problem, i.e. for $a = 1, b = -1, c = 0$.

4. Nonlocal boundary value problem

Definition 4.1. A function $y \in C^1(J, E)$ is called solution of problem $(1.3) - (1.4)$ if it satisfies the equation (1.3) on J and the condition (1.4).

In the spirit of Lemmas 3.2 and 3.3 we have the following auxiliary lemmas.

Lemma 4.2. *Let $0 < \nu \leq 1$ and let $h : [0, T] \to E$ a continuous function. The linear problem*

$$^cD^\nu y(t) = h(t), \quad t \in J$$

$$y(0) + g(y) = y_0$$

has a unique solution which is given by:

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s)ds.$$

Lemma 4.3. *Let $f : J \times E \times E \to E$ be a continuous function, then the problem* $(1.3) - (1.4)$ *is equivalent to the following functional equation*

$$y(t) = y_0 - g(y) + I^\nu H(t)$$

where $H(t) = f(t, y(t), H(t))$.

Introduce the following hypothesis:

$(H3)$ There exists $0 < \overline{K}$ such that

$$\|g(u) - g(\overline{u})\| \leq \overline{K}\|u - \overline{u}\|_\infty \text{ for any } u, \overline{u} \in C(J, E).$$

Remark 4.4. ([7]) Condition $(H3)$ is equivalent to the inequality

$$\alpha\Big(g(B)\Big) \leq \overline{K}\alpha_C(B), \text{ for any bounded set } B \subseteq C(J;E).$$

Theorem 3.5. *Assume* $(H1)-(H3)$ *hold.*
If

$$\frac{\overline{K}(1-L)\Gamma(\nu+1)+KT^{\nu}}{(1-L)\Gamma(\nu+1)} < 1 \tag{4.1}$$

and

$$\frac{KT^{\nu}}{(1-L)\Gamma(\nu+1)} + \overline{K} < 1, \tag{4.2}$$

then the BVP $(1.3)-(1.4)$ *has at least one solution defined on* J.

Our next existence result for the problem $(1.3)-(1.4)$ is based on concept of measure of noncompactness and *Mönch's* fixed point theorem.

Theorem 4.6. *Assume* $(H1)-(H3)$ *and* (4.1) *hold, where* $\overline{K} < 1$. *Then the BVP* $(1.3)-(1.4)$ *has at least one solution.*

5. Example

Example 5.1. Consider the following infinite system

$$^cD^{\frac{1}{2}}y_n(t) = \frac{(3+|y_n(t)|+|^cD^{\frac{1}{2}}y_n(t)|)}{3e^{t+2}(1+|y_n(t)||+|^cD^{\frac{1}{2}}y_n(t)|)}, \text{ for each, } t\in[0,1], \tag{5.1}$$

$$y_n(0)+y_n(1)=0. \tag{5.2}$$

Set

$$E = l^1 = \{y=(y_1,y_2,\ldots,y_n,\ldots) : \sum_{n=1}^{\infty}|y_n| < \infty\},$$

and

$$f(t,u,v) = \frac{(3+\|u\|+\|v\|)}{3e^{t+2}(1+\|u\|+\|v\|)}, \quad t\in[0,1], \ u,v\in E.$$

E is a Banach space with the norm $\|y\| = \sum_{n=1}^{\infty}|y_n|$.
Clearly, the function f is continuous.
For any $u, v, \bar{u}, \bar{v} \in E$ and $t \in [0,1]$:

$$\|f(t,u,v)-f(t,\bar{u},\bar{v})\| \leq \frac{1}{3e^2}(\|u-\bar{u}\|+\|v-\bar{v}\|).$$

Hence condition $(H2)$ is satisfied with $K = L = \frac{1}{3e^2}$.
And the conditions

$$\frac{(|b|+|a+b|)T^{\nu}K}{|a+b|\Gamma(\nu+1)(1-L)} = \frac{1}{\sqrt{\pi}(e^2-\frac{1}{3})} < 1,$$

and

$$\frac{KT^{\nu}}{(1-L)\Gamma(\nu+1)} = \frac{2}{(3e^2-1)\sqrt{\pi}} < 1$$

are satisfied with $a = b = T = 1, c = 0$ and $\nu = \frac{1}{2}$.
It follows from Theorem 3.5 that the problem $(5.1)-(5.2)$ has at least one solution on J.

Example 5.2. Consider the boundary value problem:

$$ {}^cD^{\frac{1}{2}}y_n(t) = \frac{e^{-t}}{(9+e^t)}\left[1+\frac{|y_n(t)|}{1+|y_n(t)|}-\frac{|{}^cD^{\frac{1}{2}}y_n(t)|}{1+|{}^cD^{\frac{1}{2}}y_n(t)|}\right], \quad t\in J=[0,1] \tag{5.3} $$

$$ y_n(0)+\sum_{i=1}^{m} c_i y_n(t_i) = 1, \tag{5.4} $$

where $0 < t_1 < t_2 < \ldots < t_m < 1$ and $c_i = 1,\ldots,m$ are positive constants with

$$ \sum_{i=1}^{m} c_i \le \frac{1}{3}. $$

Set

$$ E = l^1 = \{y=(y_1,y_2,\ldots,y_n,\ldots) : \sum_{n=1}^{\infty} |y_n| < \infty\}, $$

and

$$ f(t,u,v) = \frac{e^{-t}}{(9+e^t)}\left[1+\frac{\|u\|}{1+\|u\|}-\frac{\|v\|}{1+\|v\|}\right], \ t\in[0,1], \ u,v\in E. $$

E is a Banach space with the norm $\|y\| = \sum_{n=1}^{\infty}|y_n|$.
Clearly, the function f is continuous.
For each $u,\bar{u},v,\bar{v}\in E$ and $t\in[0,1]$:

$$ \begin{aligned} \|f(t,u,v)-f(t,\bar{u},\bar{v})\| &\le \frac{e^{-t}}{9+e^t}(\|u-\bar{u}\|+\|v-\bar{v}\|) \\ &\le \frac{1}{10}\|u-\bar{u}\|+\frac{1}{10}\|v-\bar{v}\|. \end{aligned} $$

Hence condition (H2) is satisfied with $K = L = \frac{1}{10}$.
On the other hand, we have for any $u,\bar{u}\in E$

$$ \|g(u)-g(\bar{u})\| \le \frac{1}{3}\|u-\bar{u}\|. $$

Hence condition $(H3)$ is satisfied with $\overline{K} = \dfrac{1}{3}$. Also, condition

$$ \frac{\overline{K}(1-L)\Gamma(\nu+1)+KT^{\nu}}{(1-L)\Gamma(\nu+1)} = \frac{9\sqrt{\pi}+6}{27\sqrt{\pi}} < 1, $$

is satisfied with $T=1$ and $\nu=\frac{1}{2}$.
It follows from Theorem 4.6 that the problem (5.3)-(5.4) has at least one solution defined on J.

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