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ON STABILITY AND HYPERSTABILITY OF AN EQUATION CHARACTERIZING MULTI-ADDITIVE MAPPINGS

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Abstract. In this paper, using the fixed point approach, we prove some stability and hyperstability results for an equation characterizing multi-additive mappings. Our results generalize some known outcomes. In particular, we give a solution of a problem concerning optimality of some estimations.
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1. Introduction

Let G be a commutative semigroup with the identity element 0 and W be a linear space. Let us recall that a function $f : G^n \to W$ is called *multi-additive* if it is additive (satisfies Cauchy's functional equation) in each variable. Some basic facts on such mappings can be found for instance in [11], where their application to the representation of polynomial functions is also presented. K. Ciepliński in [7] reduced the system of n equations defining the multi-additive mapping to obtain a single functional equation, namely he has proved (see [7] Theorem 2) that $f : G^n \to W$ is a multi-additive mapping if and only if

$$f(x_{11}+x_{12},...,x_{n1}+x_{n2}) = \sum_{i_1,...,i_n\in\{1,2\}} f(x_{1i_1},...,x_{ni_n}), \tag{1.1}$$

for all $(x_{1i},...,x_{ni}) \in G^n$, $i \in \{1,2\}$, and has shown the generalized Hyers-Ulam stability both of this system and this equation, using the direct (Hyers) method.

In this paper, using the fixed point approach and the above reduction, we prove the generalized Hyers-Ulam stability of the equation (1.1). Moreover, we show that for some natural particular forms of $\theta : V_0^{2n} \to \mathbb{R}_+ := [0,+\infty)$, where V is a linear space and $V_0 := V \setminus \{0\}$, the functional equation (1.1) is θ-hyperstable in the class of functions $f\colon V^n \to W$, where W is a normed space, i.e. each $f\colon V^n \to W$ satisfying the inequality

$$\|f(x_{11}+x_{12},...,x_{n1}+x_{n2})-\sum_{i_1,...,i_n\in\{1,2\}} f(x_{1i_1},...,x_{ni_n})\| \leqslant \theta(x_{11},...,x_{n1},x_{12},...,x_{n2}),$$

for all $x_{11},...,x_{n1},x_{12},...,x_{n2}\in V_0$, must fulfil the equation (1.1).

Our results are significant supplements and/or generalizations of some classical outcomes from [1, 13, 14, 15, 3, 4, 7, 8, 9, 10].

Speaking of the stability of a functional equation we follow the question raised in 1940 by S. M. Ulam: "when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?". The first partial answer to Ulam's question was given by D. H. Hyers (see [10]). After his result a lot of mathematicians have generalized Ulam's problem and Hyers's theorem in various directions and to other functional equations (as the words "differing slightly" and "be close" may have various meanings, different kinds of stability can be dealt with).

The term *hyperstability* has been used for the first time in [12], however it seems that the first hyperstability result was published in [2] and concerned the ring homomorphisms. For more information and numerous references on hyperstability we refer to, e.g., [4, 6].

Let us recall that a commutative semigroup G is called *uniquely divisible by* 2 provided for every $x\in G$ there exists a unique $y\in G$ (which is denoted by $\frac{x}{2}$ or $\frac{1}{2}x$) such that $x=2y$.

In the sequel, we assume that V,W are linear spaces over the rationals and

$$x^1+x^2:=(x_{11}+x_{12},...,x_{n1}+x_{n2}),$$

$$qx:=(qx_1,...,qx_n)$$

where $x:=(x_1,...,x_n), x^i:=(x_{1i},...,x_{ni})\in V^n$, $i\in\{1,2\}$, $q\in\mathbb{Q}$.

2. Auxiliary result

In this section, we present some characterization of multi-additive mappings, which will be needed in our next considerations.

Lemma 2.1. *A function $f: V^n\to W$ satisfies the equation* (1.1) *for all $x_{ji}\in V_0$, $j\in\{1,\dots,n\}, i\in\{1,2\}$ if and only if f is a multi-additive mapping.*

Proof. Assume that f satisfies (1.1) for all $x_{ji}\in V_0$, $j\in\{1,\dots,n\}, i\in\{1,2\}$.
Notice that f behaves the same way for each variable, so to prove that f is a multi-additive mapping it is enough to show that

$$f(0,...,0)=0 \quad\text{and}\quad f(0,...,0,x_{j+1},...,x_n)=0, \qquad x_{j+1},...,x_n\in V_0,$$

and f satisfies the equation

$$f(x_{11}+0,...,x_{j1}+0,x_{j+11}+x_{j+12},...,x_{n1}+x_{n2})=\sum_{i_1,...,i_n\in\{1,2\}} f(x_{1i_1},...,x_{ni_n}), \tag{2.1}$$

for $x_{11},...,x_{n1},x_{j+12},...,x_{n2}\in V_0$, $x_{12}=\cdots=x_{j2}=0$, $j\in\{1,...,n-1\}$.

Obviously,

$$f(0,...,0)=\sum_{i_1,...,i_n\in\{1,2\}} f(v_{i_1},...,v_{i_n})=2^n\sum_{i_1,...,i_n\in\{1,2\}} f(\frac{1}{2}v_{i_1},...,\frac{1}{2}v_{i_n})=2^nf(0,...,0),$$

where $v_1=v,\ \ v_2=-v$ with $v\in V_0$, which means that $f(0,...,0)=0$.
Now, we show

$$f(0,...,0,x_{j+11}+x_{j+12},...,x_{n1}+x_{n2})=2^j\sum_{i_{j+1},...,i_n\in\{1,2\}} f(0,...,0,x_{j+1i_{j+1}},...,x_{ni_n}), \tag{2.2}$$

for $x_{j+11},...,x_{n1},x_{j+12},...,x_{n2}\in V_0$, $j\in\{1,...,n-1\}$, $n\geqslant 2$.
Clearly, for $n\geqslant 2$, $j\in\{1,...,n-1\}$, $x_{j+1},...,x_n\in V_0$

$$f(0,...,0,2x_{j+1},...,2x_n)=2^{n-j}\sum_{i_1,...,i_j\in\{1,2\}} f(v_{i_1},...,v_{i_j},x_{j+1},...,x_n), \tag{2.3}$$

where $v_1=v,\ \ v_2=-v$ with $v\in V_0$.
From (2.3) and (1.1) we have for $j\in\{1,...,n-1\}$, $x_{j+1},...,x_n\in V_0$

$$\begin{aligned}
f(0,...,0,2x_{j+1},...,2x_n) &= 2^{n-j}\sum_{i_1,...,i_j\in\{1,2\}} f(v_{i_1},...,v_{i_j},x_{j+1},...,x_n)\\
&= 2^{n-j}2^n\sum_{i_1,...,i_j\in\{1,2\}} f\Big(\frac{1}{2}v_{i_1},...,\frac{1}{2}v_{i_j},\frac{1}{2}x_{j+1},...,\frac{1}{2}x_n\Big)\\
&= 2^nf(0,...,0,x_{j+1},...,x_n),
\end{aligned}$$

hence

$$f(0,...,0,2x_{j+1},...,2x_n)=2^nf(0,...,0,x_{j+1},...,x_n). \tag{2.4}$$

Using (1.1),(2.3) and (2.4) for $n\geqslant 2$, $j\in\{1,...,n-1\}$, $x_{j+11},...,x_{n1},x_{j+12},...,x_{n2}\in V_0$, we get

$$\begin{aligned}
&f(0,...,0,x_{j+11}+x_{j+12},...,x_{n1}+x_{n2})\\
&\quad= \sum_{i_1,...,i_n\in\{1,2\}} f(v_{i_1},...,v_{i_j},x_{j+1i_{j+1}},...,x_{ni_n})\\
&\quad= \frac{1}{2^{n-j}}f(0,...,0,2x_{j+1i_{j+1}},...,2x_{ni_n})\\
&\quad= 2^j\sum_{i_{j+1},...,i_n\in\{1,2\}} f(0,...,0,x_{j+1i_{j+1}},...,x_{ni_n}),
\end{aligned}$$

which completes the proof of (2.2).
Appling (2.2), for $n \geqslant 2$, $j \in \{1, ..., n-1\}$ and $x_{j+1}, ..., x_n \in V_0$, we have

$$\begin{aligned} f(0,...,0,x_{j+1},...,x_n) &= 2^{n-1}[f(0,...,0,\frac{1}{2}x_{j+1},...,\frac{1}{2}x_n) \\ &\quad +f(0,...,0,\frac{1}{2}x_{j+1},...,\frac{1}{2}x_n)], \\ f(0,...,0,x_{j+1},...,x_n) &= 2^{n-1}[f(0,...,0,\frac{3}{2}x_{j+1},\frac{1}{2}x_{j+2},...,\frac{1}{2}x_n) \\ &\quad +f(0,...,0,-\frac{1}{2}x_{j+1},\frac{1}{2}x_{j+2},...,\frac{1}{2}x_n)], \end{aligned}$$

and adding the above equality we obtain

$$2f(0,...,0,x_{j+1},...,x_n) = f(0,...,0,2x_{j+1},x_{j+2},...,x_n) + f(0,...,0,x_{j+2},...,x_n).$$

In the analogous way for every $x_{j+1}, ..., x_n \in V_0$ and each $k \in \{j+1, ..., n\}$ we obtain

$$2f(0,...,0,x_{j+1},...,x_n) = f(0,...,0,a_{j+1},...,a_n) + f(0,...,0,b_{j+1},...,b_n), \tag{2.5}$$

where $a_k = 2x_k$, $b_k = 0$ and $a_i = b_i = x_i$ for all $i \neq k$.
Using (2.4) and (2.5) for $n \geqslant 2$ and $x_n \in V_0$ we obtain

$$2^n f(0,...,0,x_n) = f(0,...,0,2x_n) = f(0,...,0,2x_n) + f(0,...,0) = 2f(0,...,0,x_n),$$

which means that

$$f(0,...,0,x_n) = 0,$$

and consequently $f(x) = 0$ for any $x \in V^n$ with exactly one component which is different from zero.

From (2.4), (2.5) and the above for $n \geqslant 3$ and $x_{n-1}, x_n \in V_0$ we have

$$\begin{aligned} 2^n f(0,...,0,x_{n-1},x_n) &= f(0,...,0,2x_{n-1},2x_n) \\ &= f(0,...,0,2x_{n-1},2x_n) + f(0,...,0,2x_n) \\ &= 2f(0,...,0,x_{n-1},2x_n) \\ &= 2[f(0,...,0,x_{n-1},2x_n) + f(0,...,0,x_{n-1},0)] \\ &= 2^2 f(0,...,0,x_{n-1},x_n), \end{aligned}$$

which means that

$$f(0,...,0,x_{n-1},x_n) = 0.$$

We continue in this fashion obtaining

$$f(0,...,0,x_{j+1},...,x_n) = 0$$

for all $x_{j+1}, ..., x_n \in V_0$ and $j \in \{1, ..., n-1\}$, and consequently $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.
Obviously, in the analogous way like in the prove of the condition (2.5), using (1.1) instead of (2.2), we obtain that for each $k \in \{1, \ldots, n\}$ and every $x_1, ..., x_n \in V_0$

$$2f(x_1,...,x_n) = f(a_1,...,a_n) + f(b_1,...,b_n) = f(a_1,...,a_n), \tag{2.6}$$

where $a_k = 2x_k$, $b_k = 0$ and $a_i = b_i = x_i$ for all $i \neq k$.

Finally, fix $l \in \{1, ..., n-1\}$, $x_{11}, ..., x_{n1}, x_{l+12}, ..., x_{n2} \in V_0^n$, and put $x_{12} = \cdots = x_{l2} = 0$. Then using (1.1), (2.6) and the fact that $f(x) = 0$ if at least one $x_j = 0$, we have

$$\begin{aligned} &f(x_{11}+0, ..., x_{l1}+0, x_{l+11}+x_{l+12}, ..., x_{n1}+x_{n2}) \\ &\quad = 2^j \sum_{i_{l+1},...,i_n \in \{1,2\}} f\Big(\frac{1}{2}x_{11}, ..., \frac{1}{2}x_{l1}, x_{l+1i_{l+1}}, ..., x_{ni_n}\Big) \\ &\quad = \sum_{i_{l+1},...,i_n \in \{1,2\}} f(x_{11}, ..., x_{l1}, x_{l+1i_{l+1}}, ..., x_{ni_n}) \\ &\quad = \sum_{i_1,...,i_n \in \{1,2\}} f(x_{1i_1}, ..., x_{ni_n}), \end{aligned}$$

which finishes the proof that f is a multi-additive mapping. The rest of the proof is clear. □

3. Stability of the equation (1.1)

In this section, we prove the generalized (in the spirit of D. G. Bourgin and P. Găvruţa) Hyers-Ulam stability of the equation (1.1). The proof is based on a fixed point result that can be derived from [5, Theorem 1].

Write for $f : V^n \to W$

$$(\Phi f)(x^1, x^2) := f(x^1 + x^2) - \sum_{i_1,...,i_n \in \{1,2\}} f(x_{1i_1}, ..., x_{ni_n}), \qquad x^1, x^2 \in V^n.$$

Theorem 3.1. *Let W be a Banach space and $D \in \{V, V_0\}$. Assume that $f : V^n \to W$, $\theta : D^{2n} \to \mathbb{R}_+$ are the mappings satisfying an estimation*

$$\|(\Phi f)(x^1, x^2)\| \leqslant \theta(x^1, x^2), \qquad x^1, x^2 \in D^n \tag{3.1}$$

and two conditions

$$\varepsilon^*(x) := \frac{1}{2^{n+n(\frac{s-1}{2})}} \sum_{j=0}^{\infty} \Big(\frac{1}{2^{sn}}\Big)^j \theta(2^{sj+\frac{s-1}{2}}x, 2^{sj+\frac{s-1}{2}}x) < \infty, \qquad x \in D^n, \tag{3.2}$$

$$\lim_{l\to\infty} \Big(\frac{1}{2^{sn}}\Big)^l \theta(2^{sl}x^1, 2^{sl}x^2) = 0, \qquad x^1, x^2 \in D^n, \tag{3.3}$$

with some $s \in \{-1, 1\}$. Then there exists a unique multi-additive mapping $F : V^n \to W$, such that

$$\|f(x) - F(x)\| \leqslant \varepsilon^*(x), \qquad x \in D^n. \tag{3.4}$$

Proof. If $s = 1$ putting in (3.1) $x^1 = x^2 = x \in D^n$ we get

$$\|f(2x) - 2^n f(x)\| \leqslant \theta(x, x),$$

hence

$$\Big\|\frac{1}{2^n} f(2x) - f(x)\Big\| \leqslant \frac{1}{2^n}\theta(x, x), \tag{3.5}$$

If $s=-1$ setting in (3.1) $x^1=x^2=\frac{1}{2}x\in D^n$ we obtain

$$\left\|f(x)-2^nf\Big(\frac{1}{2}x\Big)\right\|\leqslant\theta\Big(\frac{1}{2}x,\frac{1}{2}x\Big),$$

hence

$$\left\|2^nf\Big(\frac{1}{2}x\Big)-f(x)\right\|\leqslant\theta\Big(\frac{1}{2}x,\frac{1}{2}x\Big). \tag{3.6}$$

Let $s\in\{1,-1\}$. Write

$$\mathcal{T}\xi(x):=\tfrac{1}{2^{sn}}\xi(2^sx),\qquad \xi\colon V^n\to W, x\in V^n$$

$$\varepsilon(x):=\begin{cases}\frac{1}{2^n}\theta(x,x) & \text{if } s=1,\\ \\ \theta(\frac{1}{2}x,\frac{1}{2}x) & \text{if } s=-1,\end{cases}$$

for $x\in D^n$, then (3.5) if $s=1$, (3.6) if $s=-1$ takes the form

$$\|\mathcal{T}f(x)-f(x)\|\leqslant\varepsilon(x),\qquad x\in D^n.$$

Define

$$\Lambda\eta(x):=\frac{1}{2^{sn}}\eta(2^sx),\qquad \eta\colon D^n\to\mathbb{R}_+,\ x\in D^n.$$

The operators $\mathcal{T}$ and Λ satisfy the assumptions of Theorem 1 in [5], because for every $\xi,\mu\colon V^n\to W$, $x\in V^n$

$$\begin{aligned}\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x)\| &= \left\|\frac{1}{2^{sn}}\xi(2^sx)-\frac{1}{2^{sn}}\mu(2^sx)\right\|\\ &\leqslant \frac{1}{2^{sn}}\|\xi(2^sx)-\mu(2^sx)\|,\end{aligned}$$

and it is easy to check by induction that for every $x\in D^n$ and $j\in\mathbb{N}\cup\{0\}$

$$\begin{aligned}\Lambda^j\varepsilon(x) &= \Big(\frac{1}{2^{sn}}\Big)^j\varepsilon(2^{sj}x)\\ &= \frac{1}{2^{n+n(\frac{s-1}{2})}}\Big(\frac{1}{2^{sn}}\Big)^j\theta(2^{sj+\frac{s-1}{2}}x,2^{sj+\frac{s-1}{2}}x).\end{aligned}$$

Applying this version of the fixed point theorem we obtain that there exists a unique solution $F^*\colon D^n\to W$

$$F^*(x)=\frac{1}{2^{sn}}F^*(2^sx),\qquad x\in D^n$$

such that (3.4) holds. Moreover,

$$F^*(x):=\lim_{k\to\infty}(\mathcal{T}^kf)(x),\qquad x\in D^n.$$

We define the funtion $F:V^n\to W$ in the following way

$$F(x):=\lim_{k\to\infty}(\mathcal{T}^kf)(x),\qquad x\in V^n,$$

obviously $F(x)=F^*(x)$ for $x\in D^n$.
Now, we show that

$$\|\Phi(\mathcal{T}^lf)(x^1,x^2)\|\leqslant\Big(\frac{1}{2^{sn}}\Big)^l\theta(2^{sl}x^1,2^{sl}x^2),\qquad x^1,x^2\in D^n \tag{3.7}$$

for every $l \in \mathbb{N} \cup \{0\}$. If $l = 0$, then (3.7) is just (3.1). Assume that (3.7) holds for some $l \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
&\|\Phi(\mathcal{T}^{l+1} f)(x^1, x^2)\| \\
&= \|\mathcal{T}^{l+1} f\big(x^1 + x^2\big) - \sum_{i_1,\dots,i_n \in \{1,2\}} \mathcal{T}^{l+1} f(x_{1i_1}, \dots, x_{ni_n})\| \\
&= \Big\| \frac{1}{2^{sn}} \mathcal{T}^l f\big(2^s(x^1 + x^2)\big) - \sum_{i_1,\dots,i_n \in \{1,2\}} \frac{1}{2^{sn}} \mathcal{T}^l f(2^s x_{1i_1}, \dots, 2^s x_{ni_n}) \Big\| \\
&\leqslant \frac{1}{2^{sn}} \|\Phi(\mathcal{T}^l f)(2^s x^1, 2^s x^2)\| \\
&\leqslant \Big(\frac{1}{2^{sn}}\Big)^{l+1} \theta(2^{s(l+1)} x^1, 2^{s(l+1)} x^2).
\end{aligned}$$

Letting $l \to \infty$ in (3.7) and using (3.3) we obtain that

$$\Phi F(x^1, x^2) = 0, \qquad x^1, x^2 \in D^n,$$

which means that F satisfying the equation (1.1) for all $x^1, x^2 \in D^n$. In the case when $D = V_0$, we need to use Lemma 2.1 and finally, we get that F is a multi-additive mapping.

Now, we assume that $F' : V^n \to W$ is another function satisfying the equation (1.1) for all $x^1, x^2 \in V^n$ and the inequality (3.4). Then using (1.1) and (3.4), we have for $x \in D^n$, $l \in \mathbb{N}$

$$\begin{aligned}
\|F(x) - F'(x)\| &= \Big\| \frac{1}{2^{snl}} F(2^{sl} x) - \frac{1}{2^{snl}} F'(2^{sl} x) \Big\| \\
&\leqslant \frac{1}{2^{snl}} \big(\|F(2^{sl} x) - f(2^{sl} x)\| + \|F'(2^{sl} x) - f(2^{sl} x)\|\big) \\
&\leqslant \frac{2}{2^{snl}} \varepsilon^*(2^{sl} x) \\
&= \frac{2}{2^{n + n(\frac{s-1}{2})}} \sum_{j=l}^{\infty} \Big(\frac{1}{2^{sn}}\Big)^j \theta(2^{sj + \frac{s-1}{2}} x, 2^{sj + \frac{s-1}{2}} x),
\end{aligned}$$

whence letting $l \to \infty$ and using (3.2) we obtain $F(x) = F'(x)$ for $x \in D^n$, which with the fact that F, F' are the multi-additive mappings finishes the proof. □

After analysing the proof of Theorem 3.1 we obtain that the following theorems are valid.

Theorem 3.2. *(see* [7], *Theorem 3) Let G be a commutative semigroup with the identity element 0 and W be a Banach space. Assume that $f : G^n \to W$, $\theta : G^{2n} \to \mathbb{R}_+$ are mappings satisfying estimation*

$$\|f(x^1 + x^2) - \sum_{i_1,\dots,i_n \in \{1,2\}} f(x_{1i_1}, \dots, x_{ni_n})\| \leqslant \theta(x^1, x^2), \quad x^1, x^2 \in G^n, \tag{3.8}$$

and the conditions

$$\varepsilon^*(x) := \sum_{j=0}^{\infty} \Big(\frac{1}{2^n}\Big)^{j+1} \theta(2^j x, 2^j x) < \infty, \qquad x \in G^n, \tag{3.9}$$

$$\lim_{l\to\infty}\frac{1}{2^{nl}}\theta(2^l x^1, 2^l x^2) = 0, \qquad x^1, x^2 \in G^n. \tag{3.10}$$

Then there exists a unique multi-additive mapping $F : G^n \to W$*, such that*

$$\|f(x) - F(x)\| \leqslant \varepsilon^*(x), \qquad x \in G^n, \tag{3.11}$$

and

$$F(x) := \lim_{j\to\infty}\frac{1}{2^{nj}} f(2^j x), \qquad x \in G^n.$$

We notice that in the above Theorem 3.2 the assumptions are weaker than in Theorem 3 in [7] (compare the conditions (3.9), (3.10) and condition (7) from [7]).

Theorem 3.3. *Let* G *be a commutative semigroup uniquely divisible by* 2 *with the identity element* 0 *and* W *be a Banach space. Assume that* $f : G^n \to W$*,* $\theta : G^{2n} \to \mathbb{R}_+$ *are mappings satisfying the estimation* (3.8) *and the conditions*

$$\varepsilon^*(x) := \sum_{j=0}^{\infty} 2^{nj}\theta\Big(\frac{1}{2^{j+1}}x, \frac{1}{2^{j+1}}x\Big) < \infty, \qquad x \in G^n,$$

$$\lim_{l\to\infty} 2^{nl}\theta\Big(\frac{1}{2^l}x^1, \frac{1}{2^l}x^2\Big) = 0, \qquad x^1, x^2 \in G^n.$$

Then there exists a unique multi-additive mapping $F : G^n \to W$*, such that* (3.11) *holds and*

$$F(x) := \lim_{j\to\infty} 2^{nj} f\Big(\frac{1}{2^j}x\Big), \qquad x \in G^n.$$

Applying Theorem 3.1 for specific functions θ yields the following stability results.

Corollary 3.4. *Let* W *be a Banach space. If* $f : V^n \to W$ *satisfies an estimation*

$$\|(\Phi f)(x^1, x^2)\| \leqslant \sum_{i=1}^{n} C_i(\|x_{i1}\|^{p_i} + \|x_{i2}\|^{p_i}), \tag{3.12}$$

for $x^1, x^2 \in D^n$*,* $C_i \in (0, +\infty)$ *and* $p_i \in \mathbb{R}$ *such that* $c_i := p_i - n$ *fulfill a condition*

$$\forall_{i\in\{1,\dots,n\}} c_i < 0 \quad \textit{or} \quad \forall_{i\in\{1,\dots,n\}} c_i > 0. \tag{3.13}$$

Then there exists a unique multi-additive mapping $F : V^n \longrightarrow W$ *such that*

$$\|f(x) - F(x)\| \leqslant \sum_{i=1}^{n}\frac{2C_i}{2^n|1 - 2^{c_i}|}\|x_i\|^{p_i}, \qquad x \in D^n.$$

Proof. Put

$$\theta(x^1, x^2) := \sum_{i=1}^{n} C_i(\|x_{i1}\|^{p_i} + \|x_{i2}\|^{p_i}), \qquad x^1, x^2 \in D^n.$$

From (3.13) we get that $c_i < 0$ for all $i \in \{1, ..., n\}$ or $c_i > 0$ for all $i \in \{1, ..., n\}$. Then there exists $s \in \{1, -1\}$ such that $2^{sc_i} < 1$ ($s = 1$ if $c_i < 0$, $s = -1$ if $c_i > 0$), and for each $i \in \{1, ..., n\}$

$$\sum_{j=0}^{\infty}(2^{sc_i})^j = \frac{1}{1 - 2^{sc_i}}.$$

Using Theorem 3.1, because

$$\varepsilon^*(x) := \begin{cases} \frac{1}{2^n}\sum_{i=1}^n \frac{2C_i}{1-2^{c_i}}\|x_i\|^{p_i} & \text{if } s=1,\\ \sum_{i=1}^n \frac{2C_i 2^{-p_i}}{1-2^{-c_i}}\|x_i\|^{p_i} & \text{if } s=-1,\end{cases}$$

and

$$\lim_{l\to\infty}\sum_{i=1}^n C_i\big(2^{sc_i}\big)^l(\|x_{i1}\|^{p_i}+\|x_{i2}\|^{p_i})=0,$$

we obtain the thesis. □

Corollary 3.5. *Let W be a Banach space. If $f : V^n \to W$ satisfies an estimation*

$$\|(\Phi f)(x^1,x^2)\| \leqslant C\prod_{i=1}^n \|x_{i1}\|^{p_i}\|x_{i2}\|^{q_i}, \tag{3.14}$$

for $x^1, x^2 \in D^n$ with some $C \in (0,+\infty)$ and $p_i, q_i \in \mathbb{R}$ such that

$$d := \sum_{i=1}^n (p_i+q_i) - n \neq 0. \tag{3.15}$$

Then there exists a unique multi-additive mapping $F : V^n \longrightarrow W$ such that

$$\|f(x)-F(x)\| \leqslant \frac{C}{2^n|1-2^d|}\prod_{i=1}^n \|x_i\|^{p_i+q_i}, \qquad x\in D^n.$$

Proof. Put

$$\theta(x^1,x^2) := C\prod_{i=1}^n \|x_{i1}\|^{p_i}\|x_{i2}\|^{q_i}, \qquad x^1,x^2\in D^n.$$

From (3.15) we get that $d < 0$ or $d > 0$. Then there exists $s \in \{1,-1\}$ such that $2^{sd} < 1$ ($s = 1$ if $d < 0$, $s = -1$ if $d > 0$) and

$$\sum_{j=0}^\infty (2^{sd})^j = \frac{1}{1-2^{sd}}.$$

Using Theorem 3.1, because

$$\varepsilon^*(x) := \begin{cases} \frac{C}{2^n(1-2^d)}\prod_{i=1}^n \|x_i\|^{p_i+q_i} & \text{if } s=1,\\ \frac{C}{2^n}\frac{2^{-d}}{1-2^{-d}}\prod_{i=1}^n \|x_i\|^{p_i+q_i} & \text{if } s=-1,\end{cases}$$

and

$$\lim_{l\to\infty} C\big(2^{sd}\big)^l\prod_{i=1}^n \|x_{i1}\|^{p_i}\|x_{i2}\|^{q_i}=0,$$

we obtain the thesis. □

Putting $p_i = q_i = 0$ for all $i \in \{1,...,n\}$ in Corollary 3.4 or Corollary 3.5 we obtain the following well known result.

Corollary 3.6. *Let W be a Banach space and $\varepsilon > 0$. If $f : V^n \to W$ satisfies inequality*

$$\|(\Phi f)(x^1, x^2)\| \leqslant \varepsilon, \qquad x^1, x^2 \in V^n,$$

then there exists a unique multi-additive mapping $F : V^n \to W$ such that

$$\|f(x) - F(x)\| \leqslant \frac{\varepsilon}{2^n - 1}, \qquad x \in V^n.$$

4. θ-hyperstability of the equation (1.1)

In this section, we show that the functional equation (1.1) is θ-hyperstable in the class of functions $g\colon V^n \to W$, where

$$\theta(x^1, x^2) = \sum_{i=1}^{n} C_i(\|x_{i1}\|^{p_i} + \|x_{i2}\|^{p_i}), \qquad x^1, x^2 \in V_0^n,$$

with $C_1, ..., C_n > 0,\ p_1, ..., p_n < 0$ or

$$\theta(x^1, x^2) = C \prod_{i=1}^{n} \|x_{i1}\|^{p_i} \|x_{i2}\|^{q_i}, \qquad x^1, x^2 \in V_0,$$

with $C \geqslant 0, \sum_{i=1}^{n}(p_i + q_i) \neq n$ and $p_k + q_k < 0$ for some $k \in \{1, ..., n\}$.

Using Corollary 3.4 we can obtain the following hyperstability result.

Corollary 4.1. *Let W be a normed space and $p_1, ..., p_n < 0$, $C_1, ..., C_n > 0$. If $f : V^n \to W$ satisfies the condition (3.12) for $x^1, x^2 \in V_0^n$ then f is a multi-additive mapping.*

Proof. First we notice that without loss of generality we can assume that W is a Banach space, because otherwise we can replace it by its completion. According to the Corollary 3.4, there exists a unique multi-additive mapping $F : V^n \to W$ such that

$$\|f(x) - F(x)\| \leqslant \varphi(x), \qquad x \in V_0^n,$$

where $\varphi(x) = \sum_{i=1}^{n} \frac{2C_i}{2^n - 2^{p_i}} \|x_i\|^{p_i}$.

Observe that for every $x \in V_0^n$ and $m \in \mathbb{N}$

$$\begin{aligned} &\|f(x) - F(x)\| \\ &= \quad \|\Phi f((m+1)x, -mx) + \sum_{a_{1m}, ..., a_{nm} \in \{m+1, -m\}} (f - F)(a_{1m}x_1, ..., a_{nm}x_n)\| \\ &\leqslant \quad \sum_{i=1}^{n} C_i[(m+1)^{p_i} + m^{p_i}]\|x_i\|^{p_i} + \sum_{a_{1m}, ..., a_{nm} \in \{m+1, -m\}} \varphi(a_{1m}x_1, ..., a_{nm}x_n). \end{aligned}$$

Letting $m \to \infty$ in the above inequality and using the fact that

$$\lim_{m\to\infty} \varphi(a_{1m}x_1, ..., a_{nm}x_n) = 0, \qquad a_{1m}, ..., a_{nm} \in \{m+1, -m\},$$

we conclude that $f = F$ on V_0^n. Using Lemma 2.1 we obtain that f is a multi-additive mapping. □

We observe that putting $n = 1$ in the above Corollary 4.1 we obtain result from [4] with $X = V_0$ and $p = p_1$.

Using Corollary 3.5 we can get the following hyperstability result.

Corollary 4.2. *Let W be a normed space and $C > 0$ and $p_i, q_i \in \mathbb{R}$, $i \in \{1, ..., n\}$ be such that* (3.15) *holds and $p_k + q_k < 0$ for some $k \in \{1, ..., n\}$. If $f : V^n \to W$ satisfies the condition* (3.14) *for $x^1, x^2 \in V_0^n$, then f is a multi-additive mapping.*

Proof. According to the Corollary 3.5, there exists a unique multi-additive mapping $F : V^n \to W$ such that

$$\|f(x) - F(x)\| \leqslant \varphi(x), \qquad x \in V_0^n,$$

where $\varphi(x) := \dfrac{C}{2^n|1 - 2^d|} \prod_{i=1}^{n} \|x_i\|^{p_i+q_i}$.

Since $p_k + q_k < 0$ with some $k \in \{1, ..., n\}$, then at least one of p_k, q_k must be negative. Without loss of generality we can assume that $p_k < 0$.

We put for each $x \in V_0^n$ and $m \in \mathbb{N}$

$z_m^x := (z_{m1}^x, ..., z_{mn}^x)$ such that $z_{mk}^x = (m + 1)x_k$ and $z_{mi}^x = \frac{1}{2}x_i$ for $i \neq k$, and

$w_m^x := (w_{m1}^x, ..., w_{mn}^x)$ such that $w_{mk}^x = -mx_k$ and $w_{mi}^x = \frac{1}{2}x_i$ for $i \neq k$.

Then we obtain for every $x \in V_0^n$ and $m \in \mathbb{N}$

$$\begin{aligned} &\|f(x) - F(x)\| \\ &= \|\Phi f(z_m^x, w_m^x) + 2^{n-1}[(f - F)(z_m^x) + (f - F)(w_m^x)]\| \\ &\leqslant C(m + 1)^{p_k} m^{q_k} \left(\frac{1}{2}\right)^{\sum_{i\neq k}(p_i+q_i)} \prod_{i=1}^{n} \|x_i\|^{p_i+q_i} + 2^{n-1}[\varphi(z_m^x) + \varphi(w_m^x)]. \end{aligned}$$

Letting $m \to \infty$ in the above inequality and using the fact that

$$\lim_{m\to\infty} \varphi(z_m^x) = \lim_{m\to\infty} \varphi(w_m^x) = 0,$$

we obtain that $f = F$ on V_0^n. Finally, from Lemma 2.1 we have that f is a multi-additive mapping. □

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2**(1950), 64–66.

[2] D.G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J., **16**(1949), 385-397.

[3] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., **57**(1951), 223–237.

[4] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar., **141**(2013), 58–67.

[5] J. Brzdęk, J. Chudziak, Zs. Páles, *A fixed point approach to stability of functional equations*, Nonlinear Anal., **74**(2011), 6728–6732.

[6] J. Brzdęk, K. Ciepliński, *Hyperstability and superstability*, Abstr. Appl. Anal., **2013**(2013), Article ID 401756, 13 pp.

[7] K. Ciepliński, *Generalized stability of multi-additive mappings*, Appl. Math. Lett., **23**(2010), 1291–1294.

[8] K. Ciepliński, *Stability of multi-additive mappings in β-Banach spaces*, Nonlinear Anal., **75**(2012), 4205–4212.

[9] P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**(1994), 431–436.

[10] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27**(1941), 222–224.

[11] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Birkhäuser Verlag, Basel, 2009.

[12] Gy. Maksa, Zs. Páles, *Hyperstability of a class of linear functional equations*, Acta Math. Acad. Paedag. Nyìregyháziensis, **17**(2001), 107-112.

[13] W.-G. Park, J.-H. Bae, *On the solution of a multi-additive functional equation and its stability*, J. Appl. Math. Comput., **22**(2006), 517–522.

[14] W.-G. Park, J.-H. Bae, *Solution of a vector variable bi-additive functional equation*, Commun. Korean Math. Soc., **23**(2008), 191–199.

[15] W.-G. Park, J.-H. Bae, *Stability of a bi-additive functional equation in Banach modules over a C^*-algebra*, Discrete Dynam. Nat. Soc., **2012**(2012), Article ID 835893, 12 pp.

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