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A NONLOCAL PROBLEM AT INFINITY FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper we propose the study of a scalar integral equation of the type

$$y(t) = g(y) + \int_t^\infty (s-t)a(s)f(y(s))\,ds, \quad t \geq 0,$$

and give conditions on g, a and f that ensure the existence of solutions on $[0,\infty)$ which are asymptotically equal to $g(y)$ at ∞. As a consequence, we obtain results on the existence of solutions for a problem of the type

$$y''(t) = a(t)f(y(t)), \quad y(\infty) = g(y),$$

where $y(\infty) = \lim_{t\to\infty} y(t)$. This problem could be thought as a sort of nonlocal problem at ∞, and our conditions on f include the case of a linear equation.

Key Words and Phrases: Nonlocal problem, asymptotic behavior, integral equation, second order differential equation, Leray-Schauder type fixed point theorem.

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1. Introduction

In the present paper we propose and study a problem for second order nonlinear differential equations which could be considered as a merger of two well known problems. The first of them deals with the existence of asymptotically constant solutions for second order differential equations of the type

$$y''(t) = a(t)f(y(t)), \quad t \geq 0, \tag{1.1}$$

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that is, solutions of the problem

$$\begin{cases} y''(t) = a(t)f(y(t)), & t \geq 0, \\ y(\infty) = C, \end{cases} \tag{1.2}$$

where $C \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $a : [0, \infty) \to \mathbb{R}$. More precisely, by a solution of (1.2) we mean a solution of equation (1.1), which is defined on the whole interval $\mathbb{R}^+ = [0, \infty)$ and is asymptotically equal to C, that is, there exists the limit $y(\infty) = \lim_{t\to\infty} y(t)$ and $y(\infty) = C$.

Problem (1.2) has been studied by many authors during more than six decades and as some references we mention (apologizing in advance for the omitted ones), for instance, [3, 4, 7–12, 14–17], and the references therein.

The second problem which we are interested in is of the type

$$\begin{cases} y'(t) = F(t, y(t)), & t \in [0, T], \\ y(0) = g(y), \end{cases} \tag{1.3}$$

where $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $g : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R}$. It is known as a nonlocal initial value problem and, as far as we know, goes back to the early 90's to a paper by Byszewski and Laksmikantham [5], who studied a problem similar to (1.3) in the context of Banach spaces. After them, many papers on nonlocal first order initial value problems have been published, and this topic is still a subject of research. Let us mention, for instance, the 2010 paper by Ji and Wen [13], where much progress has been done in order to obtain a complete answer to this type of problems. Recently, Byszewski and Winiarska [6] studied a nonlocal initial value problem for a second order differential equation of the form

$$\begin{cases} y''(t) = F(t, y(t), y'(t)), & t \geq 0, \\ y(0) = y_0, \\ y'(0) = g(y). \end{cases} \tag{1.4}$$

In the present paper we propose the study of second order differential problems which are combination of problems (1.2) and (1.3). Specifically, we consider the following *nonlocal problem at* ∞,

$$\begin{cases} y''(t) = a(t)f(y(t)), & t \in [0, \infty), \\ y(\infty) = g(y)\,. \end{cases} \tag{P}$$

Again, by a solution of (P) we mean a solution of the differential equation

$$y''(t) = a(t)f(y(t))$$

on $\mathbb{R}^+$ for which, additionally, there exists the limit

$$y(\infty) = \lim_{t\to\infty} y(t) \text{ and } y(\infty) = g(y).$$

In section 3 we shall give two results on existence of solutions for (P) under different sets of assumptions, but such that both include the case of g being constant, showing in this way that problem (1.2) can also be considered as a sort of nonlocal problem at ∞. In the first case, g is assumed to be bounded, while in the second, we impose the assumption of g being a contractive mapping.

Our results will be obtained through the study of an integral equation of the type

$$y(t) = g(y) + \int_t^{\infty} (s-t)a(s)f(y(s))\,ds, \quad t \geq 0. \tag{E}$$

For this reason, in section 2 we give a result about existence of solutions for (E).

A little bit of notation and preliminary results are needed. As customary, we denote by $\mathbb{R}^+$ the set $[0,\infty)$ of nonnegative real numbers. In a Banach space X, $\overline{B}(x,r)$ denotes the closed ball in X centered at x with radius r. The space of continuous $\mathbb{R}$-valued functions defined on $\mathbb{R}^+$ is denoted by $\mathcal{C}(\mathbb{R}^+,\mathbb{R})$, while the space of bounded continuous ones is $\mathcal{C}_b(\mathbb{R}^+,\mathbb{R})$. The latter is a Banach space when endowed with the sup norm $\|\cdot\|_\infty$, (i.e., for $x \in \mathcal{C}_b(\mathbb{R}^+,\mathbb{R})$, $\|x\|_\infty = \sup\limits_{t\in\mathbb{R}^+} |x(t)|$).

We shall also be needing the following version of the Leray-Schauder Fixed Point Theorem (see, e.g., [1]): *Suppose that X is a Banach space and that $T : X \to X$ is continuous and compact (i.e., T maps bounded sets onto relatively compact ones). If T satisfies the Leray-Schauder boundary condition on some closed ball $\overline{B}(0,R)$, that is, if there exists $R > 0$ such that*

$$T(x) \neq \lambda x, \qquad \text{whenever } \|x\| = R \quad \text{and } \lambda > 1\ , \tag{LS}$$

then T has a fixed point.

In order to check that a certain operator T defined on $\mathcal{C}_b(\mathbb{R}^+,\mathbb{R})$ is compact, it will be helpful a well known version of the Arzelà-Ascoli Theorem which, in the case that occupies us, is as follows: *If F is a bounded subset of $\mathcal{C}_b(\mathbb{R}^+,\mathbb{R})$ which is equicontinuos at each $t \in \mathbb{R}^+$, then each sequence $\{u_n\} \subseteq \mathcal{F}$ has a subsequence that converges uniformly on compact subsets of $\mathbb{R}^+$ to a given function $u \in \mathcal{C}_b(\mathbb{R}^+,\mathbb{R})$.*

2. The integral equation

As we mentioned before, our results about existence of solutions for the nonlocal problem (P) will be based on the existence of solutions for the integral equation (E) associated to (P).

Theorem 2.1. *Suppose that the following set of hypotheses is satisfied:*

$$a : \mathbb{R}^+ \to \mathbb{R} \text{ is continuous, and } \int_0^\infty t|a(t)|\,dt < \infty, \tag{Ha}$$

$$f : \mathbb{R} \to \mathbb{R} \text{ is continuous,} \tag{Hf-1}$$

$$f(u) > 0 \text{ for all } u > 0, \text{ and } \int_1^\infty \frac{1}{f(u)}\,du = +\infty, \tag{Hf-2}$$

$$|f(u)| \leq f(w) \text{ for all } u, w \in \mathbb{R} \text{ with } |u| \leq w, \text{ and} \tag{Hf-3}$$

$$g : \mathcal{C}_b(\mathbb{R}^+,\mathbb{R}) \to \mathbb{R} \text{ is continuous and bounded.} \tag{Hg}$$

Then, the integral equation (E) has a bounded solution on $\mathbb{R}^+$.

Remark 2.2. Of course, this result can be adapted to obtain a similar one about existence of solutions of a nonlocal problem at ∞, but on an interval of the type $[t_0, \infty)$.

Proof. For simplicity in future arguments, we distinguish two cases. In the first place, if $a(t) = 0$ for every $t \in \mathbb{R}^+$, then equation (E) is $y(t) = g(y)$, which has a bounded solution if, and only if, there exists $c_0 \in \mathbb{R}$ such that $c_0 = g(y_{c_0})$, where, for $c \in \mathbb{R}$, $y_c : \mathbb{R}^+ \to \mathbb{R}$ denotes the function constantly equal to c. That is, equation (E) has a bounded solution if, and only if, the function $h : \mathbb{R} \to \mathbb{R}$ given as $h(t) = t - g(y_t)$ has a zero. Finally, the existence of a zero for h can be obtained as an immediate consequence of the Intermediate Value Theorem, for g is continuous and bounded and thus h is continuous and takes positive and negative values.

Suppose now that a is not the null function, what implies that $\int_0^\infty s|a(s)|ds > 0$, and let us see that (E) still has a bounded solution. With this in mind, define $T : \mathcal{C}_b(\mathbb{R}^+, \mathbb{R}) \to \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ as $T = G + S$, where

$$G(y)(t) = g(y), \quad t \geq 0,$$
$$S(y)(t) = \int_t^\infty (s - t)a(s)f(y(s))\, ds.$$

It is clear that T is well defined, and that the set of bounded solutions of (E) is just the set of fixed points for T. For that reason, our objective will be to prove that T has a fixed point, and this will be achieved by using the aforementioned Leray-Schauder Fixed Point Theorem. Consequently, we proceed to prove the following three assertions:

(a) T is compact,
(b) T is continuous, and
(c) T satisfies the Leray-Schauder boundary condition (LS) on some ball $\overline{B}(0, R)$.

Proof of (a). Let F be a bounded subset of $\mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ and let us see that $T(F)$ is relatively compact in $\mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$. To do this, it is sufficient to prove that both, $G(F)$ and $S(F)$, are relatively compact.

To prove that $G(F)$ is relatively compact, suppose that $\{y_n\}$ is any sequence in $\mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ and let us see that it has a subsequence $\{y_{n_k}\}$ such that $\{G(y_{n_k})\}$ converges in $\mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$. Having in mind that

$$\|G(x) - G(y)\|_\infty = |g(x) - g(y)|$$

for all $x, y \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$, it is enough to prove that $\{g(y_n)\}$ has a Cauchy subsequence in $\mathbb{R}$, and this is true because g is bounded.

To prove that $S(F)$ is relatively compact we shall make use of the Arzelà-Ascoli Theorem. In the first place, consider any $t_0 \in \mathbb{R}^+$ and let us see that $S(F)$ is equicontinuous at t_0. Indeed, since F is bounded as a subset of $\mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ and f is continuous on $\mathbb{R}$, there exists $M > 0$ such that $\|f \circ y\|_\infty \leq M$ for every $y \in F$. Hence, for each

$y \in F$ and $t \in \mathbb{R}^+$,

$$\begin{aligned}|S(y)(t) - S(y)(t_0)| &= \left| \int_t^\infty (s-t)\, a(s)\, f\big(y(s)\big)\, ds - \int_{t_0}^\infty (s-t_0)\, a(s)\, f\big(y(s)\big)\, ds \right| \\ &\le \left| \int_t^{t_0} (s-t)\, a(s)\, f\big(y(s)\big)\, ds \right| + \int_{t_0}^\infty |t_0 - t|\, |a(s)|\, \big|f\big(y(s)\big)\big|\, ds \\ &\le 2|t_0 - t|\, \|f \circ y\|_\infty \int_0^\infty |a(s)|\, ds \\ &\le 2|t_0 - t| M \int_0^\infty |a(s)|\, ds\,,\end{aligned}$$

which gives the equicontinuity of $S(F)$.

In the second place, $S(F)$ is bounded in $\mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ because for any $y \in F$ and any $t \ge 0$,

$$|S(y)(t)| \le \int_t^\infty (s-t)\, |a(s)|\, \big|f\big(y(s)\big)\big|\, ds \le M \int_t^\infty s\, |a(s)|\, ds \le M \int_0^\infty s\, |a(s)|\, ds < \infty. \tag{2.1}$$

To end the proof of (a), suppose that $\{u_n\}$ is a sequence in $S(F)$ and let us see that it has a convergent subsequence. Notice that this is not true just because of the Arzelà-Ascoli Theorem; instead, an additional argument using the funnel structure of the set $S(F)$ is needed. Using the Arzelà-Ascoli Theorem we obtain a subsequence of $\{u_n\}$, $\{u_{n_k}\}$, which converges uniformly on compact subsets of $\mathbb{R}^+$ to a certain function $u \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$. In order to show that $\|u_{n_k} - u\|_\infty \xrightarrow{k\to\infty} 0$, suppose that $\varepsilon > 0$ has been given, and use (Ha) to choose $t_0 \in \mathbb{R}^+$ such that

$$M \int_{t_0}^\infty s|a(s)|\, ds < \frac{\varepsilon}{2}.$$

Then, by (2.1), obtain $|u_{n_k}(t)| < \frac{\varepsilon}{2}$ for every $t \ge t_0$. Since $\{u_{n_k}\}$ converges pointwise to u, we also have $|u(t)| \le \frac{\varepsilon}{2}$ for every $t \ge t_0$ and, consequently,

$$|u_{n_k}(t) - u(t)| < \varepsilon, \quad \text{for all } t \ge t_0.$$

Now, using that $\{u_{n_k}\}$ converges uniformly to u on $[0, t_0]$, obtain $k_0 \in \mathbb{N}$ such that

$$|u_{n_k}(t) - u(t)| < \varepsilon, \quad \text{whenever } t \in [0, t_0], \text{ and } k \ge k_0\ ,$$

and hence, combining this inequality with the previous one, we finally obtain that

$$\|u_{n_k} - u\|_\infty < \varepsilon, \quad \text{for all } k \ge k_0.$$

Proof of (b). Fix $y_0 \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$. In order to prove the continuity of T at y_0, observe that for any $y \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$, and any $t \in \mathbb{R}^+$, we have

$$\big|T(y)(t) - T(y_0)(t)\big| \le \big|g(y) - g(y_0)\big| + \left| \int_t^\infty (s-t)\, a(s)\, \big[f\big(y(s)\big) - f\big(y_0(s)\big)\big]\, ds \right|. \tag{2.2}$$

Consider now a fixed $\varepsilon > 0$. Use the continuity of g at y_0 and the uniform continuity of f on $\big[\|y_0\|_\infty - 1, \|y_0\|_\infty + 1\big]$ to obtain $\delta \in (0,1)$ such that, if $y \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ with

$\|y - y_0\|_\infty < \delta$, then $|g(y) - g(y_0)| < \frac{\varepsilon}{2}$, and also

$$\big|f\big(y(t)\big) - f\big(y_0(t)\big)\big| < \frac{\varepsilon}{2}\Big(\int_0^\infty s\,|a(s)|\,ds\Big)^{-1}, \qquad \text{for all } t \geq 0\ .$$

This, together with (2.2), gives $\|T(y) - T(y_0)\|_\infty \leq \varepsilon$ for all $y \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ with $\|y - y_0\|_\infty < \delta$.

Proof of (c). Due to (Hf-1) and (Hf-2), it is allowed to define a function $F : (0, \infty) \to \mathbb{R}$ as

$$F(z) = \int_1^z \frac{1}{f(u)}\,du,$$

and it turns out that F is differentiable and strictly increasing on $(0, \infty)$. Since we also have $F(1) = 0$, and $\lim_{z\to\infty} F(z) = +\infty$, we can define a real number $R > 1$ by the expression

$$R = F^{-1}\Big(\int_1^{M_g} \frac{1}{f(u)}\,du + \int_0^\infty s\,|a(s)|\,ds\Big),$$

where $M_g > 1$ is chosen with the additional property of being $M_g > \sup\big\{|g(y)| : y \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})\big\}$.

We shall prove that T satisfies the Leray-Schauder condition (LS) on $\overline{B}(0, R)$. To do it, suppose that, for certain $\lambda > 1$ and certain $y \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$, it is true that $T(y) = \lambda y$. Let us then see that $\|y\|_\infty < R$. Indeed, for any $t \in \mathbb{R}^+$, we have

$$\begin{aligned}\big|y(t)\big| = \frac{1}{\lambda}\big|T(y)(t)\big| &\leq \frac{|g(y)|}{\lambda} + \frac{1}{\lambda}\int_t^\infty (s - t)\,\big|a(s)\big|\big|f\big(y(s)\big)\big|\,ds \\ &\leq M_g + \frac{1}{\lambda}\int_t^\infty s\,\big|a(s)\big|\big|f\big(y(s)\big)\big|\,ds.\end{aligned}$$

Next, consider the function $w : \mathbb{R}^+ \to \mathbb{R}$ defined by the right-hand side of the above inequality, that is,

$$w(t) = M_g + \frac{1}{\lambda}\int_t^\infty s\,\big|a(s)\big|\big|f\big(y(s)\big)\big|\,ds,$$

and observe that w is differentiable, $w(t) \geq M_g > 0$ and $|y(t)| \leq w(t)$ for every $t \in \mathbb{R}^+$. Then, by (Hf-3),

$$w'(t) = \frac{-1}{\lambda}\,t\,\big|a(t)\big|\,\big|f\big(y(t)\big)\big| \geq \frac{-1}{\lambda}\,t\,\big|a(t)\big|\,f\big(w(t)\big),$$

and using that $w(t) > 0$ for all $t \in \mathbb{R}^+$, and (Hf-2), obtain

$$\frac{d}{dt}F\big(w(t)\big) = \frac{w'(t)}{f\big(w(t)\big)} \geq -\frac{1}{\lambda}\,t\,\big|a(t)\big|, \quad \text{for all } t \in \mathbb{R}^+.$$

Next, integration on both sides of the above inequality gives

$$F(M_g) - F(w(t)) \geq -\frac{1}{\lambda}\int_t^\infty s\,|a(s)|\,ds, \quad t \in \mathbb{R}^+,$$

from which it follows that, for all $t \geq 0$,

$$F(w(t)) \leq F(M_g) + \frac{1}{\lambda}\int_0^\infty s\,\big|a(s)\big|\,ds := K.$$

Using that F is strictly increasing, obtain that $w(t) \leq F^{-1}(K)$, and using again the strict monotonicity of F^{-1} and that $\lambda > 1$, obtain $\|w\|_\infty \leq F^{-1}(K) < R$. This completes the proof of (c) and, with it, the whole result. □

Remark 2.3. Observe that this Theorem covers the case in which $f(u) = u$, that is, the linear case is included in this result. Furthermore, for a and g satisfying the corresponding hypotheses, (Ha) and (Hg), and for $p \in (0, 1]$, an integral equation of the type

$$y(t) = g(y) + \int_t^\infty (s-t)\, a(s) \left(y(s)\right)^p ds, \quad t \geq 0,$$

has at least one solution in $\mathbb{R}^+$.

Remark 2.4. The requirement that $\displaystyle\int_1^\infty \frac{1}{f(u)}\, du = \infty$, although it is used, it is uncertain whether it can be dropped. This requirement prevents us from considering, for instance, $f(u) = u^2$.

Despite this, the way the Theorem has been proved allows us for a small weakening of this hypothesis, namely, it just suffices to have, for

$$M_g = \max\left\{1\,,\, \sup\left\{|g(y)| \;:\; y \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})\right\}\right\},$$

$$\text{there exists } R > 1 \text{ such that } \int_1^R \frac{1}{f(z)}\, dz = \int_1^{M_g} \frac{1}{f(z)}\, dz + \int_0^\infty s\,|a(s)|\, ds, \tag{2.3}$$

in order to obtain a solution for the integral equation (E). We leave the details to the reader.

With this in mind, it can be easily checked that the integral equation

$$y(t) = g(y) + \int_t^\infty (s-t)\, \frac{1}{(s+1)^3} \left(y(s)\right)^2 ds, \tag{2.4}$$

has at least one solution in $\mathbb{R}^+$, provided $\sup\left\{|g(y)| \;:\; y \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})\right\} < 2$. Indeed, this last assumption gives $1 \leq M_g < 2$, so, having in mind that $f(u) = u^2$ and that $a(t) = (t+1)^{-3}$, obtain

$$\begin{aligned} \int_1^{M_g} \frac{1}{f(u)}\, du + \int_0^\infty s\,|a(s)|\, ds &= \int_1^{M_g} \frac{1}{u^2}\, dz + \int_0^\infty \frac{s}{(s+1)^3}\, ds \\ &= 1 - \frac{1}{M_g} + \frac{1}{2} = \frac{3}{2} - \frac{1}{M_g} \in \left[\frac{1}{2}, 1\right), \end{aligned}$$

which is in the range of the function

$$F(z) = \int_1^z \frac{1}{u^2}\, du, \; z \in [1, \infty),$$

for this function F is continuous, strictly increasing, with

$$F(1) = 0 \text{ and } \lim_{z \to \infty} F(z) = 1.$$

3. The second order differential equation

In this section we consider the differential problem (P) and use the result obtained in the previous section on the integral equation (E) to obtain now a result on the existence of solutions for (P). This will be done by means of the following lemma.

Lemma 3.1. *Under hypotheses* (Hf-1) *and* (Ha)*, the set of solutions of the nonlocal problem* (P) *equals the set of solutions of the integral equation* (E).

Remark 3.2. Observe that the solutions of either (P) or (E) must be continuous and have finite limit at ∞, so they must be bounded.

Proof. Suppose that $y : \mathbb{R}^+ \to \mathbb{R}$ is a solution of (P). To see that y satisfies (E), observe first that, if $\rho, \sigma \in \mathbb{R}^+$,

$$y'(\rho) - y'(\sigma) = \int_\sigma^\rho y''(s)\, ds = \int_\sigma^\rho a(s)\, f\big(y(s)\big)\, ds. \tag{3.1}$$

Now the facts that y is bounded and f is continuous on $\mathbb{R}$ imply that $f \circ y$ is bounded on $\mathbb{R}^+$, which together with (Ha) gives the convergence of the integral $\int_\sigma^\infty a(s)\, f(y(s))\, ds$. This fact and (3.1) yield the existence of the limit $y'(\infty) = \lim_{\rho\to\infty} y'(\rho)$ as a real number. Moreover, it must be $y'(\infty) = 0$ because otherwise it would be $\lim_{t\to\infty} y(t) = +\infty$ or $\lim_{t\to\infty} y(t) = -\infty$, and we know that none of these two possibilities can occur, since $y(\infty) = g(y) \in \mathbb{R}$. Therefore, from (3.1), as $\rho \to \infty$, obtain

$$y'(\sigma) = -\int_\sigma^\infty a(s) f\big(y(s)\big)\, ds, \quad \sigma \geq 0. \tag{3.2}$$

Integrate now in (3.2) and use Fubini's rule, or integration by parts, to arrive at

$$g(y) - y(t) = -\int_t^\infty \int_\sigma^\infty a(s)\, f\big(y(s)\big)\, ds\, d\sigma = -\int_t^\infty (s-t)\, a(s)\, f\big(y(s)\big)\, ds,$$

that is, arrive at the fact that y is a solution of the integral equation (E).

Conversely, suppose that $y : \mathbb{R}^+ \to \mathbb{R}$ is a solution of (E). Then y is bounded and, because of the shape of the integral equation, the behavior of y at ∞ is given by $y(\infty) = g(y)$. Now, by the Fundamental Theorem of Calculus, it is easy to observe that $y''(t) = a(t) f(y(t))$ for all $t \in \mathbb{R}^+$. That is, y is a solution of the nonlocal problem (P). □

Corollary 3.3. *Under the set of hypotheses of Theorem* 2.1*, the nonlocal problem* (P) *has at least one solution.*

Remark 3.4. Translating to differential problems the equations treated in Remarks 2.3 and 2.4, we first obtain that, for a and g satisfying the corresponding hypotheses, (Ha) and (Hg), the problem

$$\begin{cases} y''(t) = a(t)\,\big(y(t)\big)^p, & t \geq 0, \\ y(\infty) = g(y), \end{cases}$$

has always a solution, provided $p \in (0,1]$. Also, the problem

$$\begin{cases} y''(t) = \frac{1}{(t+1)^3}\,\big(y(t)\big)^2, & t \geq 0, \\ y(\infty) = g(y), \end{cases} \tag{3.3}$$

has always a solution, provided $\sup\big\{|g(y)| \,:\, y \in \mathcal{C}_b(\mathbb{R}^+,\mathbb{R})\big\} < 2$. We do not know whether the nonlocal problem (3.3) has always a solution for an arbitrary continuous and bounded g. It would be interesting to know about it, for it is much related to the classical Emden-Fowler equation (see, e.g., [4]).

Remark 3.5. The hypothesis on the boundedness for g cannot be dropped. As an example, we may consider the very easy example:

$$\begin{cases} y''(t) = 0, & t \geq 0, \\ y(\infty) = 1 + y(0), \end{cases} \tag{3.4}$$

This nonlocal problem (in fact, a boundary value problem) has no solution, and fulfills all the requirements, except that g, defined as $g(y) = 1 + y(0)$, is not bounded on $\mathcal{C}_b(\mathbb{R}^+,\mathbb{R})$. We could also have considered $g(y) = 1 + \limsup_{t\to\infty} y(t)$.

Remark 3.6. As for the function a, we should not expect a weakening of the hypothesis $\int_0^\infty s\,|a(s)|\,ds < \infty$ to, for instance, $\int_0^\infty |a(s)|\,ds < \infty$. As an example, consider the following Euler-Cauchy differential equation:

$$y''(t) = \frac{1}{(t+1)^2}\,y, \qquad t \geq 0. \tag{3.5}$$

Observe that $a(t) = (t+1)^{-2}$ satisfies

$$\int_0^\infty |a(s)|\,ds = \int_0^\infty (s+1)^{-2}\,ds = 1 < \infty, \qquad \text{and}$$
$$\int_0^\infty s\,|a(s)|\,ds = \int_0^\infty (s+1)^{-1} - (s+1)^{-2}\,ds = \infty.$$

Now, with the usual technique of assuming a solution of the type $y = (t+1)^r$, arrive at the general solution for (3.5) in the form

$$y(t) = C_1(t+1)^{r_1} + C_2(t+1)^{r_2}, \quad t \geq 0, \tag{3.6}$$

where $r_1 = \frac{1-\sqrt{5}}{2} < 0$ and $r_2 = \frac{1+\sqrt{5}}{2} > 0$. Now, if $y(\infty)$ is to exist and be finite, then it must be $C_2 = 0$, in which case $y = C_1(t+1)^{r_1} \to 0$ as $t \to \infty$. So we conclude that 0 is the only possible finite asymptotic value for solutions of the Equation (3.5), no other asymptotic value is allowed. Hence the corresponding nonlocal problem associated to Equation (3.5), and to a continuous and bounded g, will not have a solution unless $g(y) \equiv 0$.

The conclusion in Corollary 3.3 has been attained via a fixed point theorem of Leray-Schauder type, and this has obliged us to impose certain hypothesis on f and g. Had we wanted to use the Banach-Caccioppoli Contraction Principle (see, e.g., [1]), the conditions needed would have been of Lipschitz type, and this will be our choice in the next theorem. We shall need a previous result that we are sure has been done before, but being unable to find a reference for it, we have decided to include it for the sake of completeness.

Lemma 3.7. *Suppose that* (Ha) *is satisfied and that* f *is a Lipschitz mapping. Then, for any* $C \in \mathbb{R}$*, the equation*

$$y(t) = C + \int_t^\infty (s-t)a(s)f(y(s))\,ds \tag{E_C}$$

has a unique solution, which, of course, must be bounded with limit at ∞ *given by* C.

Proof. Our strategy, for the existence part, will be to obtain first a solution of (E_C) on a certain interval $[t_0, \infty)$ by means of the Banach-Caccioppoli Theorem, and then, using that f is Lipschitz, another solution on $[0, t_0]$ which glue well with the previous one. With this in mind, denoting by L_f the Lipschitz constant for f in $\mathbb{R}^+$, and based on hypothesis (Ha), choose a real number $t_0 > 0$ such that $L_f \int_{t_0}^\infty s|a(s)|\,ds < 1$, and consider the map $T : \mathcal{C}_b([t_0,\infty),\mathbb{R}) \to \mathcal{C}_b([t_0,\infty),\mathbb{R})$ given as

$$T(y)(t) = C + \int_t^\infty (s-t)\,a(s)\,f\big(y(s)\big)\,ds.$$

This map is well defined due to (Ha) and the fact that f maps bounded sets onto bounded sets. It is also contractive since, for any $y_1, y_2 \in \mathcal{C}_b([t_0,\infty),\mathbb{R})$ and any $t \in [t_0, \infty)$,

$$\begin{aligned} \big|T(y_1)(t) - T(y_2)(t)\big| &\le \int_t^\infty (s-t)\,\big|a(s)\big|\,\big|f(y_1(s)) - f(y_2(s))\big|\,ds \\ &\le \Big(L_f \int_{t_0}^\infty s|a(s)|\,ds\Big)\|y_1 - y_2\|_\infty. \end{aligned}$$

Hence, by the Banach-Caccioppoli Contraction Principle, T has a fixed point $x \in \mathcal{C}_b([t_0,\infty),\mathbb{R})$. This means that x is a bounded solution of equation (E_C) on $[t_0,\infty)$ and then, using Lemma 3.1 with $[0,\infty)$ replaced by $[t_0,\infty)$, x is a solution of the problem

$$\begin{cases} y''(t) = a(t)f(y(t)), & t \ge t_0, \\ y(\infty) = C. \end{cases} \tag{P^+}$$

Now notice that the problem

$$\begin{cases} y''(t) = a(t)f(y(t)), & t \in [0, t_0], \\ y(t_0) = x(t_0), \\ y'(t_0) = x'(t_0), \end{cases} \tag{P^-}$$

has a unique solution $z : [0,t_0] \to \mathbb{R}$, since (P^-) is of the form

$$y''(t) = F(t, y(t)), \qquad y(t_0) = x(t_0), \quad y'(t_0) = x'(t_0), \tag{3.7}$$

where $F(t,y) = a(t)f(y)$ is continuous on the strip $[0,t_0] \times \mathbb{R}$ and satisfies a generalized Lipschitz condition with respect to the second variable. Indeed, for any $t \in [0,t_0]$ and $y_1, y_2 \in \mathbb{R}$, we have $|F(t,y_1) - F(t,y_2)| \le L(t)|y_1 - y_2|$, where $L : [0,t_0] \to \mathbb{R}$ is the continuous function $L(t) = L_f\,|a(t)|$.

Next, the function $y : \mathbb{R}^+ \to \mathbb{R}$ given as

$$y(t) = \begin{cases} z(t), & \text{if } 0 \le t \le t_0, \\ x(t), & \text{if } t \ge t_0, \end{cases}$$

is a solution of (P) and consequently, by Lemma 3.1, it is also a solution of (E_C).

Finally, the uniqueness part follows directly from well known Gronwall-Bellman type inequalities. □

Now, we adapt the technique used in [2], in conjunction with the previous result, in order to obtain the following one.

Theorem 3.8. *Suppose, in addition to* (Ha)*, that f and g are Lipschitz mappings with Lipschitz constants L_f and L_g, respectively. If $L_g e^{L_f \int_0^\infty s|a(s)|ds} < 1$, then the nonlocal differential problem (P) has a unique solution.*

Proof. We shall prove that equation (E) has a unique solution on $\mathbb{R}^+$. Consider the operator $G : \mathcal{C}_b(\mathbb{R}^+, \mathbb{R}) \to \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ which maps each $y \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ to the unique solution $\overline{y} \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ of the equation (E_C), with $C = g(y)$, given by Lemma 3.7, that is, $\overline{y}$ is the unique function in $\mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$ which satisfies

$$\overline{y}(t) = g(y) + \int_t^\infty (s-t)\, a(s)\, f\big(\overline{y}(s)\big)\, ds \quad \text{for all } t \in \mathbb{R}^+. \tag{3.8}$$

Notice that y is a bounded solution of (E) if, and only if, $G(y) = y$, and so we only need prove that G is a contractive map. To do it, use (3.8), and the fact that f is Lipschitz, to obtain that, for any $y_1, y_2 \in \mathcal{C}_b(\mathbb{R}^+, \mathbb{R})$, and any $t \in \mathbb{R}^+$,

$$\begin{aligned} \big|G(y_1)(t) - G(y_2)(t)\big| &= |\overline{y}_1(t) - \overline{y}_2(t)| \\ &\le |g(y_1) - g(y_2)| + \int_t^\infty (s-t)\big|a(s)\big|\, \big|f\big(\overline{y}_1(s)\big) - f\big(\overline{y}_2(s)\big)\big|\, ds \\ &\le L_g \|y_1 - y_2\|_\infty + \int_t^\infty L_f s\, \big|a(s)\big|\, \big|\overline{y}_1(s) - \overline{y}_2(s)\big|\, ds. \end{aligned}$$

Again, a Gronwall-Bellman type inequality gives us that

$$\big|G(y_1)(t) - G(y_2)(t)\big| = |\overline{y}_1(t) - \overline{y}_2(t)| \le L_g\, \|y_1 - y_2\|_\infty\, e^{L_f \int_0^\infty s|a(s)|ds},$$

and this shows that G is a contractive map. □

Remark 3.9. Again, the restriction on the Lipschitz constant for g cannot be dropped, for the examples considered in Remark 3.5 are also valid for this situation: $g(y) = 1 + y(0)$, and $g(y) = 1 + \limsup_{t\to\infty} y(t)$, are Lipschitz mappings with Lipschitz constants equal to 1 in both cases.

References

[1] R. P. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge Tracts in Mathematics, vol. 141, Cambridge University Press, Cambridge, 2001.

[2] S. Aizicovici and H. Lee, *Nonlinear nonlocal Cauchy problems in Banach spaces*, Appl. Math. Lett., **18** (2005), no. 4, 401–407.

[3] F. V. Atkinson, *On second order nonlinear oscillation*, Pacific J. Math., **5** (1955), 643–647.

[4] R. Bellman, *Stability Theory in Differential Equations*, Dover Publications, Inc., New York, 1969.

[5] L. Byszewski and V. Lakshmikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal., **40** (1991), no. 1, 11–19.

[6] L. Byszewski and T. Winiarska, *An abstract nonlocal second order evolution problem*, Opuscula Math., **32** (2012), no. 1, 75–82.

[7] S. D. Dubé and A. B. Mingarelli, *Note on a non-oscillation theorem of Atkinson*, Electron. J. Diff. Equations, **2004** (2004), no. 22, 1–6.

[8] M. Ehrnström, *Positive solutions for second-order nonlinear differential equations*, Nonlinear Anal., **64** (2006), 1608–1620.

[9] M. Ehrnström, *Linear asymptotic behaviour of second order ordinary differential equations*, Glasgow Math. J., **49** (2007), 105–120.

[10] C. González and A. Jiménez-Melado, *Existence of monotonic asymptotically constant solutions for second order differential equations*, Glasgow Math. J., **49** (2007), 515–523.

[11] C. González and A. Jiménez-Melado, *Asymptotic behavior of solutions to an integral equation underlying a second-order differential equation*, Nonlinear Anal., **70** (2009), 822–829.

[12] E. Hille, *Non-oscillation theorems*, Trans. Amer. Math. Soc., **64** (1948), 234–252.

[13] S. Ji and S. Wen, *Nonlocal Cauchy problem for impulsive differential equations in Banach spaces*, Int. J. Nonlinear Sci., **10** (2010), no. 1, 88–95.

[14] T. Kusano and W. F. Trench, *Global existence theorems for solutions of nonlinear differential equations with prescribed asymptotic behaviour*, J. London Math. Soc., **31** (1985), no. 3, 478–486.

[15] J. W. Macki and J. S. W. Wong, *Oscillation of solutions to second-order nonlinear differential equations*, Pacific J. Math., **24** (1968), 111–117.

[16] O. G. Mustafa and Y. V. Rogovchenko, *Global existence of solutions with prescribed asymptotic behavior for second-order nonlinear differential equations*, Nonlinear Anal., **51** (2002), 339–368.

[17] E. Wahlén, *Positive solutions of second-order differential equations*, Nonlinear Anal., **58** (2004), 359–366.

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