

COMPARABLE LINEAR CONTRACTIONS IN ORDERED METRIC SPACES

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Abstract. In this paper, with a view to improve the g -monotonicity condition, we introduce the notion of g -comparability of a mapping defined on an ordered set and utilize the same to prove some existence and uniqueness results on coincidence points for linear contraction without g -monotonicity in ordered metric spaces. Our results extend some classical and well known results due to Ran and Reurings (Proc. Amer. Math. Soc. **132**(2004), no.5, 1435-1443), Nieto and Rodríguez-López (Acta Math. Sin. **23**(2007), no.12, 2205-2212), Turinici (Libertas Math. **31**(2011), 49-55), Turinici (Math. Student **81**(2012), no.1-4, 219-229) and Dorić *et al.* (RACSAM **108**(2014), no.2, 503-510) and similar others.

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1. INTRODUCTION

Throughout this paper, the pair (X, \preceq) , stands for a nonempty set X equipped with a partial order \preceq often called an ordered set wherein we generally write $x \succeq y$ instead of $y \preceq x$. Two elements x and y in an ordered set (X, \preceq) are said to be comparable if either $x \preceq y$ or $y \preceq x$ and denote it as $x \prec\succ y$. A subset E of an ordered set is called totally ordered if $x \prec\succ y$ for all $x, y \in E$. In respect of a pair of self-mappings (f, g) defined on an ordered set (X, \preceq) , we say that f is g -increasing (resp. g -decreasing) if for any $x, y \in X$, $g(x) \preceq g(y)$ implies $f(x) \preceq f(y)$ (resp. $f(x) \succeq f(y)$). As per standard practice, f is called g -monotone if f is either g -increasing or g -decreasing. Notice that with $g = I$ (the identity mapping), the notions of g -increasing, g -decreasing and g -monotone mappings transform into increasing, decreasing and monotone mappings respectively. Following O'Regan and Petrusel [1], the triple (X, d, \preceq) is called ordered metric space wherein X denotes a nonempty set endowed with a metric d and a partial order \preceq . If in addition, d is a complete metric on X , then we say that (X, d, \preceq) is an ordered complete metric space.

The relevant detailed discussions on basic topological properties of ordered sets are available in Milgram [2, 3], Eilenberg [4], Wolk [5, 6] and Monjardet [7]. Existence of fixed points for monotone mappings on ordered sets was first investigated by Tarski [8] and Björner [9] (on complete lattices), Abian and Brown [10], DeMarr [11], Wong [12], Pasini [13], Kurepa [14], Amann [15] and Dugundji and Granas [16] (on abstract ordered sets), Ward [17] (on ordered topological spaces), DeMarr [18] (on ordered spaces obtained from complete metric spaces), Turinici [19] (on order closed subordered metrizable uniform spaces) and Turinici [20] (on quasi-ordered complete metric spaces). In 2004, unknowingly Ran and Reurings [21] particularized a fixed point theorem proved in Turinici [20] in ordered metric spaces for continuous monotone mappings besides giving some applications to matrix equations.

Theorem 1.1 (Theorem 2.1, Ran and Reurings [21]). *Let (X, d, \preceq) be an ordered metric space and f a self-mapping on X . Suppose that the following conditions hold:*

- (a) (X, d) is complete,
- (b) f is monotone,
- (c) f is continuous,
- (d) there exists $x_0 \in X$ such that $x_0 \prec \succ f(x_0)$,
- (e) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } x \preceq y,$$

(f) every pair of elements of X has a lower bound and an upper bound.

Then f has a unique fixed point \bar{x} . Moreover, for every $x \in X$, $\lim_{n \rightarrow \infty} f^n(x) = \bar{x}$.

Thereafter, Nieto and Rodríguez-López [22, 23] slightly modified Theorem 1.1 for monotone mappings to relax the continuity requirement by assuming an additional hypothesis on ordered metric space besides observing that (owing to assumption (f)) the existence of lower bound (or upper bound) for every pair of elements of X serves our purpose which is also followed by some applications of their results to ordinary differential equations.

Theorem 1.2 (Theorem 5, Nieto and Rodríguez-López [23]). *Let (X, d, \preceq) be an ordered metric space and f a self-mapping on X . Suppose that the following conditions hold:*

- (a) (X, d) is complete,
- (b) f is monotone,
- (c) either f is continuous or (X, d, \preceq) satisfies the following property:

if $\{x_n\}$ is a sequence in X such that $x_n \xrightarrow{d} x$ whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is comparable to the limit x ,
- (d) there exists $x_0 \in X$ such that $x_0 \prec \succ f(x_0)$,
- (e) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } x \preceq y,$$
- (f) every pair of elements of X has a lower bound or an upper bound.

Then f has a unique fixed point.

In subsequent papers (cf. ([24]-[34])) many authors generalized and refined Theorem 1.2 and proved several fixed point theorems in ordered metric spaces. In all such results, the contractivity condition holds on the monotone map for only those elements

which are related by the underlying partial ordering. Thus, in the context of fixed point theorems for ordered metric spaces, the usual contraction condition is weakened but at the expense of monotonicity of the underlying mapping.

To relax the monotonicity requirement on underlying mapping, Nieto and Rodríguez-López [23] replaced this condition by preservation of comparable elements and improved Theorem 1.2 as follows:

Theorem 1.3 (Theorem 7, Nieto and Rodríguez-López [23]). *Let (X, d, \preceq) be an ordered metric space and f a self-mapping on X . Suppose that the following conditions hold:*

- (a) (X, d) is complete,
- (b) for $x, y \in X$ with $x \preceq y \Rightarrow f(x) \preceq f(y)$ or $f(x) \succeq f(y)$,
- (c) either f is continuous or (X, d, \preceq) satisfies the following property:
if $\{x_n\}$ is a sequence in X such that $x_n \xrightarrow{d} x$ whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is comparable to the limit x ,
- (d) there exists $x_0 \in X$ such that $x_0 \prec \succ f(x_0)$,
- (e) there exists $\alpha \in [0, 1)$ such that
$$d(fx, fy) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } x \preceq y,$$
- (f) for every pair $x, y \in X$ there exists $z \in X$ which is comparable to x and y .

Then f has a unique fixed point \bar{x} . Moreover, for every $x \in X$, $\lim_{n \rightarrow \infty} f^n(x) = \bar{x}$.

Here it is noticed that the assumptions (f) of Theorem 1.2 and (f) of Theorem 1.3 are equivalent (see [22]).

Turinici [35, 36] proved similar results besides observing that these results (hence Theorems 1.1 and 1.2) are particular cases of Banach Contraction Principle (cf. [37]) and its an important generalization due to Maia [38]. Following Turinici [35, 36], given $x, y \in X$, any subset $\{z_1, z_2, \dots, z_k\}$ (for $k \geq 2$) in X with $z_1 = x, z_k = y$ and $z_i \prec \succ z_{i+1}$ for each $i \in \{1, 2, \dots, k-1\}$ is called a $\prec \succ$ -chain between x and y . The class of such chains is denoted by $C(x, y, \prec \succ)$.

Theorem 1.4 (Theorem 2.1, Turinici [35]). *Let (X, d, \preceq) be an ordered metric space and f a self-mapping on X . Suppose that the following conditions hold:*

- (a) (X, d) is complete,
- (b) for $x, y \in X$ with $x \prec \succ y \Rightarrow f(x) \prec \succ f(y)$,
- (c) f is continuous,
- (d) there exists $x_0 \in X$ such that $x_0 \prec \succ f(x_0)$,
- (e) there exists $\alpha \in [0, 1)$ such that
$$d(fx, fy) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } x \preceq y,$$
- (f) $C(x, y, \prec \succ)$ is nonempty for each $x, y \in X$.

Then f has a unique fixed point z . Moreover for each $x \in X$, the sequence $\{f^n x\}$ is convergent and $\lim_{n \rightarrow \infty} f^n(x) = z$.

Theorem 1.5 (Theorem 2.1, Turinici [36]). *Let (X, d, \preceq) be an ordered metric space and f a self-mapping on X . Suppose that the following conditions hold:*

- (a) (X, d) is complete,
- (b) for $x, y \in X$ with $x \prec \succ y \Rightarrow f(x) \prec \succ f(y)$,
- (c) (X, d, \preceq) satisfies the following property:

if $\{x_n\}$ is a sequence in X such that $x_n \xrightarrow{d} x$ whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is comparable to the limit x ,

(d) there exists $x_0 \in X$ such that $x_0 \prec \succ f(x_0)$,

(e) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } x \preceq y,$$

(f) $C(x, y, \prec \succ)$ is nonempty for each $x, y \in X$.

Then f has a unique fixed point z . Moreover for each $x \in X$, the sequence $\{f^n x\}$ is convergent and $\lim_{n \rightarrow \infty} f^n(x) = z$.

Notice that the assumptions (b) of Theorems 1.4 and 1.5 are equivalents to assumption (b) of Theorems 1.3. But assumptions (f) of Theorems 1.4 and 1.5 are relatively weaker than assumption (f) of Theorems 1.3 (see details in [35, 36]).

Very recently Dorić *et al.* [39] proved the following result:

Theorem 1.6 (Corollary 2.7, Dorić *et al.* [39]). *Let (X, d, \preceq) be an ordered metric space and f a self-mapping on X . Suppose that the following conditions hold:*

(a) (X, d) is complete,

(b) for $x, y \in X$ with $x \prec \succ y \Rightarrow f(x) \prec \succ f(y)$,

(c) f is continuous or (X, d, \preceq) satisfies the following property:

if $x_n \xrightarrow{d} x$ in X then $x_n \prec \succ x$ for n sufficiently large,

(d) there exists $x_0 \in X$ such that $x_0 \prec \succ f(x_0)$,

(e) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } x \prec \succ y.$$

Then f has a fixed point in X . Moreover if

(f) for each $x, y \in X \exists z \in X$ such that $x \prec \succ z$ and $y \prec \succ z$.

Then the fixed point u is unique and for each $x \in X$ sequence $\{f^n x\}$ converges to u .

Although the property on (X, d, \preceq) in assumption (c) of Theorem 1.3 is relatively weaker than the assumption (c) of Theorem 1.6. But in Theorem 1.6, authors observed that the uniqueness of fixed point is not necessary.

As reflected in Theorems 1.3-1.6, with a view to coin a relatively weaker alternate condition to avoid the use of the monotonicity requirement, the respective authors used a common property on the involved mapping (see assumption (b) in Theorems 1.3-1.6). In this paper, we generalize this idea to a pair of mappings and utilize the same to prove some coincidence point theorems for a pair of self-mappings f and g defined on an ordered metric space X satisfying linear contractivity condition in two different directions namely: in case X is complete or alternately one of the subspaces $f(X)$ or $g(X)$ is complete while the whole space X may or may not be complete. As a consequence of our results, we also derive a corresponding fixed point theorem, which extends and improves all earlier mentioned results (*i.e.* Theorems 1.1-1.6) besides furnishing an illustrative example.

2. PRELIMINARIES

In this section, we summarize some basic definitions and auxiliary results. Throughout this paper, \mathbb{N} stands for the set of natural numbers, while \mathbb{N}_0 for the set of whole numbers (*i.e.* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

Definition 2.1 [40, 41]. Let X be a nonempty set and (f, g) a pair of self-mappings on X . Then

- (i) an element $x \in X$ is called a coincidence point of f and g if

$$g(x) = f(x),$$

- (ii) if $x \in X$ is a coincidence point of f and g and $\bar{x} \in X$ such that $\bar{x} = g(x) = f(x)$, then \bar{x} is called a point of coincidence of f and g ,
- (iii) if $x \in X$ is a coincidence point of f and g such that $x = g(x) = f(x)$, then x is called a common fixed point of f and g ,
- (iv) the pair (f, g) is said to be commuting if

$$g(fx) = f(gx) \quad \forall x \in X \text{ and}$$

- (v) the pair (f, g) is said to be weakly compatible (or partially commuting or coincidentally commuting) if f and g commute at their coincidence points, *i.e.*,

$$g(fx) = f(gx) \quad \text{whenever } g(x) = f(x).$$

Definition 2.2 [42, 43]. Let (X, d) be a metric space and (f, g) a pair of self-mappings on X . Then

- (i) the pair (f, g) is said to be weakly commuting if

$$d(gfx, fgx) \leq d(gx, fx) \quad \forall x \in X \text{ and}$$

- (ii) the pair (f, g) is said to be compatible if

$$\lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} f(x_n).$$

Clearly in a metric space, commutativity \Rightarrow weak commutativity \Rightarrow compatibility \Rightarrow weak compatibility but reverse implications are not true in general (for details see [41]-[43]).

Definition 2.3 [44]. Let (X, d) be a metric space, (f, g) a pair of self-mappings on X and $x \in X$. We say that f is g -continuous at x if for all $\{x_n\} \subset X$,

$$g(x_n) \xrightarrow{d} g(x) \Rightarrow f(x_n) \xrightarrow{d} f(x).$$

Moreover, f is called g -continuous if it is g -continuous at each point of X .

Notice that with $g = I$ (the identity mapping on X) Definition 2.3 reduces to the definition of continuity.

Definition 2.4 [34]. Let (X, \preceq) be an ordered set and $\{x_n\} \subset X$. Then

- (i) the sequence $\{x_n\}$ is said to be termwise bounded if there is an element $z \in X$ such that each term of $\{x_n\}$ is comparable with z , *i.e.*,

$$x_n \prec \succ z \quad \forall n \in \mathbb{N}_0$$

so that z is a c-bound of $\{x_n\}$ and

- (ii) the sequence $\{x_n\}$ is said to be termwise monotone if consecutive terms of $\{x_n\}$ are comparable, *i.e.*,

$$x_n \prec \succ x_{n+1} \quad \forall n \in \mathbb{N}_0.$$

Clearly all bounded above as well as bounded below sequences are termwise bounded and all monotone sequences are termwise monotone.

Let (X, d, \preceq) be an ordered metric space and $\{x_n\} \subset X$. If $\{x_n\}$ is termwise monotone and $x_n \xrightarrow{d} x$, then we denote it symbolically by $x_n \downarrow x$.

Definition 2.5 [34]. Let (X, d, \preceq) be an ordered metric space. We say that (X, d, \preceq) has *TCC* (termwise monotone-convergence-c-bound) property if every termwise monotone convergent sequence $\{x_n\}$ in X has a subsequence, which is termwise bounded by the limit of $\{x_n\}$ (as a c-bound), *i.e.*,

$$x_n \downarrow x \Rightarrow \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } x_{n_k} \prec \succ x \quad \forall k \in \mathbb{N}_0.$$

Notice that above definition is formulated using a property utilized in Nieto and Rodríguez-López [23] (see assumption (c) in Theorem 1.2).

Definition 2.6 [34]. Let (X, d, \preceq) be an ordered metric space and g a self-mapping on X . We say that (X, d, \preceq) has *g-TCC* property if every termwise monotone convergent sequence $\{x_n\}$ in X has a subsequence, whose g -image is termwise bounded by g -image of limit of $\{x_n\}$ (as a c-bound), *i.e.*,

$$x_n \downarrow x \Rightarrow \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } g(x_{n_k}) \prec \succ g(x) \quad \forall k \in \mathbb{N}_0.$$

Notice that under the restriction $g = I$, the identity mapping on X , Definition 2.6 reduces to Definition 2.5.

Very recently, Alam and Imdad [34] generalized Theorems 1.1 and 1.2 for a pair of mappings and proved the following coincidence point theorem:

Theorem 2.7 [34]. *Let (X, d, \preceq) be an ordered metric space and f and g two self-mappings on X . Suppose that the following conditions hold:*

- (a) $f(X) \subseteq g(X)$,
- (b) f is g -monotone,
- (c) there exists $x_0 \in X$ such that $g(x_0) \prec \succ f(x_0)$,
- (d) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(gx, gy) \quad \forall x, y \in X \text{ with } g(x) \prec \succ g(y),$$
- (e) (e1) (X, d) is complete,
 - (e2) (f, g) is compatible pair,
 - (e3) g is continuous,
 - (e4) either f is continuous or (X, d, \preceq) has g -TCC property,
 or alternately
- (e') (e'1) either (fX, d) or (gX, d) is complete,
 - (e'2) either f is g -continuous or f and g are continuous or (gX, d, \preceq) has TCC property.

Then f and g have a coincidence point.

We need the following known results in the proof of our main results.

Lemma 2.8 [45]. *Let X be a nonempty set and g a self-mapping on X . Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \rightarrow X$ is one-one.*

Lemma 2.9 [33]. *Let X be a nonempty set and f and g two self-mappings on X . If the pair (f, g) is weakly compatible, then every point of coincidence of f and g is also a coincidence point of f and g .*

3. RESULTS ON COINCIDENCE POINTS

Firstly, we name a property utilized in Theorems 1.3-1.6 and term the same as comparable mapping.

Definition 3.1 (see [23, 35, 36, 39]). Let (X, \preceq) be an ordered set and f a self-mapping on X . We say that f is comparable (or weakly monotone or $\prec \succ$ -preserving) if f maps comparable elements to comparable elements, i.e., for any $x, y \in X$

$$x \prec y \Rightarrow f(x) \prec f(y).$$

It is clear that every monotone mapping is comparable, but not conversely. To substantiate this view point, consider the set $X = [-\frac{1}{3}, \frac{1}{3}]$ under the natural ordering of real numbers. Define $f : X \rightarrow X$ by $f(x) = x^2$, then f is comparable but not monotone.

We extend the idea embodied in Definition 3.1 to a pair of mappings to introduce the notion of g -comparability:

Definition 3.2. Let (X, \preceq) be an ordered set and f and g two self-mappings on X . We say that f is g -comparable (or weakly g -monotone or $(g, \prec \succ)$ -preserving) if for any $x, y \in X$

$$g(x) \prec g(y) \Rightarrow f(x) \prec f(y).$$

Notice that on setting $g = I$, the identity mapping on X , Definition 3.2 reduces to Definition 3.1.

Now, we are equipped to prove our main result on coincidence points in ordered complete metric spaces which runs as follows:

Theorem 3.3. *Let (X, d, \preceq) be an ordered complete metric space and f and g two self-mappings on X . Suppose that the following conditions hold:*

- (i) $f(X) \subseteq g(X)$,
- (ii) f is g -comparable,
- (iii) (f, g) is compatible pair,
- (iv) g is continuous,
- (v) either f is continuous or (X, d, \preceq) has g -TCC property,
- (vi) there exists $x_0 \in X$ such that $g(x_0) \prec f(x_0)$,
- (vii) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(gx, gy) \quad \forall x, y \in X \text{ with } g(x) \prec g(y).$$

Then f and g have a coincidence point.

Proof. In view of assumption (vi) if $g(x_0) = f(x_0)$, then we are through. Otherwise, if $g(x_0) \neq f(x_0)$, then owing to assumption (i) (i.e. $f(X) \subseteq g(X)$), we can choose $x_1 \in X$ such that $g(x_1) = f(x_0)$. Again from $f(X) \subseteq g(X)$, we can choose $x_2 \in X$ such that $g(x_2) = f(x_1)$. Continuing this process inductively, we define a sequence $\{x_n\} \subset X$ of joint iterates such that

$$g(x_{n+1}) = f(x_n) \quad \forall n \in \mathbb{N}_0. \tag{3.1}$$

Now, we assert that $\{gx_n\}$ is a termwise monotone sequence, *i.e.*,

$$g(x_n) \prec \succ g(x_{n+1}) \quad \forall n \in \mathbb{N}_0. \quad (3.2)$$

We prove this fact by mathematical induction. On using assumption (vi) and equation (3.1) with $n = 0$, we have

$$g(x_0) \prec \succ f(x_0) = g(x_1)$$

Thus, (3.2) holds for $n = 0$. Suppose that (3.2) holds for $n = r > 0$, *i.e.*,

$$g(x_r) \prec \succ g(x_{r+1}) \quad (3.3)$$

then we have to show that (3.2) holds for $n = r + 1$. To accomplish this, we use (3.1), (3.3) and assumption (ii) so that

$$g(x_{r+1}) = f(x_r) \prec \succ f(x_{r+1}) = g(x_{r+2}).$$

Thus, by induction, (3.2) holds for all $n \in \mathbb{N}_0$.

If $g(x_{n_0}) = g(x_{n_0+1})$ for some $n_0 \in \mathbb{N}$, then using (3.1), we have $g(x_{n_0}) = f(x_{n_0})$, *i.e.*, x_{n_0} is a coincidence point of f and g and hence we are done. On the other hand, if $g(x_n) \neq g(x_{n+1})$ for each $n \in \mathbb{N}_0$, then $d(gx_n, gx_{n+1}) \neq 0$ for each $n \in \mathbb{N}_0$. On using (3.1), (3.2) and assumption (vii), we obtain

$$d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \leq \alpha d(gx_{n-1}, gx_n) \quad \forall n \in \mathbb{N}.$$

By induction, we have

$$d(gx_n, gx_{n+1}) \leq \alpha d(gx_{n-1}, gx_n) \leq \alpha^2 d(gx_{n-2}, gx_{n-1}) \leq \cdots \leq \alpha^n d(gx_0, gx_1) \quad \forall n \in \mathbb{N}.$$

so that

$$d(gx_n, gx_{n+1}) \leq \alpha^n d(gx_0, gx_1) \quad \forall n \in \mathbb{N}. \quad (3.4)$$

For $n < m$, using (3.4), we obtain

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\ &\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1})d(gx_0, gx_1) \\ &= \frac{\alpha^n - \alpha^m}{1 - \alpha} d(gx_0, gx_1) \\ &\leq \frac{\alpha^n}{1 - \alpha} d(gx_0, gx_1) \\ &\rightarrow 0 \text{ as } m, n \rightarrow \infty. \end{aligned}$$

Therefore $\{gx_n\}$ is a Cauchy sequence. As X is complete, there exists $z \in X$ such that

$$\lim_{n \rightarrow \infty} g(x_n) = z. \quad (3.5)$$

On using (3.1) and (3.5), we obtain

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_{n+1}) = z. \quad (3.6)$$

On using continuity of g in (3.5) and (3.6), we get

$$\lim_{n \rightarrow \infty} g(gx_n) = g(\lim_{n \rightarrow \infty} gx_n) = g(z) \quad (3.7)$$

$$\lim_{n \rightarrow \infty} g(fx_n) = g(\lim_{n \rightarrow \infty} fx_n) = g(z). \quad (3.8)$$

As $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_n) = z$ (due to (3.5) and (3.6)), on using compatibility of f and g , we obtain

$$\lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0. \tag{3.9}$$

Now, we show that z is a coincidence point of f and g . To accomplish this, we use assumption (v). Suppose that f is continuous. On using (3.5) and continuity of f , we obtain

$$\lim_{n \rightarrow \infty} f(gx_n) = f(\lim_{n \rightarrow \infty} gx_n) = f(z). \tag{3.10}$$

On using (3.8), (3.9), (3.10) and continuity of d , we obtain

$$\begin{aligned} d(gz, fz) &= d(\lim_{n \rightarrow \infty} gfx_n, \lim_{n \rightarrow \infty} fgx_n) \\ &= \lim_{n \rightarrow \infty} d(gfx_n, fgx_n) \\ &= 0 \end{aligned}$$

so that

$$g(z) = f(z).$$

Thus z is a coincidence point of f and g and hence we are through. Alternately, suppose that (X, d, \preceq) has g -TCC property. As $g(x_n) \downarrow z$ (due to (3.2) and (3.5)), \exists a subsequence $\{y_{n_k}\}$ of $\{gx_n\}$ such that

$$g(y_{n_k}) \prec\succ g(z) \forall k \in \mathbb{N}_0.$$

Now $\{gx_n\} \subset g(X)$ and $\{y_{n_k}\} \subset \{gx_n\}, \exists \{x_{n_k}\} \subset X$ such that $y_{n_k} = g(x_{n_k})$. Hence, we have

$$g(gx_{n_k}) \prec\succ g(z) \forall k \in \mathbb{N}_0. \tag{3.11}$$

Since $g(x_{n_k}) \rightarrow z$, so equations (3.5)-(3.10) hold for also $\{x_{n_k}\}$ instead of $\{x_n\}$.

On using (3.11) and assumption (vii), we obtain

$$d(fgx_{n_k}, fz) \leq \alpha d(ggx_{n_k}, gz) \forall k \in \mathbb{N}_0. \tag{3.12}$$

On using triangular inequality, (3.7), (3.8), (3.9) and (3.12), we get

$$\begin{aligned} d(gz, fz) &\leq d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k}) + d(fgx_{n_k}, fz) \\ &\leq d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k}) + \alpha d(ggx_{n_k}, gz) \\ &\rightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

so that

$$g(z) = f(z).$$

Thus z is a coincidence point of f and g and hence this concludes the proof.

As commutativity \Rightarrow weak commutativity \Rightarrow compatibility for a pair of mappings, therefore the following consequence of Theorem 3.3 trivially holds.

Corollary 3.4. *Theorem 3.3 remains true if we replace condition (iii) by one of the following conditions besides retaining the rest of the hypotheses:*

- (iii)' (f, g) is commuting pair,
- (iii)'' (f, g) is weakly commuting pair.

Our next result is analogous to Theorem 3.3 whenever X is not necessarily complete. Instead, we require at least, one of the subspaces $f(X)$ and $g(X)$ to be complete.

Theorem 3.5. *Let (X, d, \preceq) be an ordered metric space and f and g two self-mappings on X . Suppose that the following conditions hold:*

- (i) $f(X) \subseteq g(X)$,
- (ii) f is g -comparable,
- (iii) either (fX, d) or (gX, d) is complete,
- (iv) either f is g -continuous or f and g are continuous or (gX, d, \preceq) has TCC property,
- (v) there exists $x_0 \in X$ such that $g(x_0) \prec \succ f(x_0)$,
- (vi) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(gx, gy) \quad \forall x, y \in X \text{ with } g(x) \prec \succ g(y).$$

Then f and g have a coincidence point.

Proof. In view of assumption (v) if $g(x_0) = f(x_0)$, then x_0 is a coincidence point of f and g and hence proof is completed. Otherwise, if $g(x_0) \neq f(x_0)$, then we have $g(x_0) \prec f(x_0)$. As g is a self-mapping on X , by using Lemma 2.8, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \rightarrow X$ is one-one. Hence assumption (i) implies that $f(X) \subseteq g(E)$ so that we can choose $e_1 \in E$ such that $g(e_1) = f(x_0)$. Again, we can choose $e_2 \in E$ such that $g(e_2) = f(e_1)$. Now proceeding in the same way, we can inductively construct a sequence $\{e_n\} \subset E$ such that

$$g(e_{n+1}) = f(e_n) \quad \forall n \in \mathbb{N}. \quad (3.13)$$

Following the proof of Theorem 3.3, we can show that the sequence $\{ge_n\}$ is termwise monotone and also Cauchy. In lieu of assumption (iii), we assume that $g(X) (= g(E))$ is complete, then there exists $e \in E$ such that

$$\lim_{n \rightarrow \infty} g(e_n) = g(e).$$

In case of $f(X)$ is complete, in view of (3.13), we conclude that $\{fe_n\}$ is a Cauchy sequence in $f(X)$. Therefore, on using completeness of $f(X)$ and $f(X) \subseteq g(E)$, there exists $e \in E$ such that

$$\lim_{n \rightarrow \infty} f(e_n) = g(e).$$

Hence, in both the cases, on using (3.13), we have

$$\lim_{n \rightarrow \infty} g(e_n) = \lim_{n \rightarrow \infty} f(e_n) = g(e) \quad \text{where } e \in E. \quad (3.14)$$

Now, we show that e is a coincidence point of f and g . In view of assumption (iv), firstly, suppose that f is g -continuous, then using (3.14), we get

$$\lim_{n \rightarrow \infty} f(e_n) = f(e). \quad (3.15)$$

On using (3.14), (3.15) and uniqueness of limit, we get

$$g(e) = f(e),$$

i.e., e is a coincidence point of f and g and hence we are through.

Secondly, suppose that f and g both are continuous. Define $T : g(E) \rightarrow g(E)$ by

$$T(ga) = f(a) \quad \forall ga \in g(E) \text{ where } a \in E. \quad (3.16)$$

As $g : E \rightarrow X$ is one-one and $f(X) \subseteq g(E)$, T is well defined. Again since f and g are continuous, it follows that T is continuous. On using (3.14), (3.16) and continuity of T , we get

$$f(e) = T(ge) = T(\lim_{n \rightarrow \infty} ge_n) = \lim_{n \rightarrow \infty} T(ge_n) = \lim_{n \rightarrow \infty} f(e_n) = g(e).$$

Thus e is a coincidence point of f and g and hence we are done.

Finally, suppose that (gX, d, \preceq) has TCC property. As $\{ge_n\}$ is termwise monotone and $g(e_n) \xrightarrow{d} g(e)$, we have $g(e_n) \uparrow g(e)$. Hence by TCC property of $g(X)$, \exists a subsequence $\{ge_{n_k}\}$ of $\{ge_n\}$ such that

$$g(e_{n_k}) \prec \succ g(e) \forall k \in \mathbb{N}. \tag{3.17}$$

On using (3.17) and assumption (vi), we obtain

$$d(fe_{n_k}, fe) \leq \alpha d(ge_{n_k}, ge) \forall k \in \mathbb{N}. \tag{3.18}$$

On using (3.14), (3.18) and continuity of d , we get

$$\begin{aligned} d(ge, fe) &= d(\lim_{k \rightarrow \infty} fe_{n_k}, fe) \\ &= \lim_{k \rightarrow \infty} d(fe_{n_k}, fe) \\ &\leq \alpha \lim_{k \rightarrow \infty} d(ge_{n_k}, ge) \\ &= 0 \end{aligned}$$

so that

$$g(e) = f(e).$$

Hence e is a coincidence point of f and g . This completes the proof.

Now, we present a consequence of Theorem 3.5.

Corollary 3.6. *Theorem 3.5 remains true if we replace (iii) by one of the following conditions (iii)' and (iii)'' besides retaining the rest of the hypotheses:*

- (iii)' (X, d) is complete and one of f and g is onto,
- (iii)'' (X, d) is complete and one of $f(X)$ and $g(X)$ is closed.

Proof. If (iii)' holds, we get either $f(X) = X$ or $g(X) = X$ so that either $f(X)$ or $g(X)$ is complete, which implies that (iii) holds and hence Theorem 3.5 is applicable. If (iii)'' holds, then using the fact that closed subset of a complete metric space is complete, one of $f(X)$ and $g(X)$ is complete which implies that (iii) holds and hence Theorem 3.5 is applicable.

On combining Theorems 3.3 and 3.5, we obtain the following result:

Theorem 3.7. *Let (X, d, \preceq) be an ordered metric space and f and g two self-mappings on X . Suppose that the following conditions hold:*

- (a) $f(X) \subseteq g(X)$,
- (b) f is g -comparable,
- (c) there exists $x_0 \in X$ such that $g(x_0) \prec \succ f(x_0)$,
- (d) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(gx, gy) \forall x, y \in X \text{ with } g(x) \prec \succ g(y),$$
- (e) (e1) (X, d) is complete,
 (e2) (f, g) is compatible pair,
 (e3) g is continuous,

- (e4) either f is continuous or (X, d, \preceq) has g -TCC property, or alternately
 (e') (e'1) either (fX, d) or (gX, d) is complete,
 (e'2) either f is g -continuous or f and g are continuous or (gX, d, \preceq) has TCC property.

Then f and g have a coincidence point.

Notice that Theorem 3.7 improves Theorem 2.7 and hence in Theorem 2.7 the g -monotonicity can be alternately replaced by g -comparability, which is relatively weaker.

4. UNIQUENESS RESULTS

Recall that in order to obtain the uniqueness of fixed point in ordered metric spaces, several authors used the following alternative conditions.

(I) (X, \preceq) is totally ordered.

The preceding condition is more natural, as under this condition, results of Ran and Reurings [21] and Nieto and Rodríguez-López [22, 23] follow directly from Banach contraction principle [37] but this condition is very restrictive.

Ran and Reurings [21] used the following condition to obtain the uniqueness of fixed point in their result (see Theorem 1.1).

(II) every pair of elements of X has a lower bound and an upper bound.

Later, Nieto and Rodríguez-López [22, 23] (see Theorem 1.2) modified condition (II) by assuming relatively weaker condition as follows:

(III) every pair of elements of X has a lower bound or an upper bound,

which is equivalent (proved in [22]) to the following:

for each pair $x, y \in X$, $\exists z \in X$ such that $x \prec\succ z$ and $y \prec\succ z$.

On the lines of Jleli *et al.* [32], (X, \preceq) is called directed if it satisfies condition (III) (see Definition 2.4 [32]).

Turinici [35, 36] used the following condition (see Theorems 1.4 and 1.5):

(IV) $C(x, y, \prec\succ)$ is nonempty, for each $x, y \in X$.

Clearly, (I) \Rightarrow (II) \Rightarrow (III) \Rightarrow (IV) *i.e.* among these four conditions (IV) is the weakest one.

Inspired by Jleli *et al.* [32], we extend condition (III) to a pair of mappings which runs as follows:

Definition 4.1. Let (X, \preceq) be an ordered set and (f, g) a pair of self-mappings on X . We say that (X, \preceq) is (f, g) -directed if for each pair $x, y \in X$, $\exists z \in X$ such that $f(x) \prec\succ g(z)$ and $f(y) \prec\succ g(z)$.

In cases $f = I$ and $f = g = I$ (where I denotes identity mapping on X), (X, \preceq) is called g -directed and directed respectively.

Inspired by Turinici [35], we limit condition (IV) to an arbitrary subset rather than the whole ordered set which runs as follows:

Definition 4.2. Let (X, \preceq) be an ordered set, $E \subseteq X$ and $a, b \in E$. A subset $\{e_1, e_2, \dots, e_k\}$ of E is called $\prec\succ$ -chain between a and b in E if

- (i) $k \geq 2$,
- (ii) $e_1 = a$ and $e_k = b$,
- (iii) $e_1 \prec\succ e_2 \prec\succ \dots \prec\succ e_{k-1} \prec\succ e_k$.

Let $C(a, b, \prec, \succ, E)$ denotes the class of all \prec, \succ -chains between a and b in E . In particular for $E = X$, we write $C(a, b, \prec, \succ)$ instead of $C(a, b, \prec, \succ, X)$.

Now, we state and prove some results for uniqueness of coincidence point, point of coincidence and common fixed point corresponding to earlier results. For the sake of naturally, firstly we prove results corresponding to Theorem 3.5 and thereafter for Theorem 3.3.

Theorem 4.3. *In addition to the hypotheses of Theorem 3.5, suppose that the following condition holds:*

(u_0) $C(fx, fy, \prec, \succ, gX)$ is nonempty, for each $x, y \in X$.

Then f and g have a unique point of coincidence.

Proof. In view of Theorem 3.5, the set of the coincidence points (and hence points of coincidence) of f and g is nonempty. Let \bar{x} and \bar{y} be two points of coincidence of f and g , then $\exists x, y \in X$ such that

$$\bar{x} = g(x) = f(x) \text{ and } \bar{y} = g(y) = f(y). \tag{4.1}$$

Now, we show that

$$\bar{x} = \bar{y}. \tag{4.2}$$

As $f(x), f(y) \in f(X) \subseteq g(X)$, by (u_0) , there exists a \prec, \succ -chain $\{gz_1, gz_2, \dots, gz_k\}$ between $f(x)$ and $f(y)$ in $g(X)$, where $z_1, z_2, \dots, z_k \in X$. Owing to (4.1), without loss of generality, we can choose $z_1 = x$ and $z_k = y$. Thus we have

$$g(z_1) \prec, \succ g(z_2) \prec, \succ \dots \prec, \succ g(z_{k-1}) \prec, \succ g(z_k). \tag{4.3}$$

Define the constant sequences $z_n^1 = x$ and $z_n^k = y$, then using (4.1), we have $g(z_{n+1}^1) = f(z_n^1) = \bar{x}$ and $g(z_{n+1}^k) = f(z_n^k) = \bar{y} \forall n \in \mathbb{N}_0$. Put $z_0^2 = z_2, z_0^3 = z_3, \dots, z_0^{k-1} = z_{k-1}$. Since $f(X) \subseteq g(X)$, on the lines similar to that of Theorem 3.3, we can define sequences $\{z_n^2\}, \{z_n^3\}, \dots, \{z_n^{k-1}\}$ in X such that $g(z_{n+1}^2) = f(z_n^2), g(z_{n+1}^3) = f(z_n^3), \dots, g(z_{n+1}^{k-1}) = f(z_n^{k-1}) \forall n \in \mathbb{N}_0$. Hence, we have

$$g(z_{n+1}^i) = f(z_n^i) \forall n \in \mathbb{N}_0 \text{ and for each } i (1 \leq i \leq k). \tag{4.4}$$

Now, we claim that

$$g(z_n^1) \prec, \succ g(z_n^2) \prec, \succ \dots \prec, \succ g(z_n^{k-1}) \prec, \succ g(z_n^k) \forall n \in \mathbb{N}_0. \tag{4.5}$$

We prove this fact by the method of mathematical induction. Owing to (4.3), (4.5) holds for $n = 0$. Suppose that (4.5) holds for $n = r > 0$, i.e.,

$$g(z_r^1) \prec, \succ g(z_r^2) \prec, \succ g(z_r^3) \prec, \succ \dots \prec, \succ g(z_r^{k-1}) \prec, \succ g(z_r^k).$$

On using g -comparability of f , we obtain

$$f(z_r^1) \prec, \succ f(z_r^2) \prec, \succ f(z_r^3) \prec, \succ \dots \prec, \succ f(z_r^{k-1}) \prec, \succ f(z_r^k),$$

which on using (4.4), gives rise

$$g(z_{r+1}^1) \prec, \succ g(z_{r+1}^2) \prec, \succ g(z_{r+1}^3) \prec, \succ \dots \prec, \succ g(z_{r+1}^{k-1}) \prec, \succ g(z_{r+1}^k).$$

It follows that (4.5) holds for $n = r + 1$. Thus, by induction, (4.5) holds for all $n \in \mathbb{N}_0$. Now for all $n \in \mathbb{N}_0$, define

$$\begin{cases} t_n^1 := d(gz_n^1, gz_n^2) \\ t_n^2 := d(gz_n^2, gz_n^3) \\ \vdots \\ t_n^{k-2} := d(gz_n^{k-2}, gz_n^{k-1}) \\ t_n^{k-1} := d(gz_n^{k-1}, gz_n^k). \end{cases}$$

On using (4.4), (4.5) and assumption (vi), it can be easily shown that

$$t_{n+1}^i \leq \alpha t_n^i \quad \forall n \in \mathbb{N}_0 \text{ and for each } i (1 \leq i \leq k-1).$$

By induction, for each $i (1 \leq i \leq k-1)$, we get

$$t_{n+1}^i \leq \alpha t_n^i \leq \alpha^2 t_{n-1}^i \leq \dots \leq \alpha^{n+1} t_0^i$$

so that

$$t_{n+1}^i \leq \alpha^{n+1} t_0^i.$$

Taking the limit as $n \rightarrow \infty$ on both the sides of above inequality, we obtain

$$\lim_{n \rightarrow \infty} t_n^i = 0 \quad \text{for each } i (1 \leq i \leq k-1). \quad (4.6)$$

On using triangular inequality and (4.6), we obtain

$$d(\bar{x}, \bar{y}) \leq t_n^1 + t_n^2 + \dots + t_n^{k-1} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

\implies

$$\bar{x} = \bar{y}.$$

Hence (4.2) is proved.

Theorem 4.4. *In addition to the hypotheses of Theorem 4.3, suppose that the following condition holds:*

(u_1) *one of f and g is one-one.*

Then f and g have a unique coincidence point.

Proof. Let x and y be two coincidence points of f and g , then in view of Theorem 4.3, we have

$$g(x) = f(x) = f(y) = g(y).$$

As f or g is one-one, we have

$$x = y.$$

Theorem 4.5. *In addition to the hypotheses of Theorem 4.3, suppose that the following condition holds:*

(u_2) *(f, g) is weakly compatible pair.*

Then f and g have a unique common fixed point.

Proof. Let x be a coincidence point of f and g . Write $g(x) = f(x) = \bar{x}$, then in view of Lemma 2.9 and (u_2) , \bar{x} is also a coincidence point of f and g . It follows from Theorem 4.3 with $y = \bar{x}$ that $g(x) = g(\bar{x})$, i.e., $\bar{x} = g(\bar{x})$, which yields that

$$\bar{x} = g(\bar{x}) = f(\bar{x}).$$

Hence, \bar{x} is a common fixed point of f and g . To prove uniqueness, assume that x^* is another common fixed point of f and g . Then again from Theorem 4.3, we have

$$x^* = g(x^*) = g(\bar{x}) = \bar{x}.$$

This completes proof.

Theorem 4.6. *In addition to the hypotheses of Theorem 3.3, suppose that the condition (u_0) (of Theorem 4.3) holds, then f and g have a unique common fixed point. Proof.* We know that in a metric space, every compatible pair is weakly compatible so that (u_2) trivially holds. Hence, proceeding on the lines of the proof of Theorems 4.3 and 4.5 our result follows.

Corollary 4.7. *Theorem 4.3 (resp. Theorem 4.6) remains true if we replace the condition (u_0) by one of the following conditions (besides retaining rest of the hypotheses):*

(u_0^1) : (fX, \preceq) is totally ordered,

(u_0^2) : (X, \preceq) is (f, g) -directed.

Proof. Suppose that (u_0^1) holds, then for each pair $x, y \in X$, we have

$$f(x) \prec \succ f(y),$$

which implies that $\{fx, fy\}$ is a $\prec \succ$ -chain between $f(x)$ and $f(y)$ in $g(X)$. It follows that $C(fx, fy, \prec \succ, gX)$ is nonempty, for each $x, y \in X$, i.e., (u_0) holds and hence Theorem 4.3 (resp. Theorem 4.6) is applicable.

Next, assume that (u_0^2) holds, then for each pair $x, y \in X$, $\exists z \in X$ such that

$$f(x) \prec \succ g(z) \prec \succ f(y),$$

which implies that $\{fx, gz, fy\}$ is a $\prec \succ$ -chain between $f(x)$ and $f(y)$ in $g(X)$. It follows that $C(fx, fy, \prec \succ, gX)$ is nonempty, for each $x, y \in X$, i.e., (u_0) holds and hence Theorem 4.3 (resp. Theorem 4.6) is applicable.

5. A RELATED FIXED POINT RESULT

On setting $g = I$, the identity mapping on X , in Theorem 3.7 (together with Theorems 4.3-4.6), we get the following fixed point result.

Corollary 5.1. *Let (X, d, \preceq) be an ordered metric space and f a self-mapping on X . Suppose that the following conditions hold:*

- (a) either (fX, d) or (X, d) is complete,
- (b) f is comparable,
- (c) either f is continuous or (X, d, \preceq) has TCC property,
- (d) there exists $x_0 \in X$ such that $x_0 \prec \succ f(x_0)$,
- (e) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } x \prec \succ y.$$

Then f has a fixed point. Moreover, if we add the following one

- (f) $C(fx, fy, \prec \succ)$ is nonempty, for each $x, y \in X$.

Then we obtain uniqueness of fixed point.

Corollary 5.1 sharpens Theorems 1.1-1.6 in the following considerations:

- In Theorem 1.1 and Theorem 1.2, the monotonicity of f can be replaced by comparability of f , which is relatively weaker.

- In Theorems 1.1-1.6, the completeness of X can alternately be replaced by completeness of $f(X)$.
- In Theorem 1.6, the property on (X, d, \preceq) embodied in assumption (c) is very restrictive and can alternately be replaced by TCC property on (X, d, \preceq) , which is relatively weaker.
- All the hypotheses of Theorems 1.1-1.5 without assumption (f) guarantee the existence of fixed point and the presence of assumption (f) ensures the uniqueness of fixed point. Also assumption (f) of Corollary 5.1 is relatively weaker than each of assumptions (f) (of Theorems 1.1-1.6).

Finally, we furnish an example which demonstrates that the notion of comparable mapping is an improvement over monotonicity of the map.

Example 5.2. Let $X = [-\frac{1}{3}, \frac{1}{3}]$. Then (X, d, \preceq) is an ordered complete metric space under the usual metric and the natural partial order. Define $f : X \rightarrow X$ by $f(x) = x^2$, then f is comparable but not monotone. Also, for $x, y \in X$ with $x \preceq y$, we have

$$d(fx, fy) = |x^2 - y^2| = |x + y||x - y| \leq \frac{2}{3}d(x, y).$$

i.e. f satisfies the contractivity condition (e) of Corollary 5.1. Thus, all the conditions mentioned in Corollary 5.1 are satisfied. Notice that f has a unique fixed point in X (namely: $x = 0$).

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