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ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS WITH RESPECT TO THE BREGMAN DISTANCE IN THE INTERMEDIATE SENSE

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Abstract. The purpose of this paper is to introduce a new class of nonlinear mappings which is an extension of asymptotically quasi-nonexpansive mappings with respect to the Bregman distance in the intermediate sense. A strong convergence theorem of the shrinking projection method with the modified Mann iteration is established to find fixed points of the mappings in reflexive Banach spaces.

Key Words and Phrases: Bregman distance, Bregman projection, asymptotically quasi-nonexpansive in the intermediate sense, fixed point, Legendre function, totally convex function.
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1. Introduction

Fixed point theory is an important branch of nonlinear analysis and has been applied in numerous studies of nonlinear phenomena. Many problems in nonlinear functional analysis are related to finding fixed points of nonlinear mappings of nonexpansive types. We want to construct an iterative process to approximate fixed points of mappings of nonexpansive types. It is an important question that whether iterative schemes for mappings of nonexpansive types can be generated, modified, preferably in a simple way, so that strong convergence is guaranteed. Many authors have considered problems of iterative algorithms for mappings of nonexpansive types which converge to some fixed points. The purpose of this paper is to prove a strong convergence theorem for asymptotically quasi-nonexpansive with respect to the Bregman distance in the intermediate sense.

Let C be a nonempty subset of a real Banach space and $T : C \to C$ a mapping. A point $p \in C$ is called a *fixed point* of T if $Tp = p$. Throughout this paper, we denote by $F(T)$ the set of fixed points of T. Recall that T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in C.$$

More generally, T is said to be *asymptotically nonexpansive* (*cf.* [15]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \quad \text{for all } x, y \in C \text{ and } n \ge 1.$$

The mapping T is said to be *asymptotically nonexpansive in the intermediate sense* (*cf.* [8]) if

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{1.1}$$

If $F(T)$ is nonempty and (1.1) holds for all $x \in C$ and $y \in F(T)$, then T is said to be *asymptotically quasi-nonexpansive in the intermediate sense.* It is worth mentioning that the class of asymptotically nonexpansive mappings in the intermediate sense contains properly the class of asymptotically nonexpansive mappings, since asymptotically nonexpansive mappings in the intermediate sense are not Lipschitz continuous in general.

Takahashi, Takeuchi and Kubota [37] have introduced a new hybrid iterative scheme called a *shrinking projection method* for nonexpansive mappings in Hilbert spaces. It is an advantage of projection methods that the strong convergence of iterative sequences is guaranteed without any compact assumptions. Schu [34] has introduced a *modified Mann iteration* to approximate fixed points of asymptotically nonexpansive mappings in Banach spaces. Motivated by [34, 37], Inchan [20] has introduced a new hybrid iterative scheme by using the shrinking projection method with the modified Mann iteration for asymptotically nonexpansive mappings in Hilbert spaces. Moreover, many authors have studied iterative methods for approximating fixed points of asymptotically quasi-nonexpansive mappings in the intermediate sense in Banach spaces (see [17, 18, 26]).

In 1967, Bregman [7] has discovered an elegant and effective technique for the using of the so-called Bregman distance function in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze not only iterative algorithms for solving feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria and for computing fixed points of nonlinear mappings. Many authors have studied iterative methods for approximating fixed points of mappings of nonexpansive types with respect to the Bregman distance (see [22, 30, 31, 35]). In particular, Reich and Sabach [32] have established a strong convergence theorem with respect to the Bregman distance using the concept of the shrinking projection method. We can apply it to the solution of convex feasibility and equilibrium problems, and to finding zeroes of two different classes of nonlinear mappings.

However, it has not been studied yet for the cases of asymptotically quasi-nonexpansive with respect to the Bregman distance in the intermediate sense. From this background, we introduce a new class of nonlinear mappings which is an extension of asymptotically quasi-nonexpansive with respect to the Bregman distance in the intermediate sense. Motivated by the results above, we design a new hybrid iterative scheme for finding fixed points of a mapping in the new class by using the shrinking projection method with the modified Mann iteration in reflexive Banach spaces. We prove a new strong convergence theorem for the mapping. This theorem is an extension of results of [37, 38]. This iterative method is expected to be applied to many other problems in nonlinear functional analysis relating to the Bregman distance.

In Section 2, we present several preliminary definitions and results. In Section 3, we recall the notion of the Mosco convergence and two kinds of projection with respect to the Bregman distance. In Section 4, we introduce the new class of mappings with respect to the Bregman distance and prove closedness and convexness of the set of fixed points of mappings in the new class. In Section 5, we prove a strong convergence theorem for finding a fixed point of the mapping by using the shrinking projection method with the modified Mann iteration.

2. Preliminaries

Throughout this paper, $\mathbf{N}$ denotes the set of positive integers and $\mathbf{R}$ the set of real numbers. Moreover, E always denotes a real reflexive Banach space with the norm $\|\cdot\|$, E^* the dual space of E and $\langle \cdot, \cdot \rangle$ the pairing between E and E^*. The strong convergence of a sequence $\{x_n\}$ to x is denoted by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

Let $f : E \to (-\infty, +\infty]$ be a function. The *effective domain* of f is defined by

$$\mathrm{dom} f := \{x \in E : f(x) < +\infty\}.$$

The function f is said to be *proper* if $\mathrm{dom} f$ is nonempty. We denote by $\mathrm{int}\,\mathrm{dom} f$ the *interior* of the effective domain of f. We denote by $\mathrm{ran} f$ the *range* of f. The function f is said to be *convex* on E if it satisfies

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in E$ and $\lambda \in [0,1]$. The function f is said to be *lower semicontinuous* on E if

$$\liminf_{y \to x} f(y) \ge f(x)$$

for all $x \in E$. If f is proper, convex and lower semicontinuous on E, then f is locally Lipschitz continuous on $\mathrm{int}\,\mathrm{dom} f$ (see [6], Theorem 1.7, p. 66). The *Fenchel conjugate* function of f is the convex function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(\xi) := \sup\{\langle \xi, x\rangle - f(x) : x \in E\}.$$

The function f is said to be *cofinite* if $\mathrm{dom} f^* = E^*$. If f is proper, convex and lower semicontinuous on E, then f^* is also proper, convex and lower semicontinuous on E^* (see [2], Proposition 1.3, p. 6).

Let $f : E \to (-\infty, +\infty]$ be a proper and convex function. The *subdifferential* of f is a mapping $\partial f : E \to 2^{E^*}$ defined by

$$\partial f(x) := \{x^* \in E^* : f(y) \ge f(x) + \langle x^*, y - x\rangle,\ \forall y \in E\}$$

for all $x \in E$. We know that $x^* \in \partial f(x)$ if and only if $f(x) + f^*(x^*) = \langle x^*, x\rangle$ for $x \in E$ (see [3]).

Let $x \in \mathrm{int}\,\mathrm{dom} f$. For each $y \in E$, the *right-hand directional derivative*

$$f^\circ(x, y) := \lim_{t \downarrow 0^+} \frac{f(x+ty) - f(x)}{t}.$$

exists and defines a sublinear functional on E (see [25], Lemma 1.2, p. 2). For any $y \in E$, we define the *directional derivative of* f *at* x *in the direction* y by

$$f'(x,y) := \lim_{t \downarrow 0} \frac{f(x+ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be *Gâteaux differentiable at* x if the limit (2.1) exists for each $y \in E$. We know that f is Gâteaux differentiable at x if and only if $y \to f^\circ(x,y)$ is linear in y (see [25], p. 3). In this case, the *gradient* of f at x is the function $\nabla f(x) : E \to (-\infty, +\infty)$ defined by $\langle \nabla f(x), y \rangle = f^\circ(x,y)$ for every $y \in E$. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable at each $x \in \operatorname{int}\operatorname{dom} f$. The function f is said to be *Fréchet differentiable at* x if the limit (2.1) is attained uniformly in $\|y\| = 1$.

A function $f : E \to (-\infty, +\infty]$ is said to be *admissible* if it is proper, convex and lower semicontinuous on E and Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. Under these conditions we know that ∂f is single-valued and $\partial f = \nabla f$ (see [10], Proposition 1.1.10, p. 13). Throughout this paper, we assume that $f : E \to (-\infty, +\infty]$ is always an admissible function.

A function $f : E \to (-\infty, +\infty]$ is called *Legendre* (*cf.* [3]) if it satisfies additionally the following two conditions:

(L_1) $\operatorname{int}\operatorname{dom} f \neq \emptyset$, f is Gâteaux differentiable and $\operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f$;

(L_2) $\operatorname{int}\operatorname{dom} f^* \neq \emptyset$ f^* is Gâteaux differentiable and $\operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*$.

Let f be a Legendre function on E. Since E is reflexive, we always have $\nabla f = (\nabla f^*)^{-1}$. When this fact is combined with conditions (L_1) and (L_2), we obtain the following equalities:

$$\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^* \quad \text{and} \quad \operatorname{ran}\nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f.$$

Moreover, conditions (L_1) and (L_2) imply that f and f^* are strictly convex on the interior of their respective domains (see [3] Theorem 5.4). We know that f is Legendre if and only if f^* is Legendre (see [3], Corollary 5.5, p. 634).

Example 2.1 ([4], Example 1.1, p. 2)**.** The following functions are Legendre on $E = \mathbf{R}^n$: Let $x = (x_j)_{1 \le j \le n} \in \mathbf{R}^n$.

(i) Halved energy: $f(x) = \|x\|^2/2 = \frac{1}{2}\sum_{j=1}^n x_j^2$.

(ii) Boltzmann-Shannon entropy: $f(x) = \begin{cases} \sum_{j=1}^n (x_j \ln(x_j) - x_j), & x \geq 0; \\ +\infty, & \text{otherwise.} \end{cases}$

(iii) Burg entropy: $f(x) = \begin{cases} -\sum_{j=1}^n \ln(x_j), & x > 0; \\ +\infty, & \text{otherwise.} \end{cases}$

Note that $\operatorname{int}\operatorname{dom} f = \mathbf{R}^n$ in (i), whereas $\operatorname{int}\operatorname{dom} f = \{x \in \mathbf{R}^n : x_j > 0,\ j = 1, \dots, n\}$ in (ii) and (iii).

Given a function $f : E \to (-\infty, +\infty]$, a bifunction $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$ defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the *Bregman distance with respect to* f (*cf.* [7, 14]). In general, the Bregman distance is not a metric since it is not symmetric and does not satisfy the triangle inequality.

Example 2.2 ([4], Example 1.5, p. 3)**.** The Bregman distances corresponding to the Legendre functions of Example 2.1 are as follows: Let $x, y \in \mathbf{R}^n$.

(i) Euclidean distance: $D_f(y,x) = \|y-x\|^2/2$.
(ii) Kullback-Leibler divergence: $D_f(y,x) = \sum_{j=1}^n (y_j \ln(y_j/x_j) - y_j + x_j)$.
(iii) Itakura-Saito divergence: $D_f(y,x) = \sum_{j=1}^n (\ln(x_j/y_j) + y_j/x_j - 1)$.

Given a function $f : E \to (-\infty, +\infty]$, a *modulus of total convexity of* f *at* $x \in \operatorname{int}\operatorname{dom} f$ is a function $v_f(x,\cdot) : [0,+\infty) \to [0,+\infty]$ defined by

$$v_f(x,t) := \inf\{D_f(y,x) : y \in \operatorname{dom} f,\ \|y-x\| = t\}.$$

The function f is said to be *totally convex at* $x \in \operatorname{int}\operatorname{dom} f$ (*cf.* [9, 13]) if $v_f(x,t) > 0$ for all $t > 0$. The function f is said to be *totally convex* when it is totally convex at every point of $\operatorname{int}\operatorname{dom} f$. The function f is said to be *totally convex on bounded sets* if, for any nonempty bounded set B of E, $\inf\{v_f(x,t) : x \in B \cap \operatorname{int}\operatorname{dom} f\} > 0$ for all $t > 0$. We remark in passing that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [11], Proposition 4.2, p. 16).

Proposition 2.3 ([29], Lemma 3.1, p. 31)**.** *Let* $f : E \to \mathbf{R}$ *be a totally convex function and* $x \in \operatorname{int}\operatorname{dom} f$. *If the sequence* $\{D_f(x_n, x)\}_{n\in\mathbf{N}}$ *is bounded, then the sequence* $\{x_n\}_{n\in\mathbf{N}}$ *is also bounded.*

A function $f : E \to (-\infty, +\infty]$ is said to be *sequentially consistent* (*cf.* [12]) if

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \text{implies} \quad \lim_{n\to\infty} \|y_n - x_n\| = 0$$

for any two sequences $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ in $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom} f$, respectively, such that the first one is bounded.

Proposition 2.4 ([10], Lemma 2.1.2, p. 67)**.** *A function* $f : E \to (-\infty, +\infty]$ *is totally convex on bounded subsets of* E *if and only if it is sequentially consistent.*

3. The Bregman projections

The concept of Bregman projection was first used by Bregman [7], while the terminology is due to Censor and Lent [14]. It has been shown that this generalized projection is a good replacement for the metric projection in optimization methods and in algorithms for solving convex feasibility problems.

Let C be a nonempty, closed and convex subset of $\operatorname{dom} f$. The *left Bregman projection* proj_C^f *with respect to* f (*cf.* [7, 14]) from $\operatorname{int}\operatorname{dom} f$ onto C is defined by

$$\operatorname{proj}_C^f(x) := \arg\min_{y\in C} D_f(y,x) = \{z \in C : D_f(z,x) \le D_f(y,x),\ \forall y \in C\}$$

for all $x \in \operatorname{int}\operatorname{dom} f$.

Let $\{C_n\}_{n\in\mathbf{N}}$ be a sequence of nonempty subsets of E. We denote by s-$\mathrm{Li}_n C_n$ the set of limit points of $\{C_n\}$, that is, $x \in$ s-$\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $x_n \in C_n$ for each $n \in \mathbf{N}$ and $x_n \to x$ as $n \to \infty$. Similarly, we denote by w-$\mathrm{Ls}_n C_n$ the set of weak cluster points of $\{C_n\}$; $y \in$ w-$\mathrm{Ls}_n C_n$ if and only if there exists $\{y_{n_i}\} \subset E$ such that $y_{n_i} \in C_{n_i}$ for each $i \in \mathbf{N}$ and $y_{n_i} \rightharpoonup y$ as $i \to \infty$. Using these definitions, we define the *Mosco convergence* (*cf.* [24]) of $\{C_n\}$. If C_0 satisfies

$$\text{s-}\underset{n}{\mathrm{Li}}C_n = C_0 = \text{w-}\underset{n}{\mathrm{Ls}}C_n,$$

then we say that $\{C_n\}$ is Mosco convergent to C_0 and we write

$$C_0 = \text{M-}\lim_n C_n.$$

If $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ is Mosco convergent to $\bigcap_{n=1}^{\infty} C_n$.

Proposition 3.1 ([33], Theorem 4.5, p. 12). *Let $f : E \to (-\infty, +\infty]$ be a totally convex function which is Fréchet differentiable on* $\mathrm{int}\,\mathrm{dom} f$. *Let $\{C_n\}_{n\in\mathbf{N}}$ be a sequence of nonempty, closed and convex subsets of* $\mathrm{int}\,\mathrm{dom} f$ *and C_0 a nonempty, closed and convex subset of* $\mathrm{int}\,\mathrm{dom} f$. *Then the following statements are equivalent:*

(i) $C_0 = M\text{-}\lim_n C_n$;

(ii) $\lim_{n\to\infty} \mathrm{proj}^f_{C_n}(x) = \mathrm{proj}^f_{C_0}(x)$ *for all* $x \in \mathrm{int}\,\mathrm{dom} f$.

Let C be a nonempty, closed and convex subset of $\mathrm{int}\,\mathrm{dom} f$. The *right Bregman projection* $\overrightarrow{\mathrm{proj}}^f_C$ *with respect to* f (*cf.* [5, 23]) from $\mathrm{int}\,\mathrm{dom} f$ onto C is defined by

$$\overrightarrow{\mathrm{proj}}^f_C(x) := \arg\min_{y\in C} D_f(x,y) = \{z \in C : D_f(x,z) \le D_f(x,y),\ \forall y \in C\}$$

for all $x \in \mathrm{int}\,\mathrm{dom} f$. Since D_f is not convex in the second variable, it is not clear a priori that the right Bregman projection is well defined. However, this difficulty has already been overcome by Bauschke, Wang, Ye and Yuan [5] and Martín-Márquez, Reich and Sabach [23]. They have proved

$$\overrightarrow{\mathrm{proj}}^f_C = \nabla f^* \circ \mathrm{proj}^{f^*}_{\nabla f(C)} \circ \nabla f \tag{3.1}$$

and established several other properties of $\overrightarrow{\mathrm{proj}}^f_C$. The right Bregman projection has the following variational characterization.

Proposition 3.2 ([23], Proposition 4.11, p. 5459). *Let $f : E \to \mathbf{R}$ be a function such that f^* is admissble and totally convex. Let C be a nonempty subset of* $\mathrm{int}\,\mathrm{dom} f$ *such that $\nabla f(C)$ is closed and convex. Let $x \in \mathrm{int}\,\mathrm{dom} f$. If $\hat{x} \in C$, then the following conditions are equivalent to each other:*

(i) *The vector $\hat{x}$ is the right Bregman projection of x onto C with respect to f;*

(ii) *The vector $\hat{x}$ is a unique solution z of a variational inequality*

$$\langle \nabla f(z) - \nabla f(y), x - z\rangle \ge 0 \quad \textit{for all } y \in C;$$

(iii) *The vector $\hat{x}$ is a unique solution z of an inequality*

$$D_f(z,y) + D_f(x,z) \le D_f(x,y) \quad \textit{for all } y \in C.$$

Remark 3.3. Let $f(x) = \|x\|^2/2$ for all $x \in E$. If E is a Hilbert space H, then the right Bregman projection $\overrightarrow{\mathrm{proj}}_C^f$ is reduced to the metric projection P_C from H onto C, which is defined by

$$P_C x := \arg\min_{y \in C} \|x - y\| \quad \text{for all } x \in H.$$

4. Right Bregman asymptotically quasi-nonexpansive in the intermediate sense

In this section, we introduce and consider a new class of nonlinear mappings with respect to the Bregman distance based on asymptotically quasi-nonexpansive mappings in the intermediate sense.

Let C be a nonempty subset of $\mathrm{dom} f$ and $T : C \to \mathrm{int}\,\mathrm{dom} f$ a mapping. The mapping T is said to be *right Bregman quasi-nonexpansive* with respect to $F(T)$ (*cf.* [23]) if $F(T) \neq \emptyset$ and

$$D_f(Tx, p) \leq D_f(x, p) \quad \text{for all } p \in F(T),\ x \in C.$$

Let K be a nonempty subset of E and $R : E \to K$ a mapping. A mapping R is called a *retraction* if $Rx = x$ for each $x \in K$. A mapping R is said to be *sunny* (*cf.* [16, 27]) if $R\big(Rx + t(x - Rx)\big) = Rx$ for each $x \in E$ and all $t \geq 0$. We know that the unique sunny right Bregman quasi-nonexpansive retraction from $\mathrm{int}\,\mathrm{dom} f$ onto C is given by the right Bregman projection defined by (3.1).

Proposition 4.1 ([23], Corollary 4.6, p. 5458)**.** *Let $f : E \to \mathbf{R}$ be a Legendre, cofinite and totally convex function such that f^* is totally convex. Let C be a nonempty subset of* $\mathrm{int}\,\mathrm{dom} f$*. If $\nabla f(C)$ is closed and convex, then the right Bregman projection* (3.1) *is a unique sunny right Bregman quasi-nonexpansive retraction from* $\mathrm{int}\,\mathrm{dom} f$ *onto C.*

Let C be a nonempty subset of $\mathrm{int}\,\mathrm{dom} f$ and $T : C \to C$ a mapping. We introduce a new class of mappings: the mapping T is said to be *right Bregman asymptotically quasi-nonexpansive in the intermediate sense* (in brief, R-BAQNE) if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p \in F(T),\ x \in C} \big(D_f(T^n x, p) - D_f(x, p)\big) \leq 0. \tag{4.1}$$

Put

$$\eta_n = \max\left\{0, \sup_{p \in F(T),\ x \in C} \big(D_f(T^n x, p) - D_f(x, p)\big)\right\}.$$

The inequality (4.1) implies $\lim_{n\to\infty} \eta_n = 0$. Hence (4.1) is reduced to the following:

$$D_f(T^n x, p) \leq D_f(x, p) + \eta_n \tag{4.2}$$

for all $p \in F(T)$ and $x \in C$, where $\{\eta_n\}$ is a sequence such that $\eta_n \to 0$ as $n \to \infty$. R-BAQNE mappings are not Lipschitz continuous in general.

Example 4.2. Assume that $E = \mathbf{R}$, $C = [1/2, 3/2]$ and $T : C \to C$ defined by

$$Tx = \begin{cases} 1, & x \in [\frac{1}{2}, 1], \\ 1 - \sqrt{\frac{x-1}{2}}, & x \in (1, \frac{3}{2}]. \end{cases} \tag{4.3}$$

Note that $F(T)=\{1\}$ and $T^n x = 1$ for all $x \in C$ and $n \geq 2$. If $f : \mathbf{R} \to (-\infty, +\infty]$ is a Legendre function, then T is R-BAQNE since

$$\limsup_{n\to\infty} \sup_{x\in C} \big(D_f(T^n x, 1) - D_f(x,1)\big) \leq \limsup_{n\to\infty} \sup_{x\in C} D_f(T^n x, 1) = 0.$$

However, T is not Lipschitzian with respect to the Bregman distances in Example 2.2. Indeed, suppose that there exists $L > 0$ such that $D_f(Ty, Tx) \leq L D_f(y, x)$ for all $x, y \in C$. By Taylor's theorem, there exists $t \in (0,1)$ such that

$$D_f(y,x) = f(y) - f(x) - \langle \nabla f(x), y - x\rangle = \frac{1}{2}\nabla^2 f(x + t(y-x))(y-x)^2. \tag{4.4}$$

(i) Let $f(x) = \|x\|^2/2$ on $\mathrm{dom} f = \mathbf{R}$ and $D_f(y,x) = \|y-x\|^2/2$ for all $x, y \in \mathbf{R}$. Put $x = 1$ and $y = 1 + 1/2(L+1)$. Since $Ty = 1 - 1/2\sqrt{L+1}$, we have

$$\frac{1}{8(L+1)} = \frac{1}{2}\left\|\frac{-1}{2\sqrt{L+1}}\right\|^2 = \frac{1}{2}\|Ty - Tx\|^2 \leq \frac{L}{2}\|y-x\|^2 = \frac{L}{8(L+1)^2}.$$

This implies $L + 1 \leq L$, which is a contradiction.

(ii) Let $f(x) = x\ln(x) - x$ on $\mathrm{dom} f = [0, +\infty)$ and $D_f(y,x) = y\ln(y/x) - y + x$ for all $x \in (0, +\infty)$ and $y \in [0, +\infty)$. Note that $\nabla^2 f(x) = 1/x$. Put $x = 1$. By (4.4), we have

$$D_f(y,1) = \frac{(y-1)^2}{2(1+t(y-1))} \leq \frac{(y-1)^2}{2} \quad \text{for } y \geq 1$$

and

$$D_f(y,1) = \frac{(y-1)^2}{2(1+t(y-1))} \geq \frac{(y-1)^2}{2} \quad \text{for } 0 < y \leq 1.$$

If $y = 1 + 1/2(L+1)$, we have

$$\begin{aligned}\frac{1}{8(L+1)} = \frac{1}{2}\left(\frac{-1}{2\sqrt{L+1}}\right)^2 &\leq D_f(Ty, 1) \\ &\leq L D_f(y,1) \leq \frac{L}{2}\left(\frac{1}{2(L+1)}\right)^2 = \frac{L}{8(L+1)^2}.\end{aligned}$$

This implies $L + 1 \leq L$, which is a contradiction.

(iii) Let $f(x) = -\ln(x)$ on $\mathrm{dom} f = (0, +\infty)$ and $D_f(y,x) = \ln(x/y) + y/x - 1$ for all $x, y \in (0, +\infty)$. Note that $\nabla^2 f(x) = 1/x^2$. Put $y = 1$. By (4.4), we have

$$D_f(1,x) = \frac{(1-x)^2}{2(x+t(1-x))^2} \leq \frac{(1-x)^2}{2} \quad \text{for } x \geq 1$$

and

$$D_f(1,x) = \frac{(1-x)^2}{2(x+t(1-x))^2} \geq \frac{(1-x)^2}{2} \quad \text{for } 0 < x \leq 1.$$

If $x = 1 + 1/2(L+1)$, we have

$$\begin{aligned}\frac{1}{8(L+1)} = \frac{1}{2}\left(\frac{1}{2\sqrt{L+1}}\right)^2 &\leq D_f(1, Tx) \\ &\leq L D_f(1,x) \leq \frac{L}{2}\left(\frac{-1}{2(L+1)}\right)^2 = \frac{L}{8(L+1)^2}.\end{aligned}$$

This implies $L+1 \le L$, which is a contradiction.

Remark 4.3. Let $f(x) = \|x\|^2/2$ for all $x \in E$. If E is a Hilbert space, then R-BAQNE mappings are reduced to asymptotically quasi-nonexpansive mappings in the intermediate sense.

Theorem 4.4. *Let $f : E \to (-\infty, +\infty]$ be a Legendre and cofinite function which is totally convex on bounded subsets of E. Let $T : \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ be a closed and R-BAQNE mapping. Then $\nabla f\big(F(T)\big)$ is closed and convex subset of E^*.*

Proof. First we show that $\nabla f\big(F(T)\big)$ is convex. Let $p_1, p_2 \in F(T)$ and $p = \nabla f^*\big(t\nabla f(p_1) + (1-t)\nabla f(p_2)\big)$, where $t \in (0,1)$. It suffices to prove that $p \in F(T)$. By the definition of the Bregman distance, we have

$$\begin{aligned}
D_f(T^n p, p) &= f(T^n p) - f(p) - \langle \nabla f(p), T^n p - p\rangle \\
&= t\{f(T^n p) - f(p_1) - \langle \nabla f(p_1), T^n p - p_1\rangle\} \\
&\quad + (1-t)\{f(T^n p) - f(p_2) - \langle \nabla f(p_2), T^n p - p_2\rangle\} \\
&\quad - f(p) + tf(p_1) + (1-t)f(p_2) \\
&\quad + \langle \nabla f(p), p\rangle - t\langle \nabla f(p_1), p_1\rangle - (1-t)\langle \nabla f(p_2), p_2\rangle \\
&= tD_f(T^n p, p_1) + (1-t)D_f(T^n p, p_2) - f(p) + \langle \nabla f(p), p\rangle \\
&\quad + t(f(p_1) - \langle \nabla f(p_1), p_1\rangle) + (1-t)(f(p_2) - \langle \nabla f(p_2), p_2\rangle). \qquad (4.5)
\end{aligned}$$

It is known that $f(x) + f^*\big(\nabla f(x)\big) = \langle \nabla f(x), x\rangle$ for all $x \in E$. By (4.5), we have

$$\begin{aligned}
D_f(T^n p, p) &= tD_f(T^n p, p_1) + (1-t)D_f(T^n p, p_2) \\
&\quad + f^*\big(\nabla f(p)\big) - tf^*\big(\nabla f(p_1)\big) - (1-t)f^*\big(\nabla f(p_2)\big). \qquad (4.6)
\end{aligned}$$

By (4.2), we have

$$\begin{aligned}
&tD_f(T^n p, p_1) + (1-t)D_f(T^n p, p_2) \\
&\le tD_f(p, p_1) + (1-t)D_f(p, p_2) + \eta_n \\
&= f(p) - t\langle \nabla f(p_1), p\rangle - (1-t)\langle \nabla f(p_2), p\rangle \\
&\quad - t(f(p_1) - \langle \nabla f(p_1), p_1\rangle) - (1-t)(f(p_2) - \langle \nabla f(p_2), p_2\rangle) + \eta_n \\
&= f(p) - \langle \nabla f(p), p\rangle + tf^*\big(\nabla f(p_1)\big) + (1-t)f^*\big(\nabla f(p_2)\big) + \eta_n \\
&= -f^*\big(\nabla f(p)\big) + tf^*\big(\nabla f(p_1)\big) + (1-t)f^*\big(\nabla f(p_2)\big) + \eta_n. \qquad (4.7)
\end{aligned}$$

By (4.6) and (4.7), we have

$$\begin{aligned}
D_f(T^n p, p) &\le -f^*\big(\nabla f(p)\big) + tf^*\big(\nabla f(p_1)\big) + (1-t)f^*\big(\nabla f(p_2)\big) + \eta_n \\
&\quad + f^*\big(\nabla f(p)\big) - tf^*\big(\nabla f(p_1)\big) - (1-t)f^*\big(\nabla f(p_2)\big) \\
&= \eta_n.
\end{aligned}$$

This implies

$$\lim_{n\to\infty} D_f(T^n p, p) = \lim_{n\to\infty} \eta_n = 0.$$

By Proposition 2.4, we have $\|T^n p - p\| \to 0$ as $n \to \infty$. Since T is closed, we have

$$p = \lim_{n\to\infty} T^{n+1} p = T \lim_{n\to\infty} T^n p = Tp$$

and hence $p \in F(T)$.

Next we show that $\nabla f\big(F(T)\big)$ is closed. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in $F(T)$ such that $\nabla f(x_n) \to x^* \in E^*$ as $n \to \infty$. Since f is Legendre and cofinite, we have $\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^* = E^*$. Hence there exists $x \in E$ such that $x^* = \nabla f(x)$. It suffices to prove that $x \in F(T)$. Since $\{x_n\} \subset F(T)$ and T is R-BAQNE, we have

$$D_f(T^n x, x_n) \leq D_f(x, x_n) + \eta_n = f(x) + f^*\big(\nabla f(x_n)\big) - \langle \nabla f(x_n), x\rangle + \eta_n.$$

By assumption, f^* is continuous and $\nabla f(x_n) \to \nabla f(x)$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} D_f(T^n x, x_n) \leq f(x) + \lim_{n\to\infty}\big(f^*\big(\nabla f(x_n)\big) - \langle \nabla f(x_n), x\rangle + \eta_n\big) = 0.$$

Moreover,

$$\begin{aligned} D_f(T^n x, x) &= D_f(T^n x, x_n) + f(x_n) + \langle \nabla f(x_n), T^n x - x_n\rangle \\ &\quad - f(x) - \langle \nabla f(x), T^n x - x\rangle \\ &= D_f(T^n x, x_n) - f^*\big(\nabla f(x_n)\big) + f^*\big(\nabla f(x)\big) + \langle \nabla f(x_n) - \nabla f(x), T^n x\rangle. \end{aligned}$$

Hence $D_f(T^n x, x) \to 0$ as $n \to \infty$. By Proposition 2.4, we have $\|T^n x - x\| \to 0$ as $n \to \infty$. Since T is closed, we have $x = Tx$ and hence $x \in F(T)$. □

Theorem 4.5. *Let $f : E \to \mathbf{R}$ be a Legendre and cofinite function which is totally convex on bounded subsets of E such that f^* is totally convex. If $T : \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ is a closed and R-BAQNE mapping, then there exists a unique sunny right Bregman quasi-nonexpansive retraction from $\operatorname{int}\operatorname{dom} f$ onto $F(T)$, which is the right Bregman projection onto $F(T)$.*

Proof. By the assumption of f and T, it follows from Theorem 4.4 that $\nabla f\big(F(T)\big)$ is closed and convex in E^*. Proposition 4.1 ensures that there exists the right Bregman projection $\overrightarrow{\operatorname{proj}}^f_{F(T)}$ which is a unique sunny right Bregman quasi-nonexpansive retraction from $\operatorname{int}\operatorname{dom} f$ onto $F(T)$. □

When a mapping T is right Bregman quasi-nonexpansive, Theorems 4.4 and 4.5 can be reduced to the following existing results.

Corollary 4.6 ([23], Proposition 3.3, p. 5454)**.** *Let $f : E \to (-\infty, +\infty]$ be a Legendre and cofinite function and $T : \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ a right Bregman quasi-nonexpansive mapping. Then $\nabla f\big(F(T)\big)$ is a closed and convex subset of E^*.*

Corollary 4.7 ([23], Corollary 4.7, p. 5458)**.** *Let $f : E \to \mathbf{R}$ be a Legendre, cofinite and totally convex function such that f^* is totally convex. If $T : \operatorname{int}\operatorname{dom} f \to \operatorname{int}\operatorname{dom} f$ is a right Bregman quasi-nonexpansive mapping, then there exists a unique sunny right Bregman quasi-nonexpansive retraction from $\operatorname{int}\operatorname{dom} f$ onto $F(T)$, which is the right Bregman projection onto $F(T)$.*

5. A strong convergence theorem of R-BAQNE mappings

In this section, we prove a strong convergence theorem for finding a fixed point of an R-BAQNE mapping by the shirinking projection method with the modified Mann iteration.

Let C be a nonempty subset of E and $T : C \to C$ a mapping. The mapping T is said to be *asymptotically regular* on C if, for any bounded subset K of C,

$$\lim_{n\to\infty} \sup_{x\in K} \left\| T^{n+1}x - T^n x \right\| = 0.$$

Theorem 5.1. *Let $f : E \to \mathbf{R}$ be a Legendre and cofinite function which is totally convex on bounded subsets on E such that f^* is admissible, totally convex and Fréchet differentiable on* $\operatorname{int}\operatorname{dom} f^*$*. Let C be a nonempty subset of* $\operatorname{int}\operatorname{dom} f$ *such that $\nabla f(C)$ is closed and convex. Let $T : C \to C$ be a closed and R-BAQNE maping. Suppose that T is asymptotically regular on C and $F(T)$ is bounded. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in C generated by*

$$\begin{cases} x_0 \in \operatorname{int}\operatorname{dom} f, \ \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \overrightarrow{\operatorname{proj}}^f_{C_1} x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n) T^n x_n, \\ C_{n+1} = \{ z \in C_n : D_f(y_n, z) \le D_f(x_n, z) + \eta_n \}, \\ x_{n+1} = \overrightarrow{\operatorname{proj}}^f_{C_{n+1}} x_0, \ n \in \mathbf{N}, \end{cases} \tag{5.1}$$

where $\overrightarrow{\operatorname{proj}}^f_{C_n}$ is the right Bregman projection from $\operatorname{int}\operatorname{dom} f$ *onto C_n,*

$$\eta_n = \max\left\{ 0, \sup_{p\in F(T),\ x\in C} \big(D_f(T^n x, p) - D_f(x, p) \big) \right\}$$

and $0 \le \alpha_n \le a < 1$ for all $n \in \mathbf{N}$. Then $\{x_n\}_{n\in\mathbf{N}}$ converges strongly to $\overrightarrow{\operatorname{proj}}^f_{F(T)} x_0$, where $\operatorname{proj}^f_{F(T)}$ is the right Bregman projection from $\operatorname{int}\operatorname{dom} f$ *onto $F(T)$.*

Proof. We divide the proof into six steps.

Step 1. We show that $\nabla f(C_n)$ is closed and convex for all $n \in \mathbf{N}$. It is obvious that $\nabla f(C_1) = \nabla f(C)$ is closed and convex. Suppose that $\nabla f(C_k)$ is closed and convex for some $k \in \mathbf{N}$. We see that, for $z \in C_k$, $D_f(y_k, z) \le D_f(x_k, z) + \eta_k$ is equivalent to

$$\langle \nabla f(z), x_k - y_k \rangle \le f(x_k) - f(y_k) + \eta_k. \tag{5.2}$$

First we prove that $\nabla f(C_{k+1})$ is closed. Let $\{z_i\}_{i\in\mathbf{N}} \subset C_{k+1}$ with $\nabla f(z_i) \to z^*$ as $i \to \infty$. Since f is Legendre and cofinite, we have $\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^* = E^*$. Hence there exists $z \in E$ such that $z^* = \nabla f(z)$. It is sufficient to prove that $z \in C_{k+1}$. By (5.2), we have

$$\langle \nabla f(z), x_k - y_k \rangle = \lim_{i\to\infty} \langle \nabla f(z_i), x_k - y_k \rangle \le f(x_k) - f(y_k) + \eta_k$$

and hence $z \in C_{k+1}$. Thus $\nabla f(C_n)$ is closed for all $n \in \mathbf{N}$. Next we prove that $\nabla f(C_{k+1})$ is convex. Let $x, y \in C_{k+1}$ and $t \in (0,1)$. Define $z = \nabla f^*(t\nabla f(x) + (1-$

$t)\nabla f(y))$. We prove that $z \in C_{k+1}$. By (5.2), we have

$$\begin{aligned}\langle \nabla f(z), x_k - y_k\rangle &= \langle t\nabla f(x) + (1-t)\nabla f(y), x_k - y_k\rangle \\ &= t\langle \nabla f(x), x_k - y_k\rangle + (1-t)\langle \nabla f(y), x_k - y_k\rangle \\ &\le f(x_k) - f(y_k) + \eta_k\end{aligned}$$

and hence $z \in C_{k+1}$. Thus $\nabla f(C_n)$ is convex for all $n \in \mathbf{N}$. Therefore $\nabla f(C_n)$ is closed and convex. By Proposition 4.1, there exists a unique sunny right Bregman quasi-nonexpansive retraction from $\operatorname{int}\operatorname{dom} f$ onto C_n which is $\overrightarrow{\mathrm{proj}}^f_{C_n}$. Hence $\{x_n\}$ is well defined.

Step 2. We show that $F(T) \subset C_n$ for all $n \in \mathbf{N}$. It is obvious that $F(T) \subset C_1 = C$. Suppose that $F(T) \subset C_k$ for some $k \in \mathbf{N}$. Since f is convex, the function $D_f(\cdot, x)$ is also convex for all $x \in \operatorname{int}\operatorname{dom} f$. For any $p \in F(T)$, we have

$$\begin{aligned}D_f(y_k, p) &= D_f(\alpha_k x_k + (1-\alpha_k)T^k x_k, p) \\ &\le \alpha_k D_f(x_k, p) + (1-\alpha_n) D_f(T^k x_k, p) \\ &\le \alpha_k D_f(x_k, p) + (1-\alpha_n)(D_f(x_k, p) + \eta_k) \\ &\le D_f(x_k, p) + \eta_k. \qquad (5.3)\end{aligned}$$

and hence $p \in C_{k+1}$. Therefore $F(T) \subset C_n$ for all $n \in \mathbf{N}$. Since $F(T)$ is nonempty, C_n is a nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f$.

Step 3. We show that $\{x_n\}_{n\in\mathbf{N}}$ is bounded. Let $p \in F(T)$. By Proposition 3.2 (iii), we have

$$D_f(x_n, p) = D_f(\overrightarrow{\mathrm{proj}}^f_{C_n}(x_0), p) \le D_f(x_0, p) - D_f(x_0, \overrightarrow{\mathrm{proj}}^f_{C_n}(x_0)) \le D_f(x_0, p).$$

This implies that $\{D_f(x_n, p)\}_{n\in\mathbf{N}}$ is bounded. By Proposition 2.3, the sequence $\{x_n\}_{n\in\mathbf{N}}$ is bounded.

Step 4. Put $C_0^* := \bigcap_{n=1}^\infty \nabla f(C_n)$. We show that $\{x_n\}$ converges to $\nabla f^*(\mathrm{proj}^{f^*}_{C_0^*}\nabla f(x_0))$ as $n \to \infty$. Since $\{\nabla f(C_n)\}$ is a nonincreasing sequence with respect to inclusion of nonempty, closed and convex subsets of E^*, we have

$$\emptyset \ne \nabla f(F(T)) \subset M\text{-}\lim_n \nabla f(C_n) = \bigcap_{n=1}^\infty \nabla f(C_n) = C_0^*.$$

By Proposition 3.1, $\{\mathrm{proj}^{f^*}_{\nabla f(C_n)}\nabla f(x_0)\}$ converges strongly to $x^* = \mathrm{proj}^{f^*}_{C_0^*}\nabla f(x_0)$ as $n \to \infty$. Since E^* has a Fréchet differentiable norm, $(\nabla f)^{-1} = \nabla f^*$ is continuous. We have

$$x_n = \overrightarrow{\mathrm{proj}}^f_{C_n}(x_0) = \nabla f^* \circ \mathrm{proj}^{f^*}_{\nabla f(C_n)} \circ \nabla f(x_0) \to \nabla f^*(x^*)$$

as $n \to \infty$. To complete the proof, it is sufficient to show that $\nabla f^*(x^*) = \overrightarrow{\mathrm{proj}}^f_{F(T)}(x_0)$.

Step 5. We show that $\nabla f^*(x^*) \in F(T)$. Since $x_n = \overrightarrow{\mathrm{proj}}^f_{C_n}(x_0)$ and $x_{n+1} = \overrightarrow{\mathrm{proj}}^f_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, we have $D_f(x_0, x_n) \le D_f(x_0, x_{n+1})$. This implies that

$\{D_f(x_0,x_n)\}_{n\in\mathbf{N}}$ is nondecreasing and the limit of $D_f(x_0,x_n)$ as $n\to\infty$ exists. By Proposition 3.2 (iii), we have

$$\begin{aligned}D_f(x_n,x_{n+1}) &= D_f(\overrightarrow{\mathrm{proj}}^f_{C_n}(x_0),x_{n+1})\\ &\le D_f(x_0,x_{n+1})-D_f(x_0,\overrightarrow{\mathrm{proj}}^f_{C_n}(x_0))\\ &= D_f(x_0,x_{n+1})-D_f(x_0,x_n)\end{aligned}$$

for all $n\in\mathbf{N}$. This implies

$$\lim_{n\to\infty} D_f(x_n,x_{n+1})=0. \tag{5.4}$$

By Proposition 2.4, we have

$$\lim_{n\to\infty}\|x_n-x_{n+1}\|=0. \tag{5.5}$$

Since $x_{n+1}\in C_{n+1}$, by (5.4), we have

$$\lim_{n\to\infty} D_f(y_n,x_{n+1})\le \lim_{n\to\infty}(D_f(x_n,x_{n+1})+\eta_n)=0.$$

By Proposition 2.4, we have

$$\lim_{n\to\infty}\|y_n-x_{n+1}\|=0. \tag{5.6}$$

By the definition of y_n, we have

$$\|T^nx_n-x_{n+1}\|\le\frac{1}{1-\alpha_n}\|x_{n+1}-y_n\|+\frac{\alpha_n}{1-\alpha_n}\|x_{n+1}-x_n\|.$$

By (5.5), (5.6) and the definition of α_n, we have $\|T^nx_n-x_{n+1}\|\to 0$ as $n\to\infty$. This implies

$$\lim_{n\to\infty}T^nx_n=\lim_{n\to\infty}x_{n+1}=\nabla f^*(x^*).$$

Since

$$\left\|T^{n+1}x_n-\nabla f^*(x^*)\right\|\le\left\|T^{n+1}x_n-T^nx_n\right\|+\|T^nx_n-\nabla f^*(x^*)\|$$

and T is asymptotically regular on C, we have

$$\lim_{n\to\infty}\left\|T^{n+1}x_n-\nabla f^*(x^*)\right\|=0.$$

Hence $TT^nx_n-\nabla f^*(x^*)\to 0$ as $n\to\infty$. By the closedness of T, we have $T(\nabla f^*(x^*))=\nabla f^*(x^*)$. Therefore $\nabla f^*(x^*)\in F(T)$.

Step 6. We show that $\overrightarrow{\mathrm{proj}}^f_{F(T)}(x_0)=\nabla f^*(x^*)$. Put $z_0^*=\overrightarrow{\mathrm{proj}}^f_{F(T)}(x_0)$. Since $z_0^*\in F(T)\subset C_n$ and $x_n=\overrightarrow{\mathrm{proj}}^f_{C_n}(x_0)$, we have $D_f(x_0,x_n)\le D_f(x_0,z_0^*)$ for all $n\in\mathbf{N}$. We have

$$\begin{aligned}D_f(x_0,\nabla f^*(x^*)) &= f(x_0)-f(\nabla f^*(x^*))-\langle x^*,x_0-\nabla f^*(x^*)\rangle\\ &= \lim_{n\to\infty}(f(x_0)-f(x_n)-\langle\nabla f(x_n),x_0-x_n\rangle)\\ &= \lim_{n\to\infty}D_f(x_0,x_n)\le D_f(x_0,z_0^*).\end{aligned}$$

Therefore $z_0^*=\nabla f^*(x^*)$ and hence $\{x_n\}$ converges strongly to z_0^*. $\square$

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