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A NEGATIVE ANSWER TO POPESCU'S CONJECTURE

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Abstract. We give a negative answer to Popescu's conjecture on some fixed point theorem in complete metric spaces, which was raised in 2013.
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1. Introduction

Recently, Popescu [3] proved very interesting fixed point theorems. The following is a corollary of one of the theorems.

Theorem 1.1 (Popescu [3])**.** *Let (X, d) be a complete metric space and let T be a mapping on X. Suppose that there exist real numbers r and s such that $0 \le r < 1$, $r < s$ and*

$$d(y, Tx) \le sd(y, x) \Longrightarrow d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Then T has a fixed point.

We note that Theorem 1.1 is a generalization of the Banach contraction principle [1]. In [3], Popescu raised the following conjecture.

Conjecture 1.2 (Popescu [3])**.** Theorem 1.1 is still valid in the case where $r = s$ instead of $r < s$.

If this conjecture were true, then Theorem 1.1 would be a partial generalization of Theorem 2 in [4] because $d(y, Tx) \le rd(y, x)$ implies $(1 + r)^{-1}d(x, Tx) \le d(x, y)$. So it is very meaningful to study whether this conjecture is true. See also [2, 5].

In this paper, we give a negative answer to the above conjecture. That is, we can tell that the condition $r < s$ in Theorem 1.1 is best possible.

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2. Results

In this section, we give our main results. We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Lemma 2.1. *Define a set X by*

$$X = \{u_1, u_2, \cdots, v_1, v_2, \cdots, a, b\}. \tag{2.1}$$

Let f and g be functions from $\mathbb{N}$ into $(0, \infty)$ satisfying $f(m) \geq f(n) \geq g(n)$ for any $m, n \in \mathbb{N}$ with $m < n$. Let c be a positive real number with $c \geq f(1)$. Define a function d from $X \times X$ into $[0, \infty)$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ f(m) + g(n) & \text{if } x = u_m, y = u_n, m < n \\ f(n) & \text{if } x = u_n, y = a \\ c + g(m) + g(n) & \text{if } x = u_m, y = v_n \\ c + g(n) & \text{if } x = u_n, y = b \\ c & \text{if } x = a, y = b \\ d(u_m, u_n) & \text{if } x = v_m, y = v_n, m < n \\ d(u_n, b) & \text{if } x = v_n, y = a \\ d(u_n, a) & \text{if } x = v_n, y = b \\ d(y, x) & \text{otherwise.} \end{cases}$$

Then (X, d) is a complete metric space.

Proof. It is obvious that $d(x,y) \geq 0$, $d(x,y) = 0 \Leftrightarrow x = y$ and $d(x,y) = d(y,x)$ for any $x, y \in X$. We put u_0, u_∞, v_0 and v_∞ by $u_\infty = v_0 = a$ and $u_0 = v_\infty = b$. We also put $f(0) = c$, $f(\infty) = 0$ and $g(\infty) = 0$. In order to simplify the proof, we suppose $g(0)$ is a real number with $0 \leq g(0) \leq f(0)$. Then we note

$$d(u_i, u_j) = f(\min\{i,j\}) + g(\max\{i,j\})$$

for $i, j \in \mathbb{N} \cup \{0, \infty\}$ with $i \neq j$ and

$$d(u_n, v_t) = c + g(n) + g(t)$$

for $n, t \in \mathbb{N} \cup \{\infty\}$. So we have

$$d(u_i, u_j) \geq f(\min\{i,j\}) \geq f(j) \geq g(j) \tag{2.2}$$

and

$$d(u_i, u_j) \leq f(\min\{i,j\}) + f(\max\{i,j\}) \leq 2c \tag{2.3}$$

for $i, j \in \mathbb{N} \cup \{0, \infty\}$ with $i \neq j$. We shall prove the triangle inequality of d. Suppose $i, j, k \in \mathbb{N} \cup \{0, \infty\}$, $i < j$ and $m, n, s, t \in \mathbb{N} \cup \{\infty\}$. We also suppose that u_i, u_j, u_k, u_m, u_n, v_s and v_t are all different. We have from (2.2)

$$d(u_i, u_j) = f(i) + g(j) \leq d(u_i, u_k) + d(u_j, u_k).$$

We have from (2.3)

$$d(u_m, u_n) \leq c + c \leq d(u_m, v_t) + d(u_n, v_t).$$

Also we have from (2.2)

$$d(u_n, v_t) \leq f(n) + c + g(t) \leq d(u_n, u_m) + d(v_t, u_m)$$

and

$$d(u_n, v_t) \leq c + g(n) + f(t) \leq d(u_n, v_s) + d(v_t, v_s).$$

We have shown the triangle inequality of d. Therefore (X, d) is a metric space. Let us show that X is complete. We consider the following two cases:

- $\lim_{n \in \mathbb{N}, n \to \infty} f(n) = 0$
- $\lim_{n \in \mathbb{N}, n \to \infty} f(n) > 0$

In the first case, we let $\{x_n\}$ be a Cauchy sequence in X such that x_n are all different. Since

$$\inf\{d(u_n, v_t) : n, t \in \mathbb{N}\} = c > 0,$$

there exist $\nu \in \mathbb{N}$ and a function h from $\{n \in \mathbb{N} : n \geq \nu\}$ into $\mathbb{N}$ such that $\lim_n h(n) = \infty$ and either $x_n = u_{h(n)}$ for all $n \geq \nu$ or $x_n = v_{h(n)}$ for all $n \geq \nu$. Then $\{x_n\}$ converges to a or b. In the second case, since

$$\inf\{d(x, y) : y \in X, y \neq x\} \geq \lim_{n \in \mathbb{N}, n \to \infty} f(n) > 0$$

for any $x \in X$, there exists no Cauchy sequence $\{x_n\}$ in X such that x_n are all different. Therefore X is complete. □

Now we give an example which shows that Popescu's conjecture is false.

Example 2.2. Let $r \in (0, 1)$ and put $\varepsilon = 1 - r$ and $\kappa = 1/r$. Define a set X by (2.1). Define sequences $\{p_n\}_{n \in \mathbb{N} \cup \{0\}}$ and $\{q_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}$ by

$$p_n = r\varepsilon \frac{n + \kappa + 1}{n + \kappa + 2} \quad \text{and} \quad q_n = \frac{\varepsilon}{n + \kappa + 1}.$$

Define a metric d on X by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ r^m(1 + p_m) + r^n(1 + q_n) & \text{if } x = u_m, y = u_n, m < n \\ r^n(1 + p_n) & \text{if } x = u_n, y = a \\ 1 + p_0 + r^m(1 + q_m) + r^n(1 + q_n) & \text{if } x = u_m, y = v_n \\ 1 + p_0 + r^n(1 + q_n) & \text{if } x = u_n, y = b \\ 1 + p_0 & \text{if } x = a, y = b \\ d(u_m, u_n) & \text{if } x = v_m, y = v_n, m < n \\ d(u_n, b) & \text{if } x = v_n, y = a \\ d(u_n, a) & \text{if } x = v_n, y = b \\ d(y, x) & \text{otherwise.} \end{cases}$$

Define a mapping T on X by

$$Tx = \begin{cases} u_{n+1} & \text{if } x = u_n \\ v_{n+1} & \text{if } x = v_n \\ v_1 & \text{if } x = a \\ u_1 & \text{if } x = b. \end{cases}$$

Then (X, d) is a complete metric space, T has no fixed points and T satisfies

$$d(y, Tx) \le r d(y, x) \implies d(Tx, Ty) \le r d(x, y) \tag{2.4}$$

for any $x, y \in X$.

Proof. In order to simplify the proof, we define q_0. Put $v_0 = a$ and $u_0 = b$. We first note that $\{p_n\}$ is increasing, $p_n > 0$ and $q_n > 0$. Define functions f and g from $\mathbb{N} \cup \{0\}$ into $(0, \infty)$ by

$$f(n) = r^n(1 + p_n) \quad \text{and} \quad g(n) = r^n(1 + q_n).$$

We note that f is decreasing because

$$f(n+1) < r^{n+1}(1 + \varepsilon) = r^n(2r - r^2) < r^n < f(n)$$

for $n \in \mathbb{N} \cup \{0\}$. We also note $g(n) < f(n)$ because

$$f(n) - g(n) = \frac{r^n \varepsilon \big(r(n+1)^2 + n\big)}{(n + \kappa + 1)(n + \kappa + 2)} > 0$$

for $n \in \mathbb{N} \cup \{0\}$. By Lemma 2.1, (X, d) is a complete metric space. It is obvious that T has no fixed points. Let us prove (2.4). In the case where $n \in \mathbb{N} \cup \{0\}$, $x = u_n$ and $y = u_{n+1}$, since

$$d(u_n, u_{n+1}) = f(n) + g(n+1) = r^n(1 + r + r\varepsilon),$$

we have

$$d(Tx, Ty) = d(u_{n+1}, u_{n+2}) = r d(u_n, u_{n+1}) = r d(x, y).$$

In the case where $m, n \in \mathbb{N} \cup \{0\}$, $m + 1 < n$, $x = u_m$ and $y = u_n$, we have

$$\begin{aligned} d(y, Tx) &= r^{m+1}(1 + p_{m+1}) + r^n(1 + q_n) \\ &> r^{m+1}(1 + p_m) + r^{n+1}(1 + q_n) = r d(y, x). \end{aligned}$$

We also have

$$\begin{aligned} d(x, Ty) &= r^m(1 + p_m) + r^{n+1}(1 + q_{n+1}) \\ &> r^m(1 + p_m) + r^{n+1} \\ &= r^{m+1}(1 + p_m) + r^m\varepsilon(1 + p_m) + r^{n+1} \\ &> r^{m+1}(1 + p_m) + r^m\varepsilon + r^{n+1} \\ &> r^{m+1}(1 + p_m) + r^{n+1}(1 + \varepsilon) \\ &> r^{m+1}(1 + p_m) + r^{n+1}(1 + q_n) \\ &= r d(x, y). \end{aligned}$$

In the case where $m, n \in \mathbb{N}$, $x = u_m$ and $y = v_n$, we have

$$\begin{aligned} d(y, Tx) - rd(y, x) &= \varepsilon(1 + p_0) + \varepsilon r^n (1 + q_n) + r^{m+1}(q_{m+1} - q_m) \\ &> \varepsilon + r^{m+1}(q_{m+1} - q_m) = \varepsilon - \frac{r^{m+1}\varepsilon}{(m + \kappa + 1)(m + \kappa + 2)} > 0. \end{aligned}$$

Since $\{p_n\}$ is increasing, we note $rf(n) < f(n+1)$ for $n \in \mathbb{N} \cup \{0\}$. In the case where $n \in \mathbb{N}$, $x = u_n$ and $y = a$, we have

$$d(y, Tx) = f(n + 1) > rf(n) = rd(y, x)$$

and

$$d(x, Ty) > f(0) > f(n) = d(x, y) > rd(x, y).$$

In the case where $x = a$ and $y = b$, we have

$$d(y, Tx) = f(1) > rf(0) = rd(y, x).$$

In the other cases, we can prove (2.4) similarly. So we have shown (2.4). □

Remark 2.3. In the case where $r = 0$, Popescu's conjecture is true. Indeed the condition

$$d(y, Tx) \leq 0d(y, x) \implies d(Tx, Ty) \leq 0d(x, y)$$

implies that Tx is a fixed point of T for any $x \in X$.

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