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SELECTIONS OF GENERALIZED CONVEX SET-VALUED FUNCTIONS WITH BOUNDED DIAMETER

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Abstract. Applying the classical Banach fixed point theorem we prove that a set-valued function with bounded diameter satisfying a linear functional inclusion admits a unique selection fulfilling the corresponding functional equation. We also adopt the method of the proof for investigating the Hyers-Ulam stability of some functional equations.
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1. Introduction

Let X, Y be real vector spaces and $D \subset X$ a convex set. A set-valued function $F\colon D \to n(Y)$, where $n(Y)$ denote the family of all nonempty subsets of Y, is said to be *convex* if

$$\alpha F(x) + (1-\alpha)F(y) \subset F(\alpha x + (1-\alpha)y), \quad x, y \in D,\ \alpha \in [0,1].$$

If

$$\frac{1}{2}F(x) + \frac{1}{2}F(y) \subset F(\frac{x+y}{2}), \quad x, y \in D,$$

the set-valued function F is called *midconvex* or *Jensen convex.* A. Smajdor and W. Smajdor proved, that if Y is a topological vector space and $F\colon X \to n(Y)$ is a convex set-valued function with compact values, then there exists an affine selection of F. They have also observed that, if F is midconvex set-valued function, then for every $y_0 \in F(0)$ there exists an additive function $a\colon X \to Y$ such that $X \ni x \mapsto y_0 + a(x) \in Y$ is a selection of F (see Theorem 3 and 4 in [13]). Similar results, but in particular case, were earlier obtained by K. Nikodem in [7].

Some theorems, for set-valued functions satisfying more general inclusions were proved by K. Nikodem and D. Popa in [8]. M. Piszczek in [9] proved the following result. Let K be a convex cone in a vector space X, i.e. $\alpha K + \beta K \subset K$ for every $\alpha, \beta \geq 0$, $(Y, \|\cdot\|)$ a Banach space and $a, b, p, q > 0$. Consider a set-valued function $F\colon K \to n(Y)$ such that $F(x)$ is convex and closed for each $x \in K$ and $\sup\{\mathrm{diam} F(x),\ x \in K\} < +\infty$, where

$$\mathrm{diam} F(x) = \sup\{\|y_1 - y_2\|,\ y_1, y_2 \in F(x)\}.$$

If

$$pF(x) + qF(y) \subset F(ax + by), \quad x, y \in K,$$

then

(1) if $p+q < 1$, there exists a unique selection $f\colon K \to Y$ of F satisfying equation $pf(x) + qf(y) = f(ax + by),\ x, y \in K$,
(2) if $p + q > 1$, F is single-valued.

D. Popa in [12] investigated the existence of selections satisfying a certain functional equation for a set-valued function $F\colon X \to n(Y)$ such that

$$F(x)\Diamond F(y) \subset F(x * y), \quad x, y \in X,$$

where $(X, *)$, $(Y, \Diamond)$ are square-symmetric grupoids.

Recently D. Inoan and D. Popa in [5] proved the existence of selections of a set-valued function $F\colon G \to n(Y)$ fulfilling the inclusion

$$(1 - p)F(x) + pF(y) \subset F(x * y) + C, \quad x, y \in G,$$

where $(G, *)$ is a grupoid with a bisymmetric operation and Y is a Banach space.

2. The main theorems

Let K be a nonempty set and (Y, ϱ) a metric space. The *diameter* of the set $A \in n(Y)$ is given by formula $\mathrm{diam} A = \sup\{\varrho(a, b) :\ a, b \in A\}$. Consider a set-valued function $F\colon K \to n(Y)$. A function $f\colon K \to Y$ is a *selection* of the set-valued function F iff $f(x) \in F(x),\ x \in K$. Let

$$Sel(F) := \{f\colon K \to Y : f(x) \in F(x),\ x \in K\}.$$

It is easy to check that if $\sup\{\mathrm{diam}(F(x)),\ x \in K\} < +\infty$, then the functional

$$d(f, g) := \sup\{\varrho(f(x), g(x)),\ x \in K\} < +\infty, \quad f, g \in Sel(F)$$

is a metric in $Sel(F)$. Obviously, the convergence in the space $(Sel(F), d)$ is the uniform convergence on the set K. Moreover if (Y, ϱ) is complete and $F(x) \in cl(Y),\ x \in K$ (where $cl(Y)$ denotes the family of all closed members of $n(Y)$), then $(Sel(F), d)$ is complete (see [4], ch. 11).

Theorem 2.1. *Let $\alpha \in (-1, 1)$, $p, q > 0$ and K be a subset of a real vector space X such that $0 \in K$ and $K \subset pK$. Assume that $(Y, \|\cdot\|)$ is a real Banach space and $F\colon K \to cl(Y)$ a set-valued function with bounded diameter, i.e.*

$$\sup\{\mathrm{diam} F(x),\ x \in K\} = M < +\infty.$$

If

$$\alpha F(x) + (1 - \alpha)F(y) \subset F(px + qy), \quad x, y \in K,\ px + qy \in K, \tag{2.1}$$

then there exists a unique function $f\colon K \to Y$ such that

$$\alpha f(x) + (1 - \alpha)f(y) = f(px + qy), \quad x, y \in K,\ px + qy \in K$$

and

$$f(x) + F(0) \subset F(x),\ x \in K.$$

Proof. Fix $a \in F(0)$ and define a set-valued function $G\colon K \to cl(Y)$ by formula

$$G(x) = F(x) - a,\ x \in K.$$

Obviously, $0 \in G(0),\ \operatorname{diam} G(x) = \operatorname{diam} F(x) \le M,\ x \in K$ and

$$\alpha G(x) + (1-\alpha)G(y) \subset G(px+qy), \quad x, y \in K,\ px+qy \in K. \tag{2.2}$$

Indeed, fix $x, y \in K$ such that $px + qy \in K$. By (2.1),

$$\alpha G(x) + (1-\alpha)G(y) = \alpha F(x) + (1-\alpha)F(y) - a \subset F(px+qy) - a = G(px+qy).$$

Setting $y = 0$ and replacing x by $\frac{x}{p}$ in (2.2), we get

$$\alpha G\left(\frac{x}{p}\right) + (1-\alpha)G(0) \subset G(x), \quad x \in K.$$

Hence

$$\alpha G\left(\frac{x}{p}\right) \subset G(x), \quad x \in K. \tag{2.3}$$

Let

$$T(g)(x) := \alpha g\left(\frac{x}{p}\right),\ x \in K,\ g \in Sel(G).$$

By (2.3), $T(g) \in Sel(G),\ g \in Sel(G)$. Moreover for every $g_1, g_2 \in Sel(G)$

$$d(T(g_1), T(g_2)) = |\alpha| \sup\left\{\|g_1\left(\frac{x}{p}\right) - g_2\left(\frac{x}{p}\right)\|,\ x \in K\right\} \le |\alpha| d(g_1, g_2).$$

Thus $T\colon Sel(G) \to Sel(G)$ is contractive in the complete metric space $(Sel(G), d)$, so by the Banach theorem, it has a unique fixed point g_a and $\lim_{n\to\infty} T^n(g) = g_a$ for each $g \in Sel(G)$. Hence $g_a\colon K \to Y$ is a unique selection of the set-valued function G such that

$$g_a(x) = \alpha g_a\left(\frac{x}{p}\right),\ x \in K. \tag{2.4}$$

In particular, $g_a(0) = 0$ and $g_a(x) + a \in F(x),\ x \in K$. Fix now $g \in Sel(G)$ and $x, y \in K$ such that $px + qy \in K$. By (2.2),

$$\alpha g(x) + (1-\alpha)g(y),\ g(px+qy) \in G(px+qy), x, y \in K,\ px+qy \in K.$$

Hence

$$\|\alpha g(x) + (1-\alpha)g(y) - g(px+qy)\| \le \operatorname{diam} G(px+qy) \le M.$$

Thus for each $x, y \in K$ such that $px + qy \in K$,

$$\|\alpha T^0(g)(x) + (1-\alpha)T^0(g)(y) - T^0(g)(px+qy)\| \le |\alpha|^0 M.$$

Fix now $n \ge 0$ and assume, that

$$\|\alpha T^n(g)(x) + (1-\alpha)T^n(g)(y) - T^n(g)(px+qy)\| \le |\alpha|^n M$$

for all $x, y \in K$ with $px+qy \in K$. Let $x, y \in K$ and $px+qy \in K$. Then $\frac{x}{p}, \frac{y}{p} \in K$ and $p\frac{x}{p}+q\frac{y}{p} \in K$. Hence

$$\begin{aligned}&\|\alpha T^{n+1}(g)(x)+(1-\alpha)T^{n+1}(g)(y)-T^{n+1}(g)(px+qy)\|\\&=|\alpha|\cdot\|\alpha T^n(g)\left(\tfrac{x}{p}\right)+(1-\alpha)T^n(g)(\tfrac{y}{p})-T^n(g)(p\tfrac{x}{p}+q\tfrac{y}{p})\|\\&\le|\alpha|^{n+1}M.\end{aligned}$$

It follows that for every $x, y \in K$ such that $px+qy \in K$ and each $n \geq 0$,

$$\|\alpha T^n(g)(x)+(1-\alpha)T^n(g)(y)-T^n(g)(px+qy)\| \le |\alpha|^n M.$$

Letting $n \to \infty$ we obtain

$$\alpha g_a(x)+(1-\alpha)g_a(y)=g_a(px+qy),\ x,y\in K,\ px+qy\in K.$$

We have proved, that for every $a \in F(0)$ there exists a unique function $g_a\colon K \to Y$ such that $g_a(x)+a \in F(x),\ x \in K$ and

$$\alpha g_a\left(\frac{x}{p}\right)=g_a(x),\ x\in K.$$

Moreover,

$$\alpha g_a(x)+(1-\alpha)g_a(y)=g_a(px+qy),\ x,y\in K\ px+qy\in K.$$

For the end of the proof fix $a_1, a_2 \in F(0)$. From what has already been proved, there exist functions $g_1, g_2\colon K \to Y$ such that $g_i(x)+a_i \in F(x)$ and $\alpha g_i(\frac{x}{p}) = g_i(x)$, $x \in K$, $i = 1, 2$. Consequently, for all $x \in K$,

$$\|g_1(x)-g_2(x)\| \le \mathrm{diam}F(x)+\|a_1-a_2\| \le \mathrm{diam}F(x)+\mathrm{diam}F(0) \le 2M.$$

Hence

$$\|g_1(x)-g_2(x)\| \le 2|\alpha|^0 M,\ x\in K.$$

Assume, that for fixed $n \geq 0$

$$\|g_1(x)-g_2(x)\| \le 2|\alpha|^n M,\ x\in K$$

and fix $x \in K$. Then $\frac{x}{p} \in K$ and

$$\|g_1(x)-g_2(x)\| = |\alpha|\|g_1\left(\frac{x}{p}\right)-g_2\left(\frac{x}{p}\right)\| \le 2|\alpha|^{n+1}M.$$

Thus, for every for $x \in K$,

$$\|g_1(x)-g_2(x)\| \le 2|\alpha|^n M,\ n\ge 0,$$

what clearly forces

$$g_1(x)=g_2(x),\ x\in K.$$

It follows that there exists a unique function $f\colon K \to Y$ such that

$$\alpha f(x)+(1-\alpha)f(y)=f(px+qy),\quad x,y\in K,\ px+qy\in K$$

and

$$f(x)+F(0)\subset F(x),\ x\in K. \qquad \square$$

Remark 2.2. *Let $\alpha, \beta \neq 0$ and $p, q > 0$. Assume that X and Y are real vector spaces and K is a convex cone in X. If $f \colon K \to Y$ satisfies the equation*

$$\alpha f(x) + \beta f(y) = f(px + qy), \quad x, y \in K$$

and $f(0) = 0$, then f is additive, i.e.

$$f(x) + f(y) = f(x + y), \quad x, y \in K.$$

The proof is immediate.

Corollary 2.3. *Let $\alpha \in (-1, 2) \setminus \{0, 1\}$, $p, q > 0$. Assume that K is a convex cone in a real vector space X, $(Y, \|\cdot\|)$ a real Banach space and $F \colon K \to cl(Y)$ a set-valued function with the bounded diameter. If*

$$\alpha F(x) + (1 - \alpha)F(y) \subset F(px + qy), \quad x, y \in K,$$

then there exists a unique function $f \colon K \to Y$ such that

$$f(x) + F(0) \subset F(x), \ x \in K$$

and

$$\alpha f(x) + (1 - \alpha)f(y) = f(px + qy), \quad x, y \in K.$$

The function f is additive.

Proof. Since K is a convex cone and p, q are positive numbers,

$$K \subset pK \quad \text{and} \quad K \subset qK.$$

If $\alpha \in (-1, 1)$, then by Theorem 2.1, there exists exactly one function $f \colon K \to Y$ satisfying the equation

$$\alpha f(x) + (1 - \alpha)f(y) = f(px + qy), \quad x, y \in K$$

and such that

$$f(x) + F(0) \subset F(x), \ x \in K.$$

In particular, $f(0) + F(0) \subset F(0)$, it follows that $nf(0) + F(0) \subset F(0)$ for every positive integer n. The set $F(0)$ is nonempty and bounded, and so $f(0) = 0$. If $\alpha \in (1, 2)$, then $1 - \alpha \in (-1, 0)$, so it is enough to change variables in Theorem 2.1. □

Example 2.4. Let $K = [0, +\infty)$ and $F \colon K \to cl(\mathbb{R})$ be given by

$$F(x) = \left[1, 3 - \frac{1}{x + 1}\right], \ x \in K.$$

The set-valued function F is convex i.e.

$$\alpha F(x) + (1 - \alpha)F(y) \subset F(\alpha x + (1 - \alpha)y), \ x, y \in K, \ \alpha \in (0, 1)$$

and $\text{diam} F(x) \leq 2, \ x \in K$. The function $K \ni x \mapsto f(x) = 0 \in \mathbb{R}$ is additive and

$$f(x) + F(0) = [1, 2] \subset F(x), \ x \in K.$$

Example 2.5. Let $K = [0, +\infty)$ and $F\colon K \to cl(\mathbb{R})$ be a set-valued function defined by

$$F(x) = [x + 1, x + 2] = x + [1, 2],\ x \in K.$$

Then F is convex and $f(x) = x,\ x \in K$ is an additive function such that

$$f(x) + F(0) = x + [1, 2] \subset F(x),\ x \in K.$$

Theorem 2.6. *Let $\alpha \in (-1, 1),\ \ p \geq 1, q > 0$, K be a convex cone in a real normed space $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ a real Banach space. Assume that $F\colon K \to cl(Y)$ is a set-valued function such that*

$$K \ni x\ \mapsto\ \mathrm{diam} F(x) \in [0, +\infty)$$

maps bounded subsets of K onto bounded. If

$$\alpha F(x) + (1 - \alpha)F(y) \subset F(px + qy), \quad x, y \in K,$$

then there exists a unique function $f\colon K \to Y$ such that

$$f(x) + F(0) \subset F(x),\ x \in K$$

and fulfilling the functional equation

$$\alpha f(x) + (1 - \alpha)f(y) = f(px + qy), \quad x, y \in K.$$

Moreover, f is an additive function.

Proof. For every $n \geq 1$ define

$$K_n := \{x \in K :\ \|x\| \leq n\}$$

and a set-valued function $F_n\colon K \to cl(Y)$ by

$$F_n(x) = F(x),\ x \in K_n.$$

Then $K_n \subset pK_n$ and

$$\alpha F_n(x) + (1 - \alpha)F_n(y) \subset F_n(px + qy), \quad x, y \in K_n,\ px + qy \in K_n.$$

Moreover

$$\sup\{\mathrm{diam} F_n(x),\ x \in K_n\} < +\infty.$$

By Theorem 2.1, there exists a unique function $f_n\colon K_n \to Y$ such that

$$\alpha f_n(x) + (1 - \alpha)f_n(y) = f_n(px + qy), \quad x, y \in K_n,\ px + qy \in K_n \tag{2.5}$$

and

$$f_n(x) + F(0) \subset F(x),\ x \in K_n.$$

Since

$$F_{n+1}(x) = F_n(x),\ x \in K_n,$$

by the uniqueness of the function f_n, we have

$$f_{n+1}(x) = f_n(x),\ x \in K_n.$$

It follows that the function $f\colon K \to Y$ given by

$$f(x) = f_n(x),\ x \in K_n$$

is defined properly. Obviously $f(x) + F(0) \subset F(x),\ x \in K$. For the end fix $x, y \in K$. There $x, y, px + qy \in K_n$ for n big enough. Since $f(x) = f_n(x),\ x \in K_n$ and by (2.5),

$$\alpha f(x) + (1 - \alpha)f(y) = f(px + qy).$$

By Remark 2.2, f must be additive. □

3. Stability results

In this section we first present an application of the method used in the proof of Theorem 1 to the investigation of the Hyers-Ulam stability of the functional equation

$$\alpha f(x) + (1 - \alpha)f(y) = f(px + qy). \tag{3.1}$$

The above equation is a particular case of general linear equation

$$\alpha f(x) + \beta f(y) + \gamma = f(px + qy + r),$$

where f maps a vector space into a Banach space. The general linear equation was considered by several authors (see for example [1, 6, 10]), however they verified the cases $\gamma = 0$ and $r = 0$ or investigated the stability under some additional assumptions like $\alpha + \beta \neq 1$ in [2]. Further information can be found in [3, 11].

Theorem 3.1. *Let K be a convex cone in a vector space X, $(Y, \|\cdot\|)$ be a real Banach space and $\alpha \in (0, 1)$, $p, q > 0$ and $\varepsilon > 0$. If a function $f \colon K \to Y$ satisfies*

$$\|\alpha f(x) + (1 - \alpha)f(y) - f(px + qy)\| \leq \varepsilon, \quad x, y \in K,$$

then there exists a unique function $f_0 \colon K \to Y$ fulfilling the condition

$$\|f(x) - f_0(x) - f(0)\| \leq \frac{\varepsilon}{1 - \alpha},\ x \in K$$

and the functional equation

$$\alpha f_0(x) + (1 - \alpha)f_0(y) = f_0(px + qy), \quad x, y \in K.$$

Moreover, f_0 is an additive function.

Proof. Let $g \colon K \to Y$ be given by $g(x) = f(x) - f(0),\ x \in K$. Then $g(0) = 0$ and

$$\|\alpha g(x) + (1 - \alpha)g(y) - g(px + qy)\| \leq \varepsilon, \quad x, y \in K. \tag{3.2}$$

Hence, for all $x, y \in K$

$$\alpha g(x) + (1 - \alpha)g(y) \in g(px + qy) + B_\varepsilon, \tag{3.3}$$

where B_ε denotes the closed ball $B(0, \varepsilon)$. Setting $y = 0$ and replacing x by $\frac{x}{p}$ in (3.3) we obtain

$$\alpha g\left(\frac{x}{p}\right) \in g(x) + B_\varepsilon, x \in K.$$

Thus, for all $x \in K$

$$\alpha g\left(\frac{x}{p}\right) + \frac{\alpha}{1 - \alpha}B_\varepsilon \subset g(x) + B_\varepsilon + \frac{\alpha}{1 - \alpha}B_\varepsilon = g(x) + \frac{1}{1 - \alpha}B_\varepsilon.$$

Define a set-valued function $G \colon K \to cl(Y)$ as follows

$$G(x) = g(x) + \frac{1}{1 - \alpha}B_\varepsilon,\ x \in K.$$

Then

$$\alpha G\left(\frac{x}{p}\right) \subset G(x),\ x \in K$$

and $\operatorname{diam} G(x) = \frac{2\varepsilon}{1-\alpha},\ x \in K$. The idea of the proof is the same as before so we only give a sketch. The function $T\colon Sel(G) \to Sel(G)$ given by

$$T(h)(x) := \alpha h\left(\frac{x}{p}\right),\ x \in K,\ h \in Sel(G).$$

is contraction with the constant α. By the Banach theorem, there exists the unique function $f_0 \in Sel(G)$ such that

$$f_0(px) = \alpha f_0(x),\ x \in K$$

and $\lim_{n\to\infty} T^n(g) = f_0$. By the definition of G,

$$\|g(x) - f_0(x)\| \leq \frac{\varepsilon}{1-\alpha},\ x \in K.$$

Since g satisfies (3.2),

$$\|\alpha T(g)(x) + (1-\alpha)T(g)(y) - T(g)(px+qy)\| \leq \alpha\varepsilon$$

for all $x, y \in K$. Proceeding by induction, we get

$$\|\alpha T^n(g)(x) + (1-\alpha)T^n(g)(y) - T^n(g)(px+qy)\| \leq \alpha^n\varepsilon$$

for every $x, y \in K$ and $n \geq 1$. It follows that

$$\alpha f_0(x) + (1-\alpha)f_0(y) = f_0(px+qy),\ x, y \in K.$$

Since $f_0(0) = 0$, f_0 is additive, by Remark 2.2. Thus f_0 is the unique additive function such that $\alpha f_0(x) + (1-\alpha)f_0(y) = f_0(px+qy),\ x, y \in K$ and

$$\|f(x) - f_0(x) - f(0)\| \leq \frac{\varepsilon}{1-\alpha},\ x \in K. \qquad \square$$

Now we apply Theorem 2.6 to the proof of Rassias stability of functional equation (3.1).

Theorem 3.2. *Let K be a convex cone in a real normed space $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be a real Banach space and $\alpha \in (0,1)$, $p \geq 1, q > 0$. Consider the function $\varepsilon\colon K \times K \to [0,\infty)$ such that*

$$\varepsilon(x,0) = \varepsilon(0,x),\ x \in K, \tag{3.4}$$

$$\varepsilon(x,y) + \alpha\varepsilon(x,0) + (1-\alpha)\varepsilon(0,y) \leq \varepsilon(px+qy,0),\ x, y \in K \tag{3.5}$$

and the function $x \mapsto \varepsilon(x,0)$ is bounded on every bounded subset of K.

If a function $f\colon K \to Y$ satisfies

$$\|\alpha f(x) + (1-\alpha)f(y) - f(px+qy)\| \leq \varepsilon(x,y), \quad x, y \in K,$$

then there exists a unique function $f_0\colon K \to Y$ fulfilling the condition

$$\|f(x) - f_0(x) - f(0)\| \leq \varepsilon(x,0),\ x \in K$$

and the functional equation

$$\alpha f_0(x) + (1-\alpha)f_0(y) = f_0(px+qy), \quad x, y \in K.$$

Moreover, f_0 is an additive function.

Proof. Let $g\colon K \to Y$ be given by $g(x) = f(x) - f(0),\ x \in K$ and $F(x) := g(x) + \varepsilon(x,0)B,\ x \in K$, where B is the closed unit ball. By (3.5), $\varepsilon(0,0) = 0$. Hence $F(0) = \{0\}$ and

$$\begin{aligned}
&\alpha F(x) + (1-\alpha)F(y)\\
&\quad= [\alpha g(x) + (1-\alpha)g(y)] + \alpha\varepsilon(x,0)B + (1-\alpha)\varepsilon(0,y)B\\
&\quad\subset g(px+qy) + \varepsilon(x,y)B + \alpha\varepsilon(x,0)B + (1-\alpha)\varepsilon(0,y)B\\
&\quad\subset g(px+qy) + \varepsilon(px+qy,0)B\\
&\quad= F(px+qy).
\end{aligned}$$

According to Theorem 2.6 there exists a unique function $f_0 : K \to Y$ fulfilling equation (3.1) such that

$$f_0(x) \in F(x) = g(x) + \varepsilon(x,0)B,\ x \in K.$$

Function f_0 is additive. This completes the proof. □

Example 3.3. Let $K = [0,\infty)$, $p, q \geq 1$, $\alpha \in (0,1)$ and $\varepsilon(x,y) = x^s + y^s,\ x,y \in K$, where $s \geq 1$ is such a number that $2 - q^s \leq \alpha \leq p^s - 1$. Conditions (3.4),(3.5) hold and Theorem 3.2 may be used.

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