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APPROXIMATION OF COMMON SOLUTIONS TO PROXIMAL SPLIT FEASIBILITY PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. This paper is concerned with an algorithmic solution to the proximal split feasibility problem which is also a fixed point of a k-strictly pseudocontractive mapping in Hilbert spaces. Under some assumptions on the parameters in our iterative algorithm, we first establish a strong convergence theorem for a problem of finding a point which minimizes a proper convex lower-semicontinuous function f which is also a fixed point of a k-strictly pseudocontractive mapping such that its image under a bounded linear operator A minimizes another proper convex lower-semicontinuous function g and secondly prove another strong convergence result for a problem of finding a point which minimizes a proper convex lower-semicontinuous function f which is also a fixed point of a k-strictly pseudocontractive mapping such that its image under a bounded linear operator A minimizes locally lower semicontinuous, prox-bounded and prox-regular function g. In all our results in this work, our iterative schemes are proposed with a way of selecting the step-sizes such that their implementation does not need any prior information about the operator norm because the calculation or at least an estimate of the operator norm $||A||$ is very difficult, if it is not an impossible task. Our result complement many recent and important results in this direction.
Key Words and Phrases: proximal split feasibility problems, Moreau-Yosida approximate, prox-regularity, k-strictly pseudocontractive mapping, fixed point, strong convergence, Hilbert spaces.
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1. Introduction

In this paper, we shall assume that H is a real Hilbert space with inner product $\langle .,. \rangle$ and norm $||.||$. Let I denote the identity operator on H. A mapping $T : H \to H$ is said to be *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \ \forall \ x, y \in H, \tag{1.1}$$

and $T : H \to H$ is said to be *k-strictly pseudocontractive* (see, [2]) if for $0 \leq k < 1$,

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \ \forall \ x, y \in H. \tag{1.2}$$

It is well known that every nonexpansive mapping is strictly pseudocontractive. In a Hilbert space H, we can show that (1.2) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq ||x - y||^2 - \frac{1-k}{2}||(I - T)x - (I - T)y||^2. \tag{1.3}$$

A point $x \in H$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$. Iterative approximation of fixed points for k-strictly pseudocontractive mappings has been studied extensively by many authors (see, for example, [1, 6, 8, 9, 10, 11, 12, 13, 16, 21, 23, 20, 36] and the references contained therein).
Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.4}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [7] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [3]. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms have been invented to solve it (see, for example, [4, 17, 22, 27, 29, 30, 33] and references therein).
Note that the split feasibility problem (1.4) can be formulated as a fixed-point equation by using the fact

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*; \tag{1.5}$$

that is, x^* solves the SFP (1.4) if and only if x^* solves the fixed point equation (1.5) (see [26] for the details). This implies that we can use fixed-point algorithms (see [31, 32, 34]) to solve SFP. A popular algorithm that solves the SFP (1.4) is due to Byrne's CQ algorithm [3] which is found to be a gradient-projection method (GPM) in convex minimization. Subsequently, Byrne [4] applied Krasnoselskii-Mann iteration to the CQ algorithm, and Zhao and Yang [35] applied Krasnoselskii-Mann iteration to the perturbed CQ algorithm to solve the SFP. It is well known that the CQ algorithm and the Krasnoselskii-Mann algorithm for a split feasibility problem do not necessarily converge strongly in the infinite-dimensional Hilbert spaces.
Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous convex functions on H_1 and H_2 respectively. Define $\text{argmin} f := \{\bar{x} \in H_1 : f(\bar{x}) \leq f(x), \forall x \in H_1\}$ and $\text{argmin} g := \{\bar{y} \in H_2 : g(\bar{y}) \leq g(y), \forall y \in H_2\}$. Moudafi and Thakur [19] recently studied the following *proximal split feasibility problem*: find a minimizer x^* of f such that Ax^* minimizes g, namely

$$x^* \in \text{argmin} f \text{ such that } Ax^* \in \text{argmin}\, g. \tag{1.6}$$

We will denote the solution set of (1.6) by Γ.
Observe that if we take $f = \delta_C$ [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise], the indicator function of nonempty, closed and convex subset C of H_1 and $g = \delta_Q$, the indicator function of nonempty, closed and convex subsets Q of H_2, then Problem (1.6) reduces to (1.4).
Moudafi and Thakur [19] studied the proximal split feasibility problem (1.6) and proved weak convergence results for its solution using the following split proximal algorithm based on the idea of the algorithm introduced in *Lopez et al.* [14].

Split Proximal Algorithm 1. Given an initial point $x_1 \in H_1$. Assume that x_n has been constructed and $\theta(x_n) \neq 0$, then compute x_{n+1} via the rule

$$x_{n+1} = \text{prox}_{\lambda\mu_n f}(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n),\ n \geq 1, \tag{1.7}$$

where stepsize $\mu_n := \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.6) and the iterative process stops, otherwise, we set $n := n + 1$ and go to (1.7).

Furthermore, Moudafi and Thakur [19] assumed f to be convex and allowed the function g to be non-convex. They considered the problem of finding a minimizer $\bar{x}$ of f such that $A\bar{x}$ is a critical point of g. Thus,

$$0 \in \partial f(\bar{x}) \text{ such that } 0 \in \partial_{pg}(A\bar{x}), \tag{1.8}$$

where ∂_{pg} stands for the proximal sub-differential of g. In particular, they studied the weak convergence of the following algorithm to a solution of (1.8):

Split Proximal Algorithm 2. Given an initial point $x_1 \in H_1$. Assume that x_n has been constructed and $\theta(x_n) \neq 0$, then compute x_{n+1} via the rule

$$x_{n+1} = \text{prox}_{\lambda_n\mu_n f}(x_n - \mu_n A^*(I - \text{prox}_{\lambda_n g})Ax_n),\ n \geq 1, \tag{1.9}$$

where stepsize $\mu_n := \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.8) and the iterative process stops, otherwise, we set $n := n + 1$ and go to (1.9).

Motivating by the results of *Lopez et al.* [14], Moudafi and Thakur [19] and previous results on approximation of fixed point of k-strictly pseudocontractive mappings, our aim in this paper is to introduce new iterative schemes for solving problems (1.6) and (1.8) which is also a fixed point of a k-strictly pseudocontractive mapping and prove strong convergence of the sequences generated by our schemes in real Hilbert spaces. Our results also complement the results of Shehu [24] and Shehu and Ogbuisi [25].

2. Preliminaries

Let H be a real Hilbert space and C a nonempty, closed and convex subset of H. For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$||u - P_C u|| \leq ||u - y||,\ \forall y \in C.$$

P_C is called the *metric projection* of H onto C. We know that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq ||P_C x - P_C y||^2, \tag{2.1}$$

for all $x, y \in H$. Furthermore, $P_C x$ is characterized by the properties $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \tag{2.2}$$

for all $y \in C$.

We state the following well-known lemmas which will be used in the sequel.

Lemma 2.1. *Let H be a real Hilbert space. Then there holds the following well-known results:*

(i) $||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2, \forall x, y \in H$;

(ii) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle, \forall x, y \in H$.

Lemma 2.2. ([36]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be k-strictly pseudocontractive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.3. (Xu, [28]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \ n \ge 1,$$

where

(i) $\{a_n\} \subset [0, 1], \ \sum \alpha_n = \infty$;
(ii) $\limsup \sigma_n \le 0$;
(iii) $\gamma_n \ge 0; \ (n \ge 1), \ \Sigma\gamma_n < \infty$.

Then, $a_n \to 0$ as $n \to \infty$.

3. Main results

3.1. **Strong Convergence for Convex minimization feasibility and fixed point problem.** Let T be a k-strictly pseudocontractive mapping of H_1 into itself. Set

$$\theta(x) := \sqrt{||\nabla h(x)||^2 + ||\nabla l(x)||^2}$$

with $h(x) = \frac{1}{2}||(I - \text{prox}_{\lambda g})Ax||^2$, $l(x) = \frac{1}{2}||(I - \text{prox}_{\lambda\mu f})x||^2$ and introduce the following algorithm:

Algorithm 1. Let $u \in H_1$. Assume that x_n has been constructed and $\theta(x_n) \neq 0$, then compute x_{n+1} via the rule

$$\begin{cases} y_n = \text{prox}_{\lambda\mu_n f}(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n) \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n Ty_n - t_n(y_n - u), \ n \ge 1, \end{cases} \tag{3.1}$$

where stepsize $\mu_n := \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.6) which is also a fixed point of a k-strictly pseudocontractive mapping and the iterative process stops, otherwise, we set $n := n + 1$ and go to (3.1).

Using (3.1), we prove the following strong convergence theorem for approximation of solution of problem (1.6) which is also a fixed point of a k-strictly pseudocontractive mapping of H_1 into itself.

Theorem 3.1. *Assume that f and g are two proper convex lower-semicontinuous functions and that (1.6) is consistent (i.e., $\Gamma \neq \emptyset$). Let T be a k-strictly pseudocontractive mapping of H_1 into itself such that $F(T) \cap \Gamma \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0, 1)$, $\{\alpha_n\}$ a sequence in $(0, (1 - k)(1 - t_n)) \subset (0, 1)$. If the parameters satisfy the following conditions*

(a) $\lim_{n\to\infty} t_n = 0$;
(b) $\sum_{n=1}^{\infty} t_n = \infty$;
(c) $\epsilon \le \rho_n \le \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ *for some* $\epsilon > 0$;
(d) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1 - k$,

then sequence $\{x_n\}$ generated by (3.1) converge strongly to $x^* \in F(T) \cap \Gamma$, where $x^* = P_{F(T)\cap\Gamma}u$.

Proof. Let $x^* \in F(T) \cap \Gamma$. Using the same line of arguments and method of proof in the earlier part of proof of Theorem 2.2 of Moudafi and Thakur [19], we can show that

$$||y_n - x^*||^2 \leq ||x_n - x^*||^2 - \rho_n\Big(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \rho_n\Big)\frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)}. \tag{3.2}$$

From (3.1), we have

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||(1-\alpha_n - t_n)(y_n - x^*) + \alpha_n(Ty_n - x^*) + t_n(u - x^*)|| \\ &\leq ||(1-\alpha_n - t_n)(y_n - x^*) + \alpha_n(Ty_n - x^*)|| + t_n||u - x^*||. \end{aligned} \tag{3.3}$$

But from (1.2) and (1.3), we obtain

$$\begin{aligned} &||(1-\alpha_n - t_n)(y_n - x^*) + \alpha_n(Ty_n - x^*)||^2 \\ &= (1-\alpha_n - t_n)^2||y_n - x^*||^2 + \alpha_n^2||Ty_n - x^*||^2 \\ &+ 2(1-\alpha_n - t_n)\alpha_n\langle Ty_n - x^*, y_n - x^*\rangle \\ &\leq (1-\alpha_n - t_n)^2||y_n - x^*||^2 \\ &+ \alpha_n^2[||y_n - x^*||^2 + k||y_n - Ty_n||^2] \\ &+ 2(1-\alpha_n - t_n)\alpha_n\Big[||y_n - x^*||^2 - \frac{1-k}{2}||y_n - Ty_n||^2\Big] \\ &= (1-t_n)^2||y_n - x^*||^2 \\ &+ [k\alpha_n^2 - (1-k)(1-\alpha_n - t_n)\alpha_n]||y_n - Ty_n||^2 \\ &= (1-t_n)^2||y_n - x^*||^2 \\ &+ \alpha_n[\alpha_n - (1-t_n)(1-k)]||y_n - Ty_n||^2 \\ &\leq (1-t_n)^2||y_n - x^*||^2, \end{aligned} \tag{3.4}$$

which implies

$$||(1-\alpha_n - t_n)(y_n - x^*) + \alpha_n(Ty_n - x^*)|| \leq (1-t_n)||y_n - x^*||. \tag{3.5}$$

Therefore, it follows from (3.2), (3.3) and (3.5) that

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq (1-t_n)||y_n - x^*|| + t_n||u - x^*|| \qquad (3.6) \\ &\leq (1-t_n)||x_n - x^*|| + t_n||u - x^*|| \\ &\leq \max\{||x_n - x^*||, ||u - x^*||\}. \end{aligned}$$

By induction, we have

$$||x_n - x^*|| \leq \max\{||x_1 - x^*||, ||u - x^*||\}.$$

Hence, $\{x_n\}$ is bounded and so is $\{y_n\}$. Now, using (1.2), we have

$$\begin{aligned} & ||Tx - x^*||^2 \leq ||x - x^*||^2 + k||x - Tx||^2 \\ \Rightarrow\quad & \langle Tx - x^*, Tx - x^*\rangle \leq \langle x - x^*, x - Tx\rangle + \langle x - x^*, Tx - x^*\rangle + k||x - Tx||^2 \\ \Rightarrow\quad & \langle Tx - x^*, Tx - x\rangle \leq \langle x - x^*, x - Tx\rangle + k||x - Tx||^2 \\ \Rightarrow\quad & \langle Tx - x, Tx - x\rangle + \langle x - x^*, Tx - x\rangle \leq \langle x - x^*, x - Tx\rangle + k||x - Tx||^2 \\ \Rightarrow\quad & (1-k)||x - Tx||^2 \leq 2\langle x - x^*, x - Tx\rangle. \end{aligned} \tag{3.7}$$

Therefore, by (3.2) and Lemma 2.1 (ii), we obtain

$$\begin{aligned}
||y_{n+1}-x^*||^2 &\leq ||x_{n+1}-x^*||^2 = ||(1-\alpha_n)y_n+\alpha_n Ty_n - t_n(y_n-u)-x^*||^2 \\
&= ||(y_n-x^*)-\alpha_n(y_n-Ty_n)-t_n(y_n-u)||^2 \\
&\leq ||(y_n-x^*)-\alpha_n(y_n-Ty_n)||^2 - 2t_n\langle y_n-u, x_{n+1}-x^*\rangle \\
&= ||y_n-x^*||^2 - 2\alpha_n\langle y_n-Ty_n, y_n-x^*\rangle + \alpha_n^2||y_n-Ty_n||^2 \\
&\quad -2t_n\langle y_n-u, x_{n+1}-x^*\rangle \\
&\leq ||y_n-x^*||^2 - \alpha_n(1-k)||y_n-Ty_n||^2 + \alpha_n^2||y_n-Ty_n||^2 \\
&\quad -2t_n\langle y_n-u, x_{n+1}-x^*\rangle \\
&= ||y_n-x^*||^2 - \alpha_n[(1-k)-\alpha_n]||y_n-Ty_n||^2 \\
&\quad -2t_n\langle y_n-u, x_{n+1}-x^*\rangle \qquad (3.8)\\
&\leq ||y_n-x^*||^2 - \alpha_n[(1-k)-\alpha_n]||y_n-Ty_n||^2 \\
&\quad -2t_n\langle y_n-u, x_{n+1}-x^*\rangle.
\end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, $\exists M > 0$ such that $-2\langle x_n - u, x_{n+1} - x^*\rangle \leq M$ for all $n \geq 1$. Therefore,

$$||y_{n+1}-x^*||^2 - ||y_n-x^*||^2 + \alpha_n[(1-k)-\alpha_n]||y_n-Ty_n||^2 \leq t_n M. \qquad (3.9)$$

Now we divide the rest of the proof into two cases.

Case 1. Assume that $\{||y_n - x^*||\}$ is monotonically decreasing sequence. Then $\{||y_n - x^*||\}$ is convergent and obviously,

$$||y_{n+1}-x^*|| - ||y_n-x^*|| \to 0, n \to \infty. \qquad (3.10)$$

This together with (3.9) and the condition that $t_n \to 0$ imply that,

$$||y_n - Ty_n|| \to 0, n \to \infty. \qquad (3.11)$$

From (3.2) and (3.8), we have that

$$\begin{aligned}
&\rho_n\Big(\frac{4h(x_n)}{h(x_n)+l(x_n)} - \rho_n\Big)\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)} \leq ||x_n-x^*||^2 - ||y_n-x^*||^2 \\
\leq\ & \Big(||y_{n-1}-x^*|| + t_{n-1}||u-x^*||\Big)^2 - ||y_n-x^*||^2 \\
\leq\ & ||y_{n-1}-x^*||^2 - ||y_n-x^*||^2 + 2t_{n-1}||u-x^*||||y_{n-1}-x^*|| + t_{n-1}^2||u-x^*||^2.
\end{aligned}$$

Using condition (a) above implies that

$$\rho_n\Big(\frac{4h(x_n)}{h(x_n)+l(x_n)} - \rho_n\Big)\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)} \to 0,\ n \to \infty.$$

Hence, we obtain

$$\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)} \to 0,\ n \to \infty. \qquad (3.12)$$

If z is a weak cluster point of $\{x_n\}$, then by following the same line of arguments and method of proof in the later part of proof of Theorem 2.2 of Moudafi and Thakur [19], we can show that $z \in \Gamma$.

Since $x^* = \text{prox}_{\lambda\mu_n f}(x^* - \mu_n A^*(I - \text{prox}_{\lambda g})Ax^*)$ and $A^*(I - \text{prox}_{\lambda g})A$ is Lipschitz with constant $||A||^2$, we have from (3.1) that

$$\begin{aligned}
&||y_n - x^*||^2 \\
&= ||\text{prox}_{\lambda\mu_n f}(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n) - \text{prox}_{\lambda\mu_n f}(x^* - \mu_n A^*(I - \text{prox}_{\lambda g})Ax^*)||^2 \\
&\leq \langle (x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n) - (x^* - \mu_n A^*(I - \text{prox}_{\lambda g})Ax^*), y_n - x^* \rangle \\
&= \frac{1}{2}\Big[||(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n) - (x^* - \mu_n A^*(I - \text{prox}_{\lambda g})Ax^*)||^2 + ||y_n - x^*||^2 \\
&- ||(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n) - (x^* - \mu_n A^*(I - \text{prox}_{\lambda g})Ax^*) - (y_n - x^*)||^2\Big] \\
&\leq \frac{1}{2}\Big[(1 + \mu_n||A||^2)^2||x_n - x^*||^2 + ||y_n - x^*||^2 \\
&- ||x_n - y_n - \mu_n(A^*(I - \text{prox}_{\lambda g})Ax_n - A^*(I - \text{prox}_{\lambda g})Ax^*)||^2\Big] \\
&= \frac{1}{2}\Big[(1 + \mu_n||A||^2)^2||x_n - x^*||^2 + ||y_n - x^*||^2 \\
&- ||x_n - y_n||^2 + 2\mu_n\langle x_n - y_n, A^*(I - \text{prox}_{\lambda g})Ax_n - A^*(I - \text{prox}_{\lambda g})Ax^*)\rangle \\
&- \mu_n^2||A^*(I - \text{prox}_{\lambda g})Ax_n - A^*(I - \text{prox}_{\lambda g})Ax^*)||^2\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
||y_n - x^*||^2 \quad \leq \quad & (1 + \mu_n||A||^2)^2||x_n - x^*||^2 - ||x_n - y_n||^2 \\
& + 2\mu_n\langle x_n - y_n, A^*(I - \text{prox}_{\lambda g})Ax_n - A^*(I - \text{prox}_{\lambda g})Ax^*)\rangle \\
& - \mu_n^2||A^*(I - \text{prox}_{\lambda g})Ax_n - A^*(I - \text{prox}_{\lambda g})Ax^*)||^2. \qquad (3.13)
\end{aligned}$$

We observe that

$$0 < \mu_n < 4\frac{h(x_n) + l(x_n)}{\theta^2(x_n)} \to 0, \ n \to \infty$$

implies that $\mu_n \to 0, \ n \to \infty$. Furthermore, we obtain from (3.13) and (3.6) that

$$\begin{aligned}
&||x_n - y_n||^2 \qquad (3.14) \\
&\leq (1 + \mu_n||A||^2)^2||x_n - x^*||^2 - ||y_n - x^*||^2 \\
&+ 2\mu_n\langle x_n - y_n, A^*(I - \text{prox}_{\lambda g})Ax_n - A^*(I - \text{prox}_{\lambda g})Ax^*)\rangle \\
&= ||x_n - x^*||^2 + \mu_n||A||^2(2 + \mu_n||A||^2)||x_n - x^*||^2 - ||y_n - x^*||^2 \\
&+ 2\mu_n\langle x_n - y_n, A^*(I - \text{prox}_{\lambda g})Ax_n - A^*(I - \text{prox}_{\lambda g})Ax^*)\rangle \\
&\leq (||y_{n-1} - x^*|| + t_{n-1}||u - x^*||)^2 - ||y_n - x^*||^2 + \mu_n||A||^2(2 + \mu_n||A||^2)||x_n - x^*||^2 \\
&+ 2\mu_n\langle x_n - y_n, A^*(I - \text{prox}_{\lambda g})Ax_n - A^*(I - \text{prox}_{\lambda g})Ax^*)\rangle \\
&= ||y_{n-1} - x^*||^2 - ||y_n - x^*||^2 + 2t_{n-1}||u - x^*||||y_{n-1} - x^*|| + t_{n-1}^2||u - x^*||^2 \\
&+ \mu_n||A||^2(2 + \mu_n||A||^2)||x_n - x^*||^2 \\
&+ 2\mu_n\langle x_n - y_n, A^*(I - \text{prox}_{\lambda g})Ax_n - A^*(I - \text{prox}_{\lambda g})Ax^*)\rangle. \qquad (3.15)
\end{aligned}$$

Since $\mu_n \to 0, \ n \to \infty$ and $t_n \to 0, \ n \to \infty$, we obtain that

$$\lim_{n\to\infty}||x_n - y_n|| = 0.$$

Observe that since $\lim_{n\to\infty}||x_n - y_n|| = 0$ and $\{x_{n_j}\}$ weakly converges to z, we also have $x_{n_j} \rightharpoonup z$. Using Lemma 2.2 and (3.11), we have that $z \in F(T)$. Therefore, $z \in F(T) \cap \Gamma$.

Next, we prove that $\{x_n\}$ converges strongly to x^*, where $x^* = P_{F(T)\cap\Gamma}u$. Setting $u_n = (1-\alpha_n)y_n + \alpha_n Ty_n, n \geq 1$, then from (3.1) we have that

$$x_{n+1} = u_n - t_n(y_n - u).$$

It then follows that

$$\begin{aligned} x_{n+1} &= (1-t_n)u_n - t_n(y_n - u_n - u) \\ &= (1-t_n)u_n - t_n\alpha_n(y_n - Ty_n) + t_n u. \end{aligned} \tag{3.16}$$

Also

$$\begin{aligned} ||u_n - x^*||^2 &= ||y_n - x^* - \alpha_n(y_n - Ty_n)||^2 \\ &= ||y_n - x^*||^2 - 2\alpha_n\langle y_n - Ty_n, y_n - x^*\rangle + \alpha_n^2||y_n - Ty_n||^2 \\ &\leq ||y_n - x^*||^2 - \alpha_n[(1-k) - \alpha_n]||y_n - Ty_n||^2 \\ &\leq ||y_n - x^*||^2. \end{aligned} \tag{3.17}$$

By (3.17) and applying Lemma 2.1 (ii) to (3.16), we have

$$||y_{n+1} - x^*||^2 \tag{3.18}$$

$$\begin{aligned} &\leq ||x_{n+1} - x^*||^2 = ||(1-t_n)(u_n - x^*) - t_n\alpha_n(y_n - Ty_n) - t_n(x^* - u)||^2 \\ &\leq (1-t_n)^2||u_n - x^*||^2 - 2t_n\langle \alpha_n(y_n - Ty_n) - (x^* - u), x_{n+1} - x^*\rangle \\ &= (1-t_n)^2||u_n - x^*||^2 - 2t_n\alpha_n\langle y_n - Ty_n, x_{n+1} - x^*\rangle - 2t_n\langle x^* - u, x_{n+1} - x^*\rangle \\ &\leq (1-t_n)^2||y_n - x^*||^2 - 2t_n\alpha_n\langle y_n - Ty_n, x_{n+1} - x^*\rangle - 2t_n\langle x^* - u, x_{n+1} - x^*\rangle \\ &\leq (1-t_n)||y_n - x^*||^2 + t_n[-2\alpha_n\langle y_n - Ty_n, x_{n+1} - x^*\rangle - 2\langle x^* - u, x_{n+1} - x^*\rangle]. \end{aligned} \tag{3.19}$$

We observe that $\limsup_{n\to\infty}\Big\{-2\langle x^* - u, x_{n+1} - x^*\rangle\Big\} = -2\lim_{j\to\infty}\langle x^* - u, x_{n_j} - x^*\rangle \leq 0$ and $2\alpha_n\langle y_n - Ty_n, x_{n+1} - x^*\rangle \to 0$. Now, using Lemma 2.3 in (3.18), we have $||y_n - x^*|| \to 0$ and consequently $||x_n - x^*|| \to 0$. That is $x_n \to x^*, n \to \infty$.

Case 2. Assume that $\{||y_n - x^*||\}$ is not monotonically decreasing sequence. Set $\Gamma_n = ||y_n - x^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k < \Gamma_{k+1}\}.$$

Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} \geq 0, \ \forall n \geq n_0.$$

After a similar conclusion from (3.8), it is easy to see that

$$||y_{\tau(n)} - Ty_{\tau(n)}||^2 \leq \frac{t_{\tau(n)}M}{\alpha_{\tau(n)}[(1-k) - \alpha_{\tau(n)}]} \to 0, n \to \infty.$$

Thus,

$$||y_{\tau(n)} - Ty_{\tau(n)}|| \to 0, n \to \infty.$$

By similar argument as above in Case 1, we conclude immediately that $\{u_{\tau(n)}\}$ and $\{x_{\tau(n)}\}$ converge weakly to z as $n \to \infty$. Also, we can show that

$$\limsup_{n\to\infty} \Big\{ -2\langle x^* - u, x_{\tau(n)+1} - x^*\rangle \Big\} \le 0.$$

At the same time, we note from (3.18) that

$$\begin{aligned} ||y_{\tau(n)+1} - x^*||^2 &\le (1 - t_{\tau(n)})||y_{\tau(n)} - x^*||^2 \\ &+ t_{\tau(n)}[-2\alpha_{\tau(n)}\langle y_{\tau(n)} - Ty_{\tau(n)}, x_{\tau(n)+1} - x^*\rangle \\ &- 2\langle x^* - u, x_{\tau(n)+1} - x^*\rangle]. \end{aligned} \tag{3.20}$$

By Lemma 2.3, we have

$$\lim_{n\to\infty} ||y_{\tau(n)} - x^*|| = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0$$

Furthermore, for $n \ge n_0$, it is easy to see that $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ if $n \ne \tau(n)$ (that is $\tau(n) < n$),because $\Gamma_j \ge \Gamma_{j+1}$ for $\tau(n) + 1 \le j \le n$. As a consequence, we obtain for all $n \ge n_0$,

$$0 \le \Gamma_n \le \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}$$

Hence $\lim \Gamma_n = 0$, that is $\{x_n\}$ converges strongly to x^*. Furthermore, $\{y_n\}$ converges strongly to x^*. This completes the proof.

Corollary 3.2. *Assume that f and g are two proper convex lower-semicontinuous functions and that* (1.6) *is consistent (i.e., $\Gamma \ne \emptyset$). Let T be a nonexpansive mapping of H_1 into itself such that $F(T) \cap \Gamma \ne \emptyset$. Let $\{t_n\}$ be a sequence in* $(0, 1)$, $\{\alpha_n\}$ *a sequence in* $(0, (1 - k)(1 - t_n)) \subset (0, 1)$. *If the parameters satisfy the following conditions*

(*a*) $\lim_{n\to\infty} t_n = 0$;

(*b*) $\sum_{n=1}^{\infty} t_n = \infty$;

(*c*) $\epsilon \le \rho_n \le \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ *for some* $\epsilon > 0$;

(*d*) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

then sequence $\{x_n\}$ generated by (3.1) *converges strongly to $x^* \in F(T) \cap \Gamma$, where* $x^* = P_{F(T)\cap\Gamma}u$.

Remark 3.3. 1. We would like also to emphasize that by taking $f = \delta_C$ [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise], $g = \delta_Q$ the indicator functions of two nonempty closed convex sets C, Q of H_1 and H_2 respectively, our iterative scheme (3.1) reduces to

$$\begin{cases} y_n = P_C(x_n - \mu_n A^*(I - P_Q)Ax_n) \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n Ty_n - t_n(y_n - u), \ n \ge 1, \end{cases}$$

which approximates a solution to problem (1.4) which is also a fixed point of a k-strictly pseudocontractive mapping T.

2. It is worth mentioning that our approach in this section also works for approximation of solution to split equilibrium and split null point problems (considered in [5] and [18], respectively) which is also a fixed point of a k-strictly pseudocontractive mapping. In this case, just replace the proximal mappings of the convex functions f and g by the resolvent operators associated to two monotone equilibrium bifunctions and two maximal monotone operators, respectively.

3.2. **Strong Convergence for Non-convex Minimization Feasibility and fixed point Problem.** In this section, we introduce an iterative scheme for approximating a solution to a problem of finding a point which minimizes a proper convex lower-semicontinuous function f which also a fixed point of a k-strictly pseudocontractive mapping such that its image under a bounded linear operator A minimizes locally lower semicontinuous, prox-bounded and prox-regular function g and prove strong convergence theorem using our iterative scheme for this solution. Our result in this section is new in the sense that this is the first time (as far as we know) that an iterative scheme is introduced and strong convergence theorem is proved using the iterative scheme for common solution to non-convex minimization feasibility problem and fixed point problem for k-strictly pseudocontractive mapping in real Hilbert spaces.
Throughout this section g is assumed to be prox-regular. The following problem

$$0 \in \partial f(\bar{x}) \text{ such that } 0 \in \partial_{pg}(A\bar{x}), \tag{3.21}$$

is very general in the sense that it includes, as special cases, g convex and g lower-C^2 function which is of great importance in optimization and can be locally expressed as a difference $g - \frac{r}{2}||.||^2$, where g is a finite convex function, hence a large core of problems of interest in variational analysis and optimization. As pointed out in [15], there are a lot of non-convex problems in crystallography, astronomy and inverse scattering which need to be solved using fixed point algorithms. In what follows, we shall represent the set of solutions of (3.21) by Γ.
Our interest is in studying the strong convergence properties of the following algorithm for solving problem (3.21):
Algorithm 2. Let T be a k-strictly pseudocontractive mapping of H_1 into itself such that $F(T) \neq \emptyset$. Let $u \in H_1$. Given an initial point $x_1 \in H_1$. Assume that x_n has been constructed and $\theta(x_n) \neq 0$, then compute x_{n+1} via the rule

$$\begin{cases} y_n = \text{prox}_{\lambda_n \mu_n f}(x_n - \mu_n A^*(I - \text{prox}_{\lambda_n g})Ax_n) \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n - t_n(y_n - u), \ n \geq 1, \end{cases} \tag{3.22}$$

where stepsize $\mu_n := \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (3.21) which is also a fixed point of a a k-strictly pseudocontractive mapping T and the iterative process stops, otherwise, we set $n := n + 1$ and go to (3.22).
Theorem 3.4. *Let T be a k-strictly pseudocontractive mapping of H_1 into itself such that $F(T) \neq \emptyset$. Assume that f is a proper convex lower-semicontinuous function, g is locally lower semicontinuous at $A\bar{x}$, prox-bounded and prox-regular at $A\bar{x}$ for $\bar{v} = 0$ with $\bar{x} \in \Gamma \cap F(T) \neq \emptyset$ and A a bounded linear operator which is surjective with a dense*

domain. Let $\{t_n\}$ *be a sequence in* $(0,1)$, $\{\alpha_n\}$ *a sequence in* $(0,(1-k)(1-t_n)) \subset (0,1)$. *If the parameters satisfy the following conditions*

(*a*) $\lim_{n\to\infty} t_n = 0$;

(*b*) $\sum_{n=1}^{\infty} t_n = \infty$;

(*c*) $\epsilon \le \rho_n \le \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ *for some* $\epsilon > 0$;

(*d*) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1-k$;

(*e*) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

and if $||x_1 - \bar{x}||$ *is small enough, then sequence* $\{x_n\}$ *generated by* (3.22) *converges strongly to* $\bar{x} \in F(T)\cap\Gamma$, *where* $\bar{x} = P_{F(T)\cap\Gamma}u$.

Proof. Using the same line of arguments and method of proof given in Theorem 3.4 of Moudafi and Thakur [19], we can show that

$$||y_n-\bar{x}||^2 \le (1+M\lambda_n)||x_n-\bar{x}||^2 - \rho_n\Big(\frac{4h(x_n)}{h(x_n)+l(x_n)} - \rho_n\Big)\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}. \quad (3.23)$$

Using (3.23) in (3.6) (taking into account that $1+x \le e^x, x \ge 0$), we obtain that

$$\begin{aligned}
||x_{n+1}-\bar{x}|| &\le (1-t_n)||y_n-\bar{x}|| + t_n||u-\bar{x}|| \\
&\le (e^{M\lambda_n})^{\frac{1}{2}}\Big(||x_n-\bar{x}|| + t_n||u-\bar{x}||\Big) \\
&\le (e^{M\lambda_n})^{\frac{1}{2}}\Big(\max\Big\{||x_n-\bar{x}||, ||u-\bar{x}||\Big\}\Big) \\
&= e^{\frac{M}{2}\lambda_n}\Big(\max\Big\{||x_n-\bar{x}||, ||u-\bar{x}||\Big\}\Big) \\
&\ \ \vdots \\
&\le e^{\frac{M}{2}\sum_{n=1}^{\infty}\lambda_n}\Big(\max\Big\{||x_1-\bar{x}||, ||u-\bar{x}||\Big\}\Big).
\end{aligned}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are bounded.
Following the method of proof of Theorem 3.1, we can show that

$$\lim_{n\to\infty}(h(x_n)+l(x_n)) = 0 \Leftrightarrow \lim_{n\to\infty} h(x_n) = 0 \text{ and } \lim_{n\to\infty} l(x_n) = 0.$$

If z is a weak cluster point of $\{x_n\}$, then there exists a subsequence $\{x_{n_j}\}$ which weakly converges to z. From the proof of Theorem 3.1, we can show that

(*i*) $0 \in \partial f(z)$ such that $0 \in \partial_{pg}(Az)$,

(*ii*) $||y_n - Ty_n|| \to 0, n \to \infty$,

(*iii*) $\lim_{n\to\infty}||x_n - y_n|| = 0$ and

(*iv*) $z \in F(T)$.

Therefore, $z \in F(T)\cap\Gamma$.

Finally, from (3.22) and (3.23), we have for some $M_1 > 0$ that

$$\begin{aligned} ||y_{n+1} - z||^2 &\leq (1 + M\lambda_{n+1})||x_{n+1} - z||^2 \\ &= (1 + M\lambda_{n+1})||(1 - t_n)(u_n - z) \\ &- t_n\alpha_n(y_n - Ty_n) - t_n(z - u)||^2 \\ &\leq (1 - t_n)^2||u_n - z||^2 \\ &- 2t_n\langle \alpha_n(y_n - Ty_n) - (z - u), x_{n+1} - z\rangle + \lambda_{n+1}M_1 \\ &= (1 - t_n)^2||u_n - z||^2 \\ &- 2t_n\alpha_n\langle y_n - Ty_n, x_{n+1} - z\rangle - 2t_n\langle z - u, x_{n+1} - z\rangle + \lambda_{n+1}M_1 \\ &\leq (1 - t_n)^2||y_n - z||^2 \\ &- 2t_n\alpha_n\langle y_n - Ty_n, x_{n+1} - z\rangle - 2t_n\langle z - u, x_{n+1} - z\rangle + \lambda_{n+1}M_1 \\ &\leq (1 - t_n)||y_n - z||^2 + t_n[-2\alpha_n\langle y_n \\ &- Ty_n, x_{n+1} - z\rangle - 2\langle z - u, x_{n+1} - z\rangle] + \lambda_{n+1}M_1, \end{aligned} \tag{3.24}$$

which concludes that the sequence $\{x_n\}$ strongly converges to z using Lemma 2.3.

Corollary 3.5. *Let T be a nonexpansive mapping of H_1 into itself such that $F(T) \neq \emptyset$. Assume that f is a proper convex lower-semicontinuous function, g is locally lower semicontinuous at $A\bar{x}$, prox-bounded and prox-regular at $A\bar{x}$ for $\bar{v} = 0$ with $\bar{x} \in \Gamma \cap F(T) \neq \emptyset$ and A a bounded linear operator which is surjective with a dense domain. Let $\{t_n\}$ be a sequence in $(0, 1)$, $\{\alpha_n\}$ a sequence in $(0, (1 - t_n)) \subset (0, 1)$. If the parameters satisfy the following conditions*

(a) $\lim_{n\to\infty} t_n = 0$;
(b) $\sum_{n=1}^{\infty} t_n = \infty$;
(c) $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ *for some* $\epsilon > 0$;
(d) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(e) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

and if $||x_1 - \bar{x}||$ is small enough, then sequence $\{x_n\}$ generated by (3.22) converges strongly to $\bar{x} \in F(T) \cap \Gamma$, where $\bar{x} = P_{F(T)\cap\Gamma}u$.

Remark 3.6. 1. All the results in this paper carry over for the case when T is a quasi-nonexpansive mapping (i.e., $||Tx-Tp|| \leq ||x-p||, \forall x \in H_1, p \in F(T)$) and when T is a demicontractive mapping (i.e., $||Tx - Tp||^2 \leq ||x - p||^2 + k||(I - T)x||^2$, $\forall\, x \in H_1, p \in F(T)$).

2. It is worth mentioning here that our result in this paper is more applicable than the result of Moudafi and Thakur [19] in the sense that our result can be applied to finding an approximate common solution to proximal split feasibility problem and fixed point problem for k-strictly pseudocontractive mapping.

3. The assumption that the bounded linear operator A is surjective in Theorem 3.4 is always satisfied in inverse problems in which a priori information is available

about the representation of the target solution in a frame, see for instance [19] and the references therein.

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