Fixed Point Theory, 18(2017), No. 1, 329-350
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON A METHOD OF "CORRECTION" OF MULTI-VALUED MAPS AND ITS APPLICATIONS TO DIFFERENTIAL INCLUSIONS WITH NON-COMPACT RIGHT-HAND SIDES

ELENA PANASENKO

Tambov State University, Tambov, Russia
E-mail: panlena_t@mail.ru

Abstract. Multi-valued maps (acting in metric spaces) having arbitrary images and not necessarily continuous (or semicontinuous) with respect to the Hausdorff distance are considered. For such maps, conditions of existence and continuous dependence on parameters of fixed points are obtained. All the statements are based on the idea of replacing the initial "bad" map with a map that has closed values and is contracting in some neighborhood of a given point. The obtained results are applied then to studying the Cauchy problem for a differential inclusion in finite-dimensional space. For the case when the right-hand side of the inclusion is not necessarily compact-valued or continuous (upper semicontinuous, lower semicontinuous) in the phase variable, theorems on existence of solutions and their continuous dependence on parameters are proved.
Key Words and Phrases: Multi-valued map, fixed point, continuous dependence on parameters, differential inclusion with non-compact right-hand side.
2010 Mathematics Subject Classification: 47H04, 47H10, 34A60.

1. Introduction

Let $\mathbb{Z} \doteq \{... -2, -1, 0, 1, 2, ...\}$; $\mathbb{R}_+ \doteq [0, \infty)$; $\mathbb{R}^n$ be the n-dimensional real space with the norm $|\cdot|$. Let $\mathbb{Y} \subset \mathbb{R}^n$. By $C([a,b], \mathbb{Y})$ we denote the set of all continuous functions from $[a,b]$ into $\mathbb{Y}$, by $AC([a,b], \mathbb{Y})$ the set of all absolutely continuous functions, by $L([a,b], \mathbb{Y})$ the set of all integrable functions, and by $L_\infty([a,b], \mathbb{Y})$ the set of all essentially bounded functions from $[a,b]$ into $\mathbb{Y}$.

Let (X, ϱ_X) be a metric space. Throughout the paper we use the following notation: $S_X(x_0, r) \doteq \{x \in X : \varrho_X(x, x_0) = r\}$, $B^o_X(x_0, r) \doteq \{x \in X : \varrho_X(x, x_0) < r\}$, $B_X(x_0, r) \doteq \{x \in X : \varrho_X(x, x_0) \leqslant r\}$ are, respectively, a sphere, an open, and a closed ball of radius $r > 0$ centered at x_0 in the space X; $S_X(x_0, 0) = B_X(x_0, 0) = \{x_0\}$; $B^o_X(x_0, 0) = \varnothing$; $\overline{M} = X \setminus M$ is the complement to a set $M \subset X$; $\varrho_X(x, M) \doteq \inf_{y \in M} \varrho_X(x, y)$ is the distance from a point x to the set M in X; $d_X(M_1, M_2) \doteq \sup_{x \in M_1} \varrho_X(x, M_2)$ is the excess of a set M_1 over a set M_2;

The work is supported by the state program of the Ministry of Education and Science of the Russian Federation (project No. 1.1809.2017/DP) and by the Russian Fund for Basic Research (project No. 17-01-00553).

$\operatorname{dist}_X(M_1, M_2) = \max\{d_X(M_1, M_2); d_X(M_2, M_1)\}$ is the Hausdorff distance between the sets M_1 and M_2. By $\operatorname{clos}(X)$, $\operatorname{clbd}(X)$, and $\operatorname{comp}(X)$ we denote, respectively, the spaces of all *non-empty closed*, *non-empty closed bounded*, and *non-empty compact* subsets of X.

In the papers [4, 5], generalizations of the Nadler fixed point theorem for multi-valued maps with arbitrary images and no continuity properties were derived. Such maps arise naturally, for example, when one works on controlled systems with non-compact sets of admissible controls and comes to differential inclusions with "bad" right-hand sides. In [4, 5], the authors proposed a method of "correction" of the respective maps. The idea of the method is to construct a map which is "close" to the initial one, but has some properties that allow to study the corresponding modified inclusion, and then to show that the results obtained remain valid for the inclusion generated by the initial map. To implement this idea, one can operate, for example, in the following way (see [4]).

Suppose a multi-valued map $X \ni x \mapsto \Phi(x) \in \operatorname{clos}(X)$ which is not continuous (neither upper nor lower semicontinuous) with respect to the Hausdorff metric, moreover, we can assume that $\operatorname{dist}_X(\Phi(x_1), \Phi(x_2)) = \infty$ for any $x_1, x_2 \in X$. Consider a new "improved" map $\widetilde{\Phi} : X \to \operatorname{clos}(X)$ defined by the equality

$$\widetilde{\Phi}(x) \doteq \Phi(x) \cup \overline{B_X^o(\theta, r)}, \tag{1.1}$$

where $\theta \in X$, $r > 0$. It is obvious that for any $F, G \in \operatorname{clos}(X)$, $\theta \in X$, $r \geqslant 0$, the relations

$$\operatorname{dist}_X\left(F \cup \overline{B_X^o(\theta, r)}, G \cup \overline{B_X^o(\theta, r)}\right) < \infty, \tag{1.2}$$

$$\operatorname{dist}_X\left(F \cup \overline{B_X^o(\theta, r)}, G \cup \overline{B_X^o(\theta, r)}\right) \leqslant \operatorname{dist}_X(F, G) \tag{1.3}$$

take place, so the map $\widetilde{\Phi}$ here may possess already some continuity properties and even be Lipschitz. If it is the case, we get an opportunity to investigate the corresponding equations and inclusions using the known methods; if we can establish the existence of a fixed point $\bar{x}$ of the map $\widetilde{\Phi}$ and prove that $\bar{x} \notin \overline{B_X^o(\theta, r)}$, then we can conclude that $\bar{x}$ is also a fixed point of the initial "bad" map Φ.

In [5], a more general result is obtained: instead of equality (1.1), it is based on a slightly different way of constructing the map $\widetilde{\Phi}$. The method can be used when the images of the initial map Φ are arbitrary subsets of the space X (including an empty set and sets that are not bounded or closed). The map $\widetilde{\Phi}$ is to be defined by the equality

$$\widetilde{\Phi}(x) \doteq \big(\Phi(x) \cap \mathcal{A}(x)\big) \cup \mathcal{B}(x), \tag{1.4}$$

where $\mathcal{A}, \mathcal{B}$ are multi-valued maps acting in X and chosen in a certain manner. This method is rather universal in the sense that for any given maps $\widetilde{\Phi}$ and Φ, there exist the maps $\mathcal{A}, \mathcal{B}$ which allow to represent $\widetilde{\Phi}$ in form (1.4). In the present paper, such a way of "correction" of maps is applied to prove the statements about continuous dependence of fixed points on parameters of the map Φ.

Using the theorems about fixed points of the map Φ based on replacing this map with map (1.4), one can get the statements about solvability of the Cauchy problem for a differential inclusion of the form

$$\dot{x} \in f(t,x) \tag{1.5}$$

with not necessarily compact-valued (in $\mathbb{R}^n$) or continuous right-hand side. The key point here is also replacing the map f, with the help of multi-valued maps α and β, by a map $\widetilde{f}$ defined for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ as

$$\widetilde{f}(t,x) \doteq \big(f(t,x) \cap \alpha(t,x)\big) \cup \beta(t,x). \tag{1.6}$$

In [5], the conditions of solvability of (1.5) were derived by means of construction (1.6) with

$$\alpha(t,x) = B^o_{\mathbb{R}^n}(\theta(t), R_0(t)),\ \ \beta(t,x) = S_{\mathbb{R}^n}(\theta(t), R_0(t)), \tag{1.7}$$

here θ, R_0 are some integrable functions. But such a choice of α and β does not allow to study, for example, the problem

$$\begin{aligned} \dot{x} &= \operatorname{sign} x,\ \ t \geqslant 0, \\ x(0) &= 0, \end{aligned} \tag{1.8}$$

which obviously has solutions. In this work, the conditions of solvability of the Cauchy problem for inclusion (1.5) are studied in a more general situation, when α and β can be chosen quite freely. The obtained result (applicable, in particular, to the problems of type (1.8)) has allowed, in its turn, to prove the theorem about continuous dependence of solutions to (1.5) on parameters.

2. Existence and continuous dependence on parameters of fixed points for a multi-valued map with non-compact images

First, recall some facts about multi-valued maps acting in metric spaces.

Let (Ω, ϱ_Ω), (X, ϱ_X) be metric spaces. We denote by $\Phi : \Omega \multimap X$ a multi-valued map $\Omega \ni \omega \mapsto \Phi(\omega) \subset X$. We also write $\Phi : \Omega \to \operatorname{clos}(X)$, $\Phi : \Omega \to \operatorname{clbd}(X)$, $\Phi : \Omega \to \operatorname{comp}(X)$ if for any $\omega \in \Omega$, the set $\Phi(\omega)$ is, respectively, non-empty closed, non-empty bounded closed, non-empty compact.

Given $q \geqslant 0$ and $\Omega_0 \subset \Omega$, a map $\Phi : \Omega \multimap X$ is called *q-Lipschitz* (or *Lipschitz with the constant q*) *on the set* Ω_0, if

$$\operatorname{dist}_X \big(\Phi(\omega_1), \Phi(\omega_2)\big) \leqslant q\varrho_\Omega(\omega_1, \omega_2) \ \ \forall\, \omega_1, \omega_2 \in \Omega_0. \tag{2.1}$$

In the case when $\Omega = X$ and $q < 1$, a map $\Phi : X \multimap X$ satisfying condition (2.1) is said to be *contracting* or *q-contracting on* $\Omega_0 \subset X$. In these definitions, the set Ω_0 is usually omitted if it coincides with the space X. Given a map $\Phi : X \multimap X$, an element $x \in X$ such that $x \in \Phi(x)$ is called *a fixed point of the map* Φ.

In the following theorem, the method of maps "improvement" mentioned above helps to establish the existence of fixed points for multi-valued maps which are not necessarily contracting.

Theorem 2.1. [5] *Let X be a complete metric space and $\Phi : X \multimap X$. Suppose there exist maps $\mathcal{A} : X \multimap X$, $\mathcal{B} : X \multimap X$ such that for the multi-valued map*

$$X \ni x \mapsto \widetilde{\Phi}(x) \doteq \big(\Phi(x) \cap \mathcal{A}(x)\big) \cup \mathcal{B}(x) \subset X, \tag{2.2}$$

there are $x_0 \in X$, $q \in [0,1)$, *and* $r_0 > 0$ *satisfying the following conditions:*

a) the set $\widetilde{\Phi}(x)$ *is not empty and closed in* X *for every* $x \in B^o_X(x_0, r_0)$;

b) the map $\widetilde{\Phi}$ *is* q*-contracting on the ball* $B^o_X(x_0, r_0)$;

c) $(1-q)^{-1}\varrho_X\big(x_0, \widetilde{\Phi}(x_0)\big) < r_0$.

If for any $x \in B^o_X(x_0, r_0)$, *the set* $\mathcal{B}(x)$ *is either empty or satisfies the inequality*

$$\varrho_X\big(x_0, \mathcal{B}(x)\big) \geqslant r_0, \tag{2.3}$$

then for every r *such that*

$$(1-q)^{-1}\varrho_X\big(x_0, \widetilde{\Phi}(x_0)\big) < r \leqslant r_0, \tag{2.4}$$

the map F *has a fixed point* $\bar{x} \in B^o_X(x_0, r)$.

As it was mentioned above, equality (2.2) defines the most general way of "correction" of maps since for any given map Φ, under an appropriate choice of $\mathcal{A}$, $\mathcal{B}$, this formula gives a required map $\widetilde{\Phi}$. In the paper [5], some recipes for defining maps $\mathcal{A}$, $\mathcal{B}$ were offered; they preserved the inequality

$$\operatorname{dist}_X\big(\widetilde{\Phi}(x_1), \widetilde{\Phi}(x_2)\big) \leqslant \operatorname{dist}_X\big(\Phi(x_1), \Phi(x_2)\big) \quad \forall x_1, x_2 \tag{2.5}$$

(in this case, obviously, the map $\widetilde{\Phi}$ can be called an "improvement" of the initial map Φ).

If X is a linear normed space, then it is convenient to choose $\mathcal{A}(x) \equiv B_X(\theta, r)$, $\mathcal{B}(x) \equiv S_X(\theta, r)$, where $\theta \in X$, $r > 0$; the map

$$\widetilde{\Phi} : X \multimap X, \quad \widetilde{\Phi}(x) \doteq \big(\Phi(x) \cap B_X(\theta, r)\big) \cup S_X(\theta, r), \tag{2.6}$$

will satisfy inequality (2.5). Moreover, if $\Phi(x) \in \operatorname{clos}(X)$, then $\widetilde{\Phi}(x) \in \operatorname{clbd}(X)$. In [4], there was considered an example of the map $\Phi : \mathbb{R} \to \operatorname{clos}(\mathbb{R})$ such that $\operatorname{dist}_{\mathbb{R}}\big(\Phi(x_1), \Phi(x_2)\big) = \infty$ for any $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$; still map (2.6) was contracting, and this allowed to prove the existence of a fixed point for the map Φ.

In the case of an arbitrary metric space X, for inequality (2.5) to be held, it is enough to set $\mathcal{A}(x) \equiv X$, $\mathcal{B}(x) \equiv \mathcal{B} = \text{const}$. For example, letting $\mathcal{B} \doteq \overline{B^o_X(\theta, r)}$, $\theta \in X$, $r > 0$, one can get (see [5]) the conditions of existence of a fixed point obtained in [2].

In [5], it was shown how Theorem 2.1 can be used for studying inclusions in the space of essentially bounded functions. Here we give an example illustrating how Theorem 2.1 can be applied to functional inclusions in the space of continuous functions. In this example, the existence of fixed points is established for a map which is neither continuous no bounded and, moreover, is undefined at some points of the space of continuous functions.

Example 2.2. Let functions $h : [0,1] \to [0,1]$, $g : [0,1] \to \mathbb{R}$ be continuous and $\gamma \geq 0$. Define the multi-valued map

$$\varphi : [0,1] \times \mathbb{R} \multimap \mathbb{R}, \quad \varphi(t,x) = \{y : \ g(t) - \ln|\cos x| \leq y \leq g(t) - \ln|\cos x| + \gamma\}, \tag{2.7}$$

and consider the inclusion

$$x(t) \in \varphi\big(t, x(h(t))\big), \quad t \in [0,1], \tag{2.8}$$

with respect to the unknown function $x \in C\big([0,1],\mathbb{R}\big)$ (in the case of $\gamma = 0$, multi-valued map (2.7) becomes single-valued and inclusion (2.8) becomes an equation). Let us prove that *provided the inequality*

$$\min_{t\in[0,1]}\big|\cos g(h(t))\big| > \sqrt[4]{4/5}\,, \tag{2.9}$$

inclusion (2.8) *has a solution.*

In the space $C \doteq C\big([0,1],\mathbb{R}\big)$ of continuous functions, assume the usual metric $\varrho_C(x_1,x_2) \doteq \max_{t\in[0,1]} |x_1(t) - x_2(t)|$.

Denote $\chi \doteq \big\{\frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}\big\}$ and for any continuous function x, define the set $T_\chi(x) \doteq \{t : x(t) \in \chi\}$. Next, split C into two classes: $\mathcal{C}^0_\chi$ containing the functions for which $T_\chi(x)$ is empty, and $\mathcal{C}^+_\chi$ containing the functions with nonempty $T_\chi(x)$. Define the maps

$$F : C \multimap C,\ \ F(x) = \begin{cases} \big\{\, y :\ y(t) \in \varphi\big(t, x(t)\big)\ \ \forall\, t \,\big\}, & \text{for } x \in \mathcal{C}^0_\chi,\\ \varnothing, & \text{for } x \in \mathcal{C}^+_\chi; \end{cases}$$

$$S_h : C \to C,\ \ (S_h x)(t) = x(h(t));\quad \Phi : C \multimap C,\ \ \Phi = F S_h.$$

Now we can replace inclusion (2.8) by the equivalent inclusion $x \in Fx$ and apply Theorem 2.1 to the latter in order to study its solvability.

Let $\vartheta \doteq \big\{\, x \in \mathbb{R} :\ |\cos x| \le \sqrt{4/5}\,\big\}$. Define the map

$$\psi : [0,1]\times\mathbb{R} \to \operatorname{comp}(\mathbb{R}),\ \ \psi(t,x) = \begin{cases} \varphi(t,x) \cap \big(-\infty,\ g(t) + \ln\sqrt{5/4}\,\big], & \text{for } x \notin \vartheta,\\ \big\{\, g(t) + \ln\sqrt{5/4}\,\big\}, & \text{for } x \in \vartheta, \end{cases}$$

the corresponding Nemytskii operator

$$\Psi : C \to \operatorname{clos}(C),\quad \Psi x = \big\{ y \in C :\ y(t) \in \psi(t, x(t)),\ \forall\, t \in [0,1]\,\big\},$$

and the map $\widetilde{\Phi} : C \to \operatorname{clos}(C)$, $\widetilde{\Phi} = \Psi S_h$ (the graph of the multi-valued map $\psi(t,\cdot)$ is shown in Figure 1).

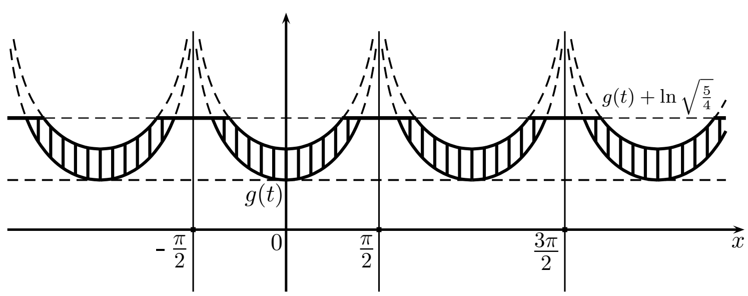

Figure 1. The graph of the multi-valued map $x \to \psi(t,x)$.

Now, we represent the map $\widetilde{\Phi}$ in form (2.2). Let $T_\vartheta(x) \doteq \{t : x(t) \in \vartheta\}$ for any $x \in C$ and suppose two sets of continuous functions:

$$\mathcal{C}^0_\vartheta = \{x \in C :\ T_\vartheta(x) = \varnothing\},\quad \mathcal{C}^+_\vartheta = \{x \in C :\ T_\vartheta(x) \ne \varnothing\}.$$

Define the maps

$$A : C \multimap C,\ \ A(x) = \begin{cases} \{\, y:\ y(t) \le g(t) + \ln\sqrt{5/4} \quad \forall\, t \,\}, & \text{for } x \in \mathcal{C}^0_\vartheta, \\ \varnothing, & \text{for } x \in \mathcal{C}^+_\vartheta; \end{cases}$$

$$B : C \multimap C,\ \ B(x) = \begin{cases} \varnothing, & \text{for } x \in \mathcal{C}^0_\vartheta, \\ \{\, y:\ y(t) \in \psi(t,x) \quad \forall\, t \,\}, & \text{for } x \in \mathcal{C}^+_\vartheta. \end{cases}$$

It is easy to see that the map $\widetilde{\Phi}$ satisfies equality (2.2) with

$$\mathcal{A}, \mathcal{B} : C \multimap C,\ \ \mathcal{A} = AS_h,\ \ \mathcal{B} = BS_h.$$

Note that $\mathcal{B}(x) = \varnothing$ for any $x \in C$ such that $|\cos x(h(t))| > \sqrt{4/5}$, $t \in [0,1]$. If $|\cos x(h(t_0))| \le \sqrt{4/5}$ for some $t_0 \in [0,1]$, then $\mathcal{B}(x) \ne \varnothing$ and $y(t_0) = g(t_0) + \ln\sqrt{5/4}$ for every $y \in \mathcal{B}(x)$.

Let us show that the map $\widetilde{\Phi}$ is contracting with the coefficient $q = 1/2$. First of all, for every $x \notin \vartheta$, the estimate

$$\left| \frac{d}{dt} \ln|\cos x| \right| = |\mathrm{tg}\, x| = \sqrt{\cos^{-2} x - 1} \le \frac{1}{2}$$

takes place, from which it follows that the map $\psi(t,\cdot) : \mathbb{R} \to \mathrm{comp}(\mathbb{R}^n)$ is 1/2-contracting for every $t \in [0,1]$. Pick any $x_1, x_2 \in C$ and denote $\tilde{r} \doteq \varrho_C(x_1,x_2)$. Next, let $y_1 \in \widetilde{\Phi}(x_1)$, i.e., $y_1(t) \in \psi\big(t, x_1(h(t))\big)$ on $[0,1]$. Since the map $\psi(t,\cdot)$ is 1/2-contracting, the set

$$\Gamma(t) \doteq B_{\mathbb{R}}\left(y_1(t), 2^{-1}\tilde{r}\right) \cap \psi\big(t, x_2(h(t))\big) \subset \mathbb{R}$$

is not empty compact and convex for every $t \in [0,1]$; moreover, the map $t \mapsto \Gamma(t)$ is continuous. So there exists a continuous selection, say $y_2(\cdot)$, of Γ such that $\varrho_C(y_1,y_2) \le 2^{-1}\tilde{r}$ and $y_2(t) \in \psi\big(t, x_2(h(t))\big)$ for all $t \in [0,1]$. This means that $\mathrm{dist}_C\left(\widetilde{\Phi}(x_1), \widetilde{\Phi}(x_2)\right) \le 2^{-1}\tilde{r}$, and hence the map $\widetilde{\Phi}$ is 1/2-contracting.

Put $x_0 = g$, then

$$\begin{aligned}(1-q)^{-1}\varrho_C\big(x_0, \widetilde{\Phi}(x_0)\big) &= 2 \max_{t\in[0,1]} \big(\ln\big|\cos g(h(t))\big|^{-1}\big) \\ &= 2\ln\Big(\max_{t\in[0,1]} \big|\cos g(h(t))\big|^{-1}\Big) < \ln\sqrt{5/4}.\end{aligned}$$

Following this estimate, we can take $r_0 = \ln\sqrt{5/4}$. At the same time, as it was shown before, for a continuous function $x : [0,1] \to \mathbb{R}$, the set $\mathcal{B}(x)$ is either empty or satisfies the relation $\varrho_C\big(x_0, \mathcal{B}(x)\big) = \ln\sqrt{5/4}$, so inequality (2.3) takes place. Thus, all the assumptions of Theorem 2.1 are verified, the solvability of inclusion (2.8) is proved: it is shown that for every r such that

$$(1-q)^{-1}\varrho_C\big(x_0, \widetilde{\Phi}(x_0)\big) < r \leqslant \ln\sqrt{5/4},$$

there exists a solution $\bar{x}$ satisfying the inequality $\varrho_C(g, \bar{x}) < r$.

Now we formulate the conditions of continuous dependence of solutions on parameters. First, give some additional notation. Let Σ be a topological space and $\sigma_0 \in \Sigma$; denote $\Sigma^* \doteq \Sigma \backslash \{\sigma_0\}$. Given a map x from Σ^* into a metric space X, we will write

"$x(\sigma) \to x_0$ as $\sigma \to \sigma_0$" meaning that for every $\varepsilon > 0$, there exists a neighborhood $U(\sigma_0) \subset \Sigma$ of σ_0 such that $\varrho_X\big(x(\sigma), x_0\big) < \varepsilon$ for every $\sigma \in U(\sigma_0)$, $\sigma \neq \sigma_0$.

Theorem 2.3. *Suppose Σ is a topological space, X is a complete metric space, $\sigma_0 \in \Sigma$, $x_0 \in X$, and $\Phi : X \times \Sigma \multimap X$. Let maps $\mathcal{A} : X \times \Sigma \multimap X$, $\mathcal{B} : X \times \Sigma \multimap X$ be such that for the map*

$$X \times \Sigma \ni (x, \sigma) \mapsto \widetilde{\Phi}(x, \sigma) \doteq \big(\Phi(x, \sigma) \cap \mathcal{A}(x, \sigma)\big) \cup \mathcal{B}(x, \sigma) \subset X \tag{2.10}$$

and for every $\sigma \in \Sigma^*$, there exist $q(\sigma) \in [0, 1)$ and $r_0(\sigma) > 0$ satisfying the following conditions:

a) the set $\widetilde{\Phi}(x, \sigma)$ is not empty and closed in X for every $x \in B^o_X\big(x_0, r_0(\sigma)\big)$;

b) the map $\widetilde{\Phi}(\cdot, \sigma) : X \multimap X$ is $q(\sigma)$-contracting on the ball $B^o_X\big(x_0, r_0(\sigma)\big)$;

c) $\big(1 - q(\sigma)\big)^{-1} \varrho_X\big(x_0, \widetilde{\Phi}(x_0, \sigma)\big) < r_0(\sigma)$.

Suppose also that for each $\sigma \in \Sigma^*$ and every $x \in B^o_X\big(x_0, r_0(\sigma)\big)$, the set $\mathcal{B}(x, \sigma)$ is either empty or satisfies the inequality

$$\varrho_X\big(x_0, \mathcal{B}(x, \sigma)\big) \geqslant r_0(\sigma). \tag{2.11}$$

Then, provided

$$\big(1 - q(\sigma)\big)^{-1} \varrho_X\big(x_0, \widetilde{\Phi}(x_0, \sigma)\big) \to 0 \tag{2.12}$$

as $\sigma \to \sigma_0$, the map $\Phi(\cdot, \sigma)$, for every $\sigma \in \Sigma^*$, has a fixed point $\bar{x}(\sigma)$ such that $\bar{x}(\sigma) \to x_0$ as $\sigma \to \sigma_0$.

Proof. From assumption c) and relation (2.12) it follows that for every $\sigma \in \Sigma^*$, there exists $r(\sigma) > 0$ such that

$$\big(1 - q(\sigma)\big)^{-1} \varrho_X\big(x_0, \widetilde{\Phi}(x_0, \sigma)\big) < r(\sigma) \leqslant r_0(\sigma)$$

and $r(\sigma) \to 0$ as $\sigma \to \sigma_0$. According to Theorem 2.1, for every $\sigma \in \Sigma^*$, the map $\Phi(\sigma, \cdot) : X \multimap X$ has a fixed point $\bar{x}(\sigma) \in B^o_X\big(x_0, r(\sigma)\big)$. Thus, $\bar{x}(\sigma) \to x_0$ as $\sigma \to \sigma_0$. $\square$

If $r_0(\sigma) = \text{const}$, then assumption c) in Theorem 2.3 is a direct consequence of (2.12) (it is enough to consider some neighborhood $B_\Sigma(\sigma_0) \subset \Sigma$ of the point σ_0 instead of the whole of Σ). Moreover, if the contraction constant does not depend on σ, then the multiplier $(1 - q(\sigma))^{-1}$ in (2.12) does not affect the convergence. So we get the following statement which is straightforward.

Corollary 2.4. *Suppose Σ is a topological space, X is a complete metric space, $\sigma_0 \in \Sigma$, $x_0 \in X$, and $\Phi : X \times \Sigma \multimap X$. Let there exist maps $\mathcal{A} : X \times \Sigma \multimap X$, $\mathcal{B} : X \times \Sigma \multimap X$, and numbers $q \in [0, 1)$, $r_0 > 0$ such that the multi-valued map $\widetilde{\Phi} : X \times \Sigma \multimap X$ defined by* (2.10)*, for every $\sigma \in \Sigma^*$ satisfies the following conditions:*

a) the set $\widetilde{\Phi}(x, \sigma)$ is not empty and closed in X for any $x \in B^o_X\big(x_0, r_0\big)$;

b) the map $\widetilde{\Phi}(\cdot, \sigma) : X \multimap X$ is q-contracting on the ball $B^o_X\big(x_0, r_0\big)$.

Suppose also that for every $\sigma \in \Sigma^*$ and each $x \in B^o_X\big(x_0, r_0\big)$, the set $\mathcal{B}(x, \sigma)$ is either empty or satisfies the inequality

$$\varrho_X\big(x_0, \mathcal{B}(x, \sigma)\big) \geqslant r_0. \tag{2.13}$$

Then given

$$\varrho_X\big(x_0, \widetilde{\Phi}(x_0,\sigma)\big) \to 0 \tag{2.14}$$

as $\sigma \to \sigma_0$, *there exists a neighborhood* $B_\Sigma(\sigma_0)$ *of* $\sigma_0 \in \Sigma$ *such that for every* $\sigma \in B_\Sigma(\sigma_0)$, $\sigma \neq \sigma_0$, *the map* $\Phi(\sigma,\cdot)$ *has a fixed point* $\bar{x}(\sigma)$ *satisfying* $\bar{x}(\sigma) \to x_0$ *as* $\sigma \to \sigma_0$.

Proof. According to (2.14), there exists a neighborhood $B_\Sigma(\sigma_0)$ of the element $\sigma_0 \in \Sigma$ such that for every $\sigma \in B_\Sigma(\sigma_0)$, $\sigma \neq \sigma_0$, the inequality $(1-q)^{-1}\varrho_X\big(x_0, \widetilde{\Phi}(x_0,\sigma)\big) < r_0$ holds. Hence all the conditions of Theorem 2.3 will be fulfilled with the subspace $B_\Sigma(\sigma_0)$ in place of the topological space Σ. □

Theorem 2.3 allows to draw a conclusion on how the set of fixed points of the map $\Phi(\cdot,\sigma)$ depends on the parameter σ. In order to formulate the corresponding corollary, we recall (see, e.g., [1]) that a multi-valued map $\Upsilon : \Sigma \multimap X$ is said to be *lower semicontinuous at a point* $x_0 \in \Sigma$ if for any open set $V \subset X$ satisfying $\Upsilon(\sigma_0) \cap V \neq \varnothing$, there exists a neighborhood $\Sigma(\sigma_0)$ of σ_0 such that $\Upsilon(\sigma) \cap V \neq \varnothing$ for every $\sigma \in \Sigma(\sigma_0)$.

Corollary 2.5. *Let* $H(\sigma)$ *denote the set of fixed points of the map* $\Phi(\cdot,\sigma)$, $\sigma \in \Sigma$. *If* $H(\sigma_0) \neq \varnothing$ *and the assumptions of Theorem 2.3 are complied for every* $x_0 \in H(\sigma_0)$, *then the map* $\sigma \to H(\sigma)$ *is lower semicontinuous at* σ_0.

Let us illustrate how Theorem 2.3 and Corollary 2.4 can be applied to functional inclusions.

Example 2.6. We continue to work on inclusion (2.8) from Example 2.2 and prove the well-posedness of its solutions. The multi-valued maps considered in this example are defined by means of the continuous functions h, g and the number γ. We will get the conditions under which a solution of (2.8) depends continuously on the listed parameters.

So, define the multi-valued map $\varphi : [0,1] \times \mathbb{R} \times C\big([0,1],\mathbb{R}\big) \times \mathbb{R}_+ \multimap \mathbb{R}$,

$$\varphi(t,x,g,\gamma) = \{y : \ g(t) - \ln|\cos x| \le y \le g(t) - \ln|\cos x| + \gamma\}$$

and consider the inclusion

$$x(t) \in \varphi\big(t, x(h(t)), g, \gamma\big), \quad t \in [0,1], \tag{2.15}$$

with respect to the unknown function $x \in C\big([0,1],\mathbb{R}\big)$.

Suppose that for the given set of parameters $\sigma_0 = (h_0, g_0, \gamma_0)$, the inequality

$$\min_{t\in[0,1]} \big|\cos g_0(h_0(t))\big| > \sqrt[4]{4/5} \tag{2.16}$$

takes place, and suppose that $\bar{x}(\sigma_0) \in C\big([0,1],\mathbb{R}\big)$ is a solution of

$$x(t) \in \varphi\big(t, x(h_0(t)), g_0, \gamma_0\big), \quad t \in [0,1],$$

for which $\varrho_C\big(g_0, \bar{x}(\sigma_0)\big) < \ln\sqrt{5/4}$ (the existence of such a solution was established in Example 2.2). Show that *there exists a* $\delta > 0$ *such that for any set of parameters* $\sigma = (h, g, \gamma)$ *satisfying the relations* $\varrho_C(h, h_0) < \delta$ *and* $\varrho_C(g, g_0) < \delta$, *inclusion* (2.15) *has a solutions; moreover, if* $\varrho_C(h, h_0) \to 0$, $\varrho_C(g, g_0) \to 0$, $\gamma \to \gamma_0$, *then for every* $\sigma =$

(h, g, γ), *there exists a solution* $\bar{x}(\sigma)$ *of inclusion* (2.15) *such that* $\varrho_C\big(\bar{x}(\sigma), \bar{x}(\sigma_0)\big) \to 0$.

Put $r_0 = 3^{-1}\left(\ln\sqrt{5/4} - \varrho_C\big(g_0, \bar{x}(\sigma_0)\big)\right)$ and define the metric space

$$\Sigma = \big\{\sigma = (h, g, \gamma) \in C\big([0,1],[0,1]\big) \times C\big([0,1],\mathbb{R}\big) \times \mathbb{R}_+ \ : \ \varrho_C(g, g_0) \le r_0, \ \min_{t\in[0,1]}\big|\cos g(h(t))\big| > \sqrt[4]{4/5}\,\big\}$$

with the distance given by

$$\varrho_\Sigma(\sigma, \bar{\sigma}) \doteq \varrho_C(h, \bar{h}) + \varrho_C(g, \bar{g}) + |\gamma - \bar{\gamma}| = \max_{t\in[0,1]}|h(t) - \bar{h}(t)| + \max_{t\in[0,1]}|g(t) - \bar{g}(t)| + |\gamma - \bar{\gamma}|$$

for any $\sigma = (h, g, \gamma)$, $\bar{\sigma} = (\bar{h}, \bar{g}, \bar{\gamma}) \in \Sigma$. Obviously, Σ is not empty; moreover, we claim that there exists a $\delta > 0$ such that for every $\gamma \in \mathbb{R}_+$ and any $h \in C\big([0,1],[0,1]\big)$, $g \in C\big([0,1],\mathbb{R}\big)$ satisfying the inequalities $\varrho_C(h, h_0) < \delta$, $\varrho_C(g, g_0) < \delta$, one has $\sigma = (h, g, \gamma) \in \Sigma$. Indeed, since $\big|\cos g_0(h_0(t))\big| > \sqrt[4]{4/5}$ for every $t \in [0,1]$, there exists a positive ε_0 such that $\big|\cos\xi\big| > \sqrt[4]{4/5}$ whenever $|\xi - g_0(h_0(t))| < \varepsilon_0$. From the uniform continuity of the function g_0 it follows that there is a $\delta_0 > 0$ such that $|g_0(t_2) - g_0(t_1)| < 2^{-1}\varepsilon_0$ for any t_1, t_2 satisfying the inequality $|t_2 - t_1| < \delta_0$. So, taking $\delta \doteq \min\{\delta_0, 2^{-1}\varepsilon_0\}$, for all $h \in B_C(h_0, \delta)$, $g \in B_C(g_0, \delta)$ we have

$$|g(h(t)) - g_0(h_0(t))| \le |g(h(t)) - g_0(h(t))| + |g_0(h(t)) - g_0(h_0(t))| < \varepsilon_0,$$

and hence $\big|\cos g(h(t))\big| > \sqrt[4]{4/5}$, $\sigma = (h, g, \gamma) \in \Sigma$.

We use the notation from Example 2.2 to denote the sets χ, $\mathcal{C}^0_\chi$, $\mathcal{C}^+_\chi$, ϑ, $\mathcal{C}^0_\vartheta$, $\mathcal{C}^+_\vartheta$, and the map S_h.

Let the map $\Phi : C\big([0,1],\mathbb{R}\big) \times \Sigma \multimap C\big([0,1],\mathbb{R}\big)$ be given by

$$\Phi(x, \sigma) = \begin{cases} \big\{\, y : \ y(t) \in \varphi\big(t, x(h(t)), g, \gamma\big) \ \ \forall\, t \,\big\}, & \text{if } S_h x \in \mathcal{C}^0_\chi, \\ \varnothing, & \text{if } S_h x \in \mathcal{C}^+_\chi. \end{cases}$$

Then inclusion (2.15) can be written in the form $x \in \Phi(x, \sigma)$. Next, define the maps $\psi : [0,1] \times \mathbb{R} \times C\big([0,1],[0,1]\big) \times \mathbb{R}_+ \to \operatorname{comp}(\mathbb{R})$, $\widetilde{\Phi} : C\big([0,1],\mathbb{R}\big) \times \Sigma \to \operatorname{clos}\big(C\,([0,1],\mathbb{R})\big)$ as follows:

$$\psi(t, x, g, \gamma) = \begin{cases} \varphi(t, x, g, \gamma) \cap \big(-\infty,\ g(t) + \ln\sqrt{5/4}\,\big], & \text{for } x \notin \vartheta, \\ \{\, g(t) + \ln\sqrt{5/4}\,\}, & \text{for } x \in \vartheta; \end{cases}$$

$$\widetilde{\Phi}(x, \sigma) = \big\{\, y : \ y(t) \in \psi\big(t, x(h(t)), g, \gamma\big) \ \ \forall\, t \,\big\}.$$

With the maps $\mathcal{A}, \mathcal{B} : C\big([0,1],\mathbb{R}\big) \times \Sigma \multimap C\big([0,1],\mathbb{R}\big)$ given by

$$\mathcal{A}(x, \sigma) = \begin{cases} \big\{\, y : \ y(t) \le g(t) + \ln\sqrt{5/4} \ \ \forall\, t \,\big\}, & \text{if } S_h x \in \mathcal{C}^0_\vartheta, \\ \varnothing, & \text{if } S_h x \in \mathcal{C}^+_\vartheta; \end{cases}$$

$$\mathcal{B}(x, \sigma) = \begin{cases} \varnothing, & \text{if } S_h x \in \mathcal{C}^0_\vartheta, \\ \big\{\, y : \ y(t) \in \psi\big(t, x(h(t)), g, \gamma\big) \ \ \forall\, t \,\big\}, & \text{if } S_h x \in \mathcal{C}^+_\vartheta, \end{cases}$$

the map $\widetilde{\Phi}$ satisfies (2.10).

So we have (see Example 2.2):

a) the set $\widetilde{\Phi}(x,\sigma)$ is closed and bounded in $C\big([0,1],\mathbb{R}\big)$ for any $x \in C\big([0,1],\mathbb{R}\big)$, $\sigma \in \Sigma$;

b) the map $\widetilde{\Phi}(\cdot,\sigma) : C\big([0,1],\mathbb{R}\big) \to \operatorname{clbd}\big(C\big([0,1],\mathbb{R}\big)\big)$ is contracting with the coefficient $q = 1/2$ for every $\sigma \in \Sigma$.

Next, for every $x \in B^o_C\left(\bar{x}(\sigma_0), r_0\right)$, if the set $\mathcal{B}(x,\sigma)$ is not empty, then $\varrho_C(y,g) = \ln\sqrt{5/4}$ for every $y \in \mathcal{B}(x,\sigma)$. Therefore,

$$\begin{aligned}\varrho_C\big(\bar{x}(\sigma_0),\mathcal{B}(x,\sigma)\big) &\geq \varrho_C\big(g,\mathcal{B}(x,\sigma)\big) - \varrho_C(g,g_0) - \varrho_C\big(g_0,\bar{x}(\sigma_0)\big)\\ &\geq \ln\sqrt{5/4} - r_0 - \varrho_C\big(g_0,\bar{x}(\sigma_0)\big)\\ &= \frac{2}{3}\Big(\ln\sqrt{5/4} - \varrho_C\big(g_0,\bar{x}(\sigma_0)\big)\Big) = 2r_0 > r_0,\end{aligned}$$

i.e., inequality (2.13) is satisfied. Finally, if $\varrho_C(h,h_0) \to 0$, $\varrho_C(g,g_0) \to 0$, and $\gamma \to \gamma_0$, then $\operatorname{dist}\big(\widetilde{\Phi}(\bar{x}(\sigma_0),\sigma), \widetilde{\Phi}(\bar{x}(\sigma_0),\sigma_0)\big) \to 0$. So, taking into account the inclusion $\bar{x}(\sigma_0) \in \widetilde{\Phi}(\bar{x}(\sigma_0),\sigma_0)$, we get $\varrho_C\big(\bar{x}(\sigma_0), \widetilde{\Phi}(\bar{x}(\sigma_0),\sigma)\big) \to 0$ as $\sigma \to \sigma_0$. Thus relation (2.14) is satisfied; all the assumptions of Corollary 2.4 are complied.

3. The Cauchy problem for a differential inclusion with non-compact right-hand side

Let there be given a multi-valued map $\varphi : [t_0,\infty)\times\mathbb{R}^n \multimap \mathbb{R}^n$ and a vector $\chi_0 \in \mathbb{R}^n$. Consider the Cauchy problem for the differential inclusion

$$\dot{x} \in \varphi(t,x), \;\; t \geq t_0, \tag{3.1}$$

$$x(t_0) = \chi_0. \tag{3.2}$$

We say that the Cauchy problem *is solvable on the interval* $[t_0,t_0+\tau]$, $\tau > 0$, if there exists a function $x \in AC\big([t_0,t_0+\tau],\mathbb{R}^n\big)$ satisfying condition (3.2) and inclusion (3.1) a.e. on $[t_0,t_0+\tau]$; such a function x is called *a solution* of (3.1), (3.2) *defined on the interval* $[t_0,t_0+\tau]$.

Problem (3.1), (3.2) is equivalent to the inclusion

$$y(t) \in \varphi\left(t,\chi_0 + \int_{t_0}^t y(s)\,ds\right), \quad t \geq t_0, \tag{3.3}$$

with respect to the unknown integrable function $y = \dot{x}$. Theorem 2.1 allows to investigate the solvability of this integral inclusion in situations when the set $\varphi(t,x)$ is defined quite freely (it may be unbounded, non-closed, or even empty). In the work [5], "correction" (2.6) of the multi-valued map generating the right-hand side of inclusion (3.3) was used to study problem (3.1), (3.2). Here we offer the existence result based on more general formula (2.2).

Given $t_1 > t_0$ and functions $R_0 \in L\big([t_0,t_1],\mathbb{R}_+\big)$, $\theta \in L\big([t_0,t_1],\mathbb{R}^n\big)$, let

$$x^*(t) \doteq \chi_0 + \int_{t_0}^t \theta(s)ds, \;\; \gamma(t) \doteq \int_{t_0}^t R_0(s)ds, \;\; t \in [t_0,t_1].$$

We are concerned with the existence of a solution to problem (3.1), (3.2), say $\bar{x}$, such that $\dot{\bar{x}} \in B_{\mathbb{R}^n}(\theta(t),R_0(t))$ for a.e. t. We would like to avoid a trivial situation when the function x^* is itself a solution of (3.1), (3.2), so we assume that

$$\exists\, E \subset [t_0, t_1],\ \mu(E) > 0\ :\ \ \theta(t) \notin \varphi(t, x^*(t))\ \forall\, t \in E. \tag{3.4}$$

Theorem 3.1. *Let there exist maps $\alpha, \beta : [t_0, t_1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ satisfying the following conditions:*

1) *for a.e. $t \in [t_0, t_1]$ and any $x \in B_{\mathbb{R}^n}\,(x^*(t), \gamma(t))$, the set $\beta(t, x)$ is either empty or such that $\varrho_{\mathbb{R}^n}(\theta(t), \beta(t, x)) \geqslant R_0(t)$;*

2) *the map $f : [t_0, t_1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ defined by*

$$f(t, x) \doteq \big(\varphi(t, x) \cap \alpha(t, x)\big) \cup \beta(t, x), \tag{3.5}$$

has non-empty bounded images for a.e. $t \in [t_0, t_1]$ and any $x \in B_{\mathbb{R}^n}\,(x^*(t), \gamma(t))$;

3) *the map $g : [t_0, t_1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ defined by*

$$g(t, p) \doteq f\big(t, x^*(t) + p\gamma(t)\big),$$

is measurable in t for every $p \in B_{\mathbb{R}^n}(0, 1)$;

4) *there holds the estimate*

$$q^* \doteq 1 - \operatorname*{ess\,sup}_{t\in[t_0,t_1]} \frac{\varrho_{\mathbb{R}^n}\big(\theta(t), f(t, x^*(t))\big)}{R_0(t)} > 0; \tag{3.6}$$

5) *there exists a function $k \in L\big([t_0, t_1], \mathbb{R}_+\big)$ such that the function $\nu : [t_0, t_1] \to \mathbb{R}_+$ defined by the equality $\nu(t) \doteq k(t)/R_0(t)$, is essentially bounded and for a.e. $t \in [t_0, t_1]$,*

$$\operatorname{dist}_{\mathbb{R}^n}\big(f(t, x_1), f(t, x_2)\big) \leqslant k(t)|x_1 - x_2| \quad \forall\, x_1, x_2 \in B_{\mathbb{R}^n}\big(x^*(t), \gamma(t)\big). \tag{3.7}$$

Then for any $q \in (0, q^*)$, there is a $\tau > 0$ such that problem (3.1),(3.2) *has a solution $\bar{x}$ defined on the interval $[t_0,\ t_0 + \tau]$ and satisfying the inequality*

$$\operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{|\dot{\bar{x}}(t) - \theta(t)|}{R_0(t)} < \frac{1}{1-q} \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{\varrho_{\mathbb{R}^n}\big(\theta(t), f(t, x^*(t))\big)}{R_0(t)}. \tag{3.8}$$

Proof. Take any $\widetilde{q}, q \in (0, q^*)$ such that $\widetilde{q} < q$. According to the essential boundedness of the function ν and to the property of absolute continuity of the Lebesgue integral, there exists a $\tau \in (0, t_1 - t_0]$ such that

$$\operatorname*{ess\,sup}_{t\in[t_0,\, t_0+\tau]} \Big(\nu(t)\,\gamma(t)\Big) \leqslant \widetilde{q}. \tag{3.9}$$

Consider the functional space

$$\mathcal{L} \doteq \left\{ y : [t_0, t_0 + \tau] \to \mathbb{R}^n\ :\ \operatorname*{ess\,sup}_{t\in[t_0,\, t_0+\tau]} \frac{|y(t) - \theta(t)|}{R_0(t)} < \infty \right\}. \tag{3.10}$$

We stress that an integrable function y belongs to $\mathcal{L}$ if and only if there is a number $\lambda \geqslant 0$ such that $|y(t) - \theta(t)| \leqslant \lambda R_0(t)$ for a.e. $t \in [t_0, t_0 + \tau]$. From this inequality it follows that $\mathcal{L} \subset L\big([t_0, t_0 + \tau], \mathbb{R}^n\big)$. Define a metric in $\mathcal{L}$ by

$$\varrho_{\mathcal{L}}(y, z) \doteq \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{|y(t) - z(t)|}{R_0(t)} \quad \forall\, y, z \in \mathcal{L}\,. \tag{3.11}$$

The metric space $\mathcal{L}$ is isometric to the space of essentially bounded functions $L_\infty\big([t_0, t_0+\tau], \mathbb{R}^n\big)$; the isometry is given by the relations

$$\mathcal{L} \ni y \leftrightarrow z \in L_\infty\big([t_0, t_0+\tau], \mathbb{R}^n\big), \quad z(t) = \frac{y(t)-\theta(t)}{R_0(t)}, \quad y(t) = \theta(t) + z(t)R_0(t).$$

So the space $\mathcal{L}$ is complete. Moreover, it can be easily checked that the convergence $\varrho_{\mathcal{L}}(y_i, y) \to 0$ implies the convergence $|y_i(t) - y(t)| \to 0$ a.e. on $[t_0, t_0+\tau]$. We also note that for an arbitrary function $y \in \mathcal{L}$, if $\varrho_{\mathcal{L}}(y, \theta) < 1$, then $y(t) \in B^o_{\mathbb{R}^n}\big(\theta(t), R_0(t)\big)$ for a.e. $t \in [t_0, t_0+\tau]$.

Consider the following operators: the Nemytskii operator $N : C([t_0, t_0+\tau], \mathbb{R}^n) \multimap \mathcal{L}$ generated by the restriction of the map φ to the interval $[t_0, t_0+\tau]$,

$$Nx \doteq \Big\{y \in \mathcal{L} : y(t) \in \varphi(t, x(t)) \text{ for a.e. } t \in [t_0, t_0+\tau]\Big\}, \tag{3.12}$$

the Nemytskii operator $\widetilde{N} : C([t_0, t_0+\tau], \mathbb{R}^n) \multimap \mathcal{L}$ generated by the restriction of the map f to the interval $[t_0, t_0+\tau]$,

$$\widetilde{N}x \doteq \Big\{y \in \mathcal{L} : y(t) \in f(t, x(t)) \text{ for a.e. } t \in [t_0, t_0+\tau]\Big\}, \tag{3.13}$$

and the integral operator $I : B_{\mathcal{L}}(\theta, 1) \to C([t_0, t_0+\tau], \mathbb{R}^n)$,

$$(Iz)(t) \doteq \chi_0 + \int_{t_0}^t z(s)ds, \quad t \in [t_0, t_0+\tau].$$

Now, our aim is to show that Theorem 2.1 can be applied to the map $\Phi = NI : B_{\mathcal{L}}(\theta, 1) \multimap \mathcal{L}$ with the superposition $\widetilde{N}I : B_{\mathcal{L}}(\theta, 1) \multimap \mathcal{L}$ taken as the map $\widetilde{\Phi}$.

We start by proving that $\widetilde{N}I$ is well-defined and has closed images in $\mathcal{L}$. Suppose the restriction x^*_τ of the function x^* to the interval $t \in [t_0, t_0+\tau]$. Denote $\mathfrak{C} \doteq \big\{x \in C\big([t_0, t_0+\tau], \mathbb{R}^n\big) : x(t) \in B_{\mathbb{R}^n}(x^*_\tau(t), \gamma(t)),\ t \in [t_0, t_0+\tau]\big\}$. It is obvious that the set $\mathfrak{C}$ is not empty and that any function $x \in \mathfrak{C}$ can be defined by the equalities $x(t_0) = \chi_0$ and

$$x(t) = x^*_\tau(t) + p(t)\gamma(t),\ t \in (t_0, t_0+\tau], \tag{3.14}$$

where $p : (t_0, t_0+\tau] \to \mathbb{R}^n$ is continuous and satisfies the estimate $|p(t)| \leqslant 1$ for all t. In what follows it is convenient to extend the function p to the interval $[t_0, t_0+\tau]$ by putting $p(t_0) = 0$; then equality (3.14) will be true for all $t \in [t_0, t_0+\tau]$ and the function $p : [t_0, t_0+\tau] \to \mathbb{R}^n$ will be measurable.

From the definitions of the space $\mathcal{L}$ and the set $\mathfrak{C}$ it follows directly that $I(B_{\mathcal{L}}(\theta, 1)) \subset \mathfrak{C}$. Indeed, if $z \in B_{\mathcal{L}}(\theta, 1)$, then $x = Iz$ satisfies the relations

$$\begin{aligned}|x(t) - x^*_\tau(t)| &= \Big|\int_{t_0}^t z(s)ds - \int_{t_0}^t \theta(s)ds\Big| \leqslant \int_{t_0}^t \frac{|z(s)-\theta(s)|}{R_0(s)} R_0(s)ds \\ &\leqslant \operatorname*{ess\,sup}_{t \in [t_0, t_0+\tau]} \frac{|z(t)-\theta(t)|}{R_0(t)} \int_{t_0}^t R_0(s)ds \leqslant \int_{t_0}^t R_0(s)ds = \gamma(t)\end{aligned}$$

for all $t \in [t_0, t_0+\tau]$.

Next, take an arbitrary function $x \in \mathfrak{C}$ and show that the map $f(\cdot, x(\cdot))$ on the interval $[t_0, t_0+\tau]$ has a selection that belongs to $\mathcal{L}$. From condition (3.7) and the definition of g it follows that, for a.e. $t \in [t_0, t_0+\tau]$, the map $g(t,\cdot) : B_{\mathbb{R}^n}(0,1) \to \operatorname{clos}(\mathbb{R}^n)$ is continuous, moreover, according to assumption 3), the map $g(\cdot,p) : [t_0, t_0+\tau] \to \operatorname{clos}(\mathbb{R}^n)$ is measurable for every $p \in B_{\mathbb{R}^n}(0,1)$. Then for the function $p : [t_0, t_0+\tau] \to \mathbb{R}^n$, continuous on $(t_0, t_0+\tau]$ and satisfying equality (3.14) and the inclusion $p(t) \in B_{\mathbb{R}^n}(0,1)$, $t \in [t_0, t_0+\tau]$, there exists a measurable selection $y : [t_0, t_0+\tau] \to \mathbb{R}^n$ of the map $g(\cdot, p(\cdot))$ (and, as a consequence, of the map $f(\cdot, x(\cdot))$) such that

$$|\theta(t) - y(t)| = \varrho_{\mathbb{R}^n}\big(\theta(t), g(t,p(t))\big) = \varrho_{\mathbb{R}^n}\big(\theta(t), f(t,x(t))\big) \tag{3.15}$$

for a.e. $t \in [t_0, t_0+\tau]$ (see, e.g., [3]). According to (3.7), we have the estimate

$$\operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{\varrho_{\mathbb{R}^n}\big(\theta(t), f(t,x(t))\big)}{R_0(t)} \leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{\varrho_{\mathbb{R}^n}\big(\theta(t), f(t,x^*_\tau(t))\big)}{R_0(t)} + \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{k(t)\,\gamma(t)}{R_0(t)}. \tag{3.16}$$

The function $\gamma(t)$ is bounded on $[t_0, t_0+\tau]$ and $\nu(t) = k(t)/R_0(t)$ is essentially bounded, so taking into account (3.6), from (3.15) and (3.16) we get that

$$\operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{|\theta(t) - y(t)|}{R_0(t)} < \infty.$$

Thus, $y \in \mathcal{L}$, i.e., $\widetilde{N}x \neq \varnothing$ for every $x \in \mathfrak{C}$.

To prove that the set $\widetilde{N}x$ is closed in $\mathcal{L}$ for every $x \in \mathfrak{C}$, consider a sequence $\{y_i\}_{i=1}^{\infty} \subset \widetilde{N}x$ such that $\varrho_{\mathcal{L}}(y_i, y) \to 0$, $i \to \infty$. Then $|y_i(t) - y(t)| \to 0$ for a.e. $t \in [t_0, t_0+\tau]$, and since the set $f(t, x(t))$ is closed in $\mathbb{R}^n$, we have $y(t) \in f(t, x(t))$. This means that $y \in \widetilde{N}x$, hence the set $\widetilde{N}x$ is closed.

Our next move is to verify that the superposition $\widetilde{N}I : B_{\mathcal{L}}(\theta, 1) \to \operatorname{clos}\big(\mathcal{L}\big)$ is q-contracting. Let $x_1, x_2 \in \mathfrak{C}$. Pick an arbitrary $y_1 \in \widetilde{N}x_1$ and consider the ball $B_{\mathbb{R}^n}(y_1(t), r_\varepsilon(t))$ of radius $r_\varepsilon(t) = k(t)|x_1(t) - x_2(t)| + \varepsilon$, where $\varepsilon > 0$. From (3.7) is follows that for a.e. $t \in [t_0, t_0+\tau]$, the set $B_{\mathbb{R}^n}(y_1(t), r_\varepsilon(t)) \cap f(t, x_2(t))$ is not empty. The map $t \mapsto B_{\mathbb{R}^n}(y_1(t), r_\varepsilon(t)) \cap f(t, x_2(t))$ is measurable, therefore it has a measurable selection, say y_2^ε. So, we have $y_2^\varepsilon \in \widetilde{N}x_2$ and $|y_1(t) - y_2^\varepsilon(t)| \leqslant r_\varepsilon(t)$ for a.e. $t \in [t_0, t_0+\tau]$. Then

$$\varrho_{\mathcal{L}}(y_1, y_2^\varepsilon) = \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{|y_1(t) - y_2^\varepsilon(t)|}{R_0(t)} \leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{r_\varepsilon(t)}{R_0(t)}.$$

Analoguosly, for any $y_2 \in \widetilde{N}x_2$, there exists $y_1^\varepsilon \in \widetilde{N}x_1$ such that

$$\varrho_{\mathcal{L}}(y_1^\varepsilon, y_2) \leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{r_\varepsilon(t)}{R_0(t)}.$$

Since $\varepsilon > 0$ has been chosen arbitrary, from these inequalities it follows that

$$\operatorname{dist}_{\mathcal{L}}\big(\widetilde{N}x_1, \widetilde{N}x_2\big) \leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{k(t)|x_1(t) - x_2(t)|}{R_0(t)}. \tag{3.17}$$

Now for any $z_1, z_2 \in B_{\mathcal{L}}(\theta, 1)$, we get the relations

$$\begin{aligned}\operatorname{dist}_{\mathcal{L}}\left(\widetilde{N}Iz_1,\, \widetilde{N}Iz_2\right) &\leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{k(t)}{R_0(t)}\left|\int_{t_0}^{t}(z_1(s)-z_2(s))ds\right| \\ &\leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{k(t)}{R_0(t)}\int_{t_0}^{t} R_0(s)\frac{|z_1(s)-z_2(s)|}{R_0(s)}ds \\ &= \varrho_{\mathcal{L}}(z_1,z_2)\operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{k(t)}{R_0(t)}\int_{t_0}^{t} R_0(s)ds,\end{aligned}$$

and, taking into account estimate (3.9), the inequality

$$\operatorname{dist}_{\mathcal{L}}\left(\widetilde{N}Iz_1, \widetilde{N}Iz_2\right) \leqslant q\varrho_{\mathcal{L}}(z_1,z_2). \tag{3.18}$$

Thus, the map $\widetilde{N}I : B_{\mathcal{L}}(\theta,1) \to \operatorname{clos}(\mathcal{L})$ is contracting.

According to definitions (3.5), (3.13) of the map f and the corresponding Nemytskii operator $\widetilde{N}$, the equality

$$\widetilde{N}(x) = \big(N(x) \cap \mathcal{A}(x)\big) \cup \mathcal{B}(x)$$

holds for any $x \in \mathfrak{C}$, where $\mathcal{A}(x) = \Big\{ y \in \mathcal{L} : y(t) \in \alpha(t, x(t)) \text{ for a.e. } t \in [t_0, t_0+\tau] \Big\}$ and $\mathcal{B}(x)$ is the set of functions having the following property: for $z \in \widetilde{N}(x)$, the inclusion $z \in \mathcal{B}(x)$ is true if and only if there exists a set $E(z) \subset [t_0, t_0+\tau]$ of measure $\mu\big(E(z)\big) > 0$ such that $z(t) \in \beta(t, x(t))$ for a.e. $t \in E(z)$. Let us prove that for any $y \in B_{\mathcal{L}}(\theta,1)$, if $\mathcal{B}Iy \neq \varnothing$, then $\varrho(\theta, \mathcal{B}Iy) \geqslant 1$. Take arbitrary $y \in B_{\mathcal{L}}(\theta,1)$ and $z \in \mathcal{B}(x)$, where $x = Iy$, and estimate $\varrho_{\mathcal{L}}(\theta, z)$. From the definition of the operator $\mathcal{B}$ it follows that

$$\varrho_{\mathcal{L}}(\theta, z) \doteq \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{|\theta(t)-z(t)|}{R_0(t)} \geqslant \operatorname*{ess\,sup}_{t\in E(z)} \frac{|\theta(t)-z(t)|}{R_0(t)}.$$

Since x satisfies the inequality $\varrho_{\mathbb{R}^n}\big(\theta(t), \beta(t, x(t))\big) \geqslant R_0(t)$ a.e. on $[t_0, t_0+\tau]$ (see assumption 1)), we get the relations

$$\operatorname*{ess\,sup}_{t\in E(z)} \frac{|\theta(t)-z(t)|}{R_0(t)} \geqslant \operatorname*{ess\,sup}_{t\in E(z)} \frac{\varrho_{\mathbb{R}^n}(\theta(t), \beta(t,x(t))}{R_0(t)} \geqslant \operatorname*{ess\,sup}_{t\in E(z)} \frac{R_0(t)}{R_0(t)} = 1.$$

Thus, $\varrho_{\mathcal{L}}(\theta, \mathcal{B}Iy) \geqslant 1$ for every $y \in B_{\mathcal{L}}(\theta,1)$.

Let us verify the inequality $(1-\widetilde{q})^{-1}\varrho_{\mathcal{L}}\big(\theta,\, \widetilde{N}I\theta\big) < 1$. The function $\theta(\cdot)$ is measurable, the function $f(\cdot, x^*_\tau(\cdot)) = g(\cdot, 0)$ is measurable with closed values, so there exists a measurable function $u : [t_0, t_0+\tau] \to \mathbb{R}^n$ such that $u(t) \in f(t, x^*_\tau(t))$ and $|\theta(t)-u(t)| = \varrho_{\mathbb{R}^n}\big(\theta(t), f(t, x^*_\tau(t))\big)$ for a.e. $t \in [t_0, t_0+\tau]$. Considering this, inequality (3.6), and the choice of $\widetilde{q}$, we get that $\varrho_{\mathcal{L}}(\theta, u) < 1-\widetilde{q}$. Thus, $u \in \widetilde{N}x^*_\tau = \widetilde{N}I\theta$, and thereby

$$\varrho_{\mathcal{L}}(\theta, \widetilde{N}I\theta) = \inf_{z\in\widetilde{N}I\theta} \varrho_{\mathcal{L}}(\theta, z) \leqslant \varrho_{\mathcal{L}}(\theta, u) < 1-\widetilde{q},$$

i.e., the required inequality holds true.

So, taken $r_0 = 1$, all the assumptions of Theorem 2.1 are satisfied. Denote $\mathfrak{r} \doteq \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \Big[\varrho_{\mathbb{R}^n}\big(\theta(t), f(t,x^*(t))\big)\big(R_0(t)\big)^{-1}\Big]$. According to condition (3.4), $\mathfrak{r} \neq 0$. Then, by Theorem 2.1, for $\varepsilon = \mathfrak{r}(q-\widetilde{q})\big((1-\widetilde{q})(1-q)\big)^{-1}$ and r satisfying the relations

$$r \leqslant 1, \quad \frac{1}{1-\widetilde{q}}\,\varrho_{\mathcal{L}}\big(\theta,\, \widetilde{N}I\theta\big) < r < \frac{1}{1-\widetilde{q}}\,\varrho_{\mathcal{L}}\big(\theta,\, \widetilde{N}I\theta\big) + \varepsilon,$$

there exists a fixed point of the map NI, say $\bar{y}$, such that

$$\operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{|\bar{y}(t)-\theta(t)|}{R_0(t)} < r. \tag{3.19}$$

The function $\bar{x} = I\bar{y}$ is a solution of problem (3.1), (3.2) on the interval $[t_0, t_0+\tau]$. From estimate (3.19) and relation

$$\varrho_{\mathcal{L}}\big(\theta,\, \widetilde{N}I\theta\big) \leqslant \operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{\varrho_{\mathbb{R}^n}\big(\theta(t), f(t,x^*(t))\big)}{R_0(t)}$$

it follows that this solution satisfies the inequality

$$\operatorname*{ess\,sup}_{t\in[t_0,t_0+\tau]} \frac{|\dot{\bar{x}}(t)-\theta(t)|}{R_0(t)} < \frac{\mathfrak{r}}{1-\widetilde{q}} + \frac{\mathfrak{r}(q-\widetilde{q})}{(1-\widetilde{q})(1-q)} = \frac{\mathfrak{r}}{1-q}. \qquad \square$$

Remark 3.2. *Let us point out the major differences of proved Theorem* 3.1 *from the corresponding result of the paper* [5]*. First, it is rather free choice of the maps* α *and* β*; in* [5]*, these maps do not depend on* x *and are defined, for all* (t,x)*, by the equalities* $\alpha(t,x) = B^o_{\mathbb{R}^n}(\theta(t), R_0(t))$*,* $\beta(t,x) = S_{\mathbb{R}^n}(\theta(t), R_0(t))$*. Secondly, assumptions* 1), 2), 5) *of Theorem* 3.1 *are supposed to be held for* x *changing in the ball* $B_{\mathbb{R}^n}(x^*(t), \gamma(t))$ *which center and radius depend on* t*, moreover,* $\gamma(t) \to 0$ *as* $t \to t_0$*; in* [5]*, the analogous assumptions should hold for* x *changing in the ball* $B_{\mathbb{R}^n}(\chi_0, \delta)$ *of constant center and radius, and this, obviously, gives more severe restrictions. The latter refinement is particularly substantial; to confirm this, we consider the following example.*

Example 3.3. Let the map $\varphi : [0,\infty) \times \mathbb{R} \multimap \mathbb{R}$ be given by the equality $\varphi(t,x) \doteq \{\operatorname{sign} x\}$ (so φ can be viewed as a single-valued map). We show that Theorem 3.1 can be applied to the Cauchy problem

$$\dot{x} \in \{\operatorname{sign} x\},\ t \geqslant 0,$$
$$x(0) = 0.$$

Take any $t_1 \in \big(0, \ln(3/2)\big)$ and put $\theta(t) = e^t,\ R_0(t) = 2^{-1}$. Then

$$x^*(t) = e^t - 1,\ \gamma(t) = 2^{-1}t\,, \quad t \in [0, t_1].$$

Next, for $t \in [0, t_1]$, $x \in B_{\mathbb{R}}\big(x^*(t), \gamma(t)\big)$, define

$$\alpha(t,x) = \mathbb{R},\ \beta(t,x) = \varnothing.$$

Since $(e^t - 1) \pm 2^{-1}t > 0$, $t \in (0, t_1]$, we have:

$$f(t,x) = \{1\}, \quad t \in (0, t_1],\ x \in B_{\mathbb{R}}\big(x^*(t), \gamma(t)\big),$$
$$g(t,p) = f(t, e^t - 1 + 2^{-1}pt) = \{1\}, \quad t \in (0, t_1],\ p \in B_{\mathbb{R}}(0,1) = [-1,1].$$

It is obvious that conditions 2) and 3) of Theorem 3.1 are fulfilled. Besides, for all $t \in [0,t_1]$, $x \in B_{\mathbb{R}}\big(x^*(t),\gamma(t)\big)$, the set $\beta(t,x)$ is empty, the relation

$$\operatorname*{ess\,sup}_{t\in[t_0,t_1]} \frac{\varrho_{\mathbb{R}}\big(\theta(t), f(t,x^*(t))\big)}{R_0(t)} = \operatorname*{ess\,sup}_{t\in(0,t_1]} \frac{\varrho_{\mathbb{R}}\big(e^t,\{1\}\big)}{2^{-1}} = 2\big(e^{t_1}-1\big) < 2\Big(\frac{3}{2}-1\Big) = 1$$

takes place, and for all $t \in (0,t_1]$, $x_1,x_2 \in B_{\mathbb{R}}\big(x^*(t),\gamma(t)\big)$, the equality

$$\operatorname{dist}_{\mathbb{R}}\big(f(t,x_1), f(t,x_2)\big) = 0$$

holds true. So, conditions 1), 4), 5) of Theorem 3.1 are also satisfied.

On the other hand, if we try to use here the results of the paper [5] and consider, instead of the ball $B_{\mathbb{R}}\big(e^t-1, 2^{-1}t\big)$, the ball $B_{\mathbb{R}}(0,\delta)$ of any arbitrary small radius $\delta > 0$, then we get $f(t,x) = \{\operatorname{sign} x\}$ for $(t,x) \in [0,t_1] \times B_{\mathbb{R}}(0,\delta)$, i.e., the map $x \to f(t,x)$, for any $t \in [0,t_1]$, is not even continuous on $B_{\mathbb{R}}(0,\delta)$; thus, assumption 5) does not hold.

Now we turn to the problem of continuous dependence of solutions to the Cauchy problem on parameters. Let Σ be a topological space, $\varphi : [t_0,\infty) \times \mathbb{R}^n \times \Sigma \multimap \mathbb{R}^n$, and $\chi_0 : \Sigma \to \mathbb{R}^n$. Consider the problem

$$\dot{x} \in \varphi(t,x,\sigma),\ t \geqslant t_0, \tag{3.20}$$

$$x(t_0) = \chi_0(\sigma). \tag{3.21}$$

Let the interval $[t_0,T] \subset \mathbb{R}$ be fixed, $\sigma_0 \in \Sigma$, $\chi_0^* \in \mathbb{R}^n$ be given, and let the functions $R_0 \in L\big([t_0,T],\mathbb{R}_+\big)$, $\theta \in L\big([t_0,T],\mathbb{R}^n\big)$, $x_0^*(t) \doteq \chi_0^* + \int_{t_0}^t \theta(s)ds$, $t \in [t_0,T]$, be defined. We are concerned with the existence of a solution to problem (3.20), (3.21), say $\bar{x}(\sigma)$, defined on the interval $[t_0,T]$ and such that $\bar{x}(\sigma) \to x_0^*$ in $AC\big([t_0,T],\mathbb{R}^n\big)$ as $\sigma \to \sigma_0$.

Theorem 3.4. *Let there exist maps $\alpha : [t_0,T] \times \mathbb{R}^n \times \Sigma \multimap \mathbb{R}^n$, $\beta : [t_0,T] \times \mathbb{R}^n \times \Sigma \multimap \mathbb{R}^n$ and a number $\delta > 0$ satisfying the following conditions:*

1) *for a.e. $t \in [t_0,T]$, any $x \in B_{\mathbb{R}^n}(x_0^*(t),\delta)$ and $\sigma \in \Sigma^* = \Sigma\backslash\{\sigma_0\}$, the set $\beta(t,x,\sigma)$ is either empty or such that $\varrho_{\mathbb{R}^n}\big(\theta(t),\beta(t,x,\sigma)\big) \geqslant R_0(t)$;*

2) *the map $f : [t_0,T] \times \mathbb{R}^n \times \Sigma \multimap \mathbb{R}^n$ defined by the equality*

$$f(t,x,\sigma) \doteq \big(\varphi(t,x,\sigma) \cap \alpha(t,x,\sigma)\big) \cup \beta(t,x,\sigma),$$

has non-empty closed images for a.e. $t \in [t_0,T]$ and any $x \in B_{\mathbb{R}^n}(x_0^*(t),\delta)$, $\sigma \in \Sigma^*$;

3) *the map $g : [t_0,T] \times \mathbb{R}^n \times \Sigma \multimap \mathbb{R}^n$ defined by the equality*

$$g(t,p,\sigma) \doteq f\big(t, x_0^*(t)+p, \sigma\big),$$

is measurable in t for any $p \in B_{\mathbb{R}^n}(0,\delta)$ and $\sigma \in \Sigma^*$;

4)

$$\operatorname*{ess\,sup}_{t\in[t_0,T]} \frac{\varrho_{\mathbb{R}^n}\big(\theta(t), f(t,x_0^*(t),\sigma)\big)}{R_0(t)} \to 0, \quad \chi_0(\sigma) \to \chi_0^* \tag{3.22}$$

as $\sigma \to \sigma_0$;

5) *there exists a function* $k \in L\big([t_0,T],\mathbb{R}_+\big)$ *such that the function* $\nu : [t_0,T] \to \mathbb{R}_+$ *defined as* $\nu(t) \doteq k(t)/R_0(t)$, *is essentially bounded, and for a.e.* $t \in [t_0,T]$ *and any* $\sigma \in \Sigma^*$, *there holds the relation*

$$\mathrm{dist}_{\mathbb{R}^n}\,\big(f(t,x_1,\sigma), f(t,x_2,\sigma)\big) \leqslant k(t)|x_1 - x_2| \quad \forall\, x_1, x_2 \in B_{\mathbb{R}^n}(x_0^*(t),\delta). \tag{3.23}$$

Then there exists a neighborhood of σ_0 *such that for any* $\sigma \neq \sigma_0$ *from this neighborhood, there is a solution* $\bar{x}(\sigma)$ *of problem* (3.20), (3.21) *on the interval* $[t_0,T]$ *such that* $\bar{x}(\sigma) \to x_0^*$ *in* $AC\big([t_0,T],\mathbb{R}^n\big)$ *as* $\sigma \to \sigma_0$, *moreover,*

$$\operatorname*{ess\,sup}_{t\in[t_0\,,T]} \frac{|\dot{\bar{x}}(t,\sigma) - \theta(t)|}{R_0(t)} \to 0. \tag{3.24}$$

Proof. Without loss of generality, assume $\delta < 1$; take any $q \in (0,1)$, $q^* \in (q,1)$, and $\varepsilon > 0$ satisfying the inequalities

$$\varepsilon < \frac{\delta}{2}, \quad \varepsilon \operatorname*{ess\,sup}_{t\in[t_0,T]} \nu(t) < \frac{1-q^*}{2}. \tag{3.25}$$

Let $Q \doteq (1-q)^{-1} \operatorname*{ess\,sup}_{t\in[t_0,T]} \nu(t)$ (note that by the second estimate in (3.25), it follows that $Q < (2\varepsilon)^{-1}$). According to the essential boundedness of the function ν and to the property of absolute continuity of the Lebesgue integral, there exists a $\tau \in (0, T-t_0]$ such that for any $t \in [t_0,T]$, the following relations take place:

$$\operatorname*{ess\,sup}_{s\in[t,\,t+\tau]\cap[t_0,T]} \left(\nu(s)\int_t^s R_0(\xi)\,d\xi\right) \leqslant q, \tag{3.26}$$

$$\int\limits_{[t,\,t+\tau]\cap[t_0,T]} R_0(\xi)d\xi \leqslant \varepsilon, \tag{3.27}$$

$$1 + Q < 2^{\frac{T-t_0}{\tau}}. \tag{3.28}$$

Denoted $\mathfrak{m} \doteq \min\big\{\, m \in \mathbb{N} : \; m \geqslant \tau^{-1}(T-t_0)\big\}$, we split the interval $[t_0,T]$ into $\mathfrak{m}$ parts by the points $t_i \doteq t_0 + i\tau$, $i = 0,1,...,\mathfrak{m}-1$, $t_{\mathfrak{m}} \doteq T$. For any $i = 0,1,...,\mathfrak{m}-1$, consider the "auxiliary" Cauchy problem

$$\dot{x} \in \varphi(t,x,\sigma), \; t \in [t_i, t_{i+1}], \tag{3.29}$$

$$x(t_i) = \chi. \tag{3.30}$$

We prove that there exists a neighborhood, say $W(\sigma_0)$, of the point σ_0 such that for every $\sigma \in W(\sigma_0)$, $\sigma \neq \sigma_0$, and each $\chi \in B_{\mathbb{R}^n}(x_0^*(t_i),\varepsilon)$, problem (3.29), (3.30) has a solution defined on $[t_i, t_{i+1}]$.

Denote $x^*(t) \doteq \chi + \int_{t_i}^t \theta(s)ds$, $t \in [t_1, t_{i+1}]$. First of all, for arbitrary $\sigma \in \Sigma^*$, $t \in [t_i, t_{i+1}]$ and any $x \in B_{\mathbb{R}^n}\left(x^*(t), \int_{t_i}^t R_0(s)ds\right)$, from estimates (3.25) and (3.27) it follows that

$$|x - x_0^*(t)| \leqslant \big|x - x^*(t)\big| + \big|x^*(t) - x_0^*(t)\big| \leqslant \int_{t_i}^t R_0(s)ds + |\chi - x_0^*(t_i)| < 2\varepsilon = \delta,$$

i.e., we have the inclusion

$$B_{\mathbb{R}^n}\left(x^*(t), \int_{t_i}^{t} R_0(s)ds\right) \subset B_{\mathbb{R}^n}(x_0^*(t),\delta), \quad t\in[t_i,t_{i+1}],\ \sigma\in\Sigma^*.$$

Let $W(\sigma_0)$ be a neighborhood of σ_0 such that

$$\operatorname*{ess\,sup}_{t\in[t_0,T]} \frac{\varrho_{\mathbb{R}^n}\Big(\theta(t), f\big(t,x_0^*(t),\sigma\big)\Big)}{R_0(t)} < \frac{1-q^*}{2} \quad \forall\sigma\in W(\sigma_0),\ \sigma\neq\sigma_0$$

(by (3.22), such a neighborhood does exist). Then, taking into account the Lipschitz condition (3.23) and estimates (3.25), for any $\sigma\in W(\sigma_0)$, $\sigma\neq\sigma_0$, we get

$$\begin{aligned}
\operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]} \frac{\varrho_{\mathbb{R}^n}\Big(\theta(t), f\big(t,x^*(t),\sigma\big)\Big)}{R_0(t)} &\leqslant \operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]} \frac{\varrho_{\mathbb{R}^n}\Big(\theta(t), f\big(t,x_0^*(t),\sigma\big)\Big)}{R_0(t)} \\
&\quad + \operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]} \frac{\operatorname{dist}_{\mathbb{R}^n}\Big(f\big(t,x_0^*(t),\sigma\big), f\big(t,x^*(t),\sigma\big)\Big)}{R_0(t)} \\
&< \frac{1-q^*}{2} + \operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]} \frac{k(t)}{R_0(t)}|x_0^*(t_i)-\chi| \\
&\leqslant \frac{1-q^*}{2} + \frac{1-q^*}{2} = 1-q^*.
\end{aligned}$$

So for every $\sigma\in W(\sigma_0)$, $\sigma\neq\sigma_0$, problem (3.29), (3.30) satisfies all the conditions of Theorem 3.1. This guarantees the existence of a solution $\bar{x}(\sigma)$ to (3.29), (3.30) defined on the interval $[t_i,t_{i+1}]$ of length τ and satisfying the estimate

$$\operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]} \frac{|\dot{\bar{x}}(t,\sigma)-\theta(t)|}{R_0(t)} < \frac{1}{1-q} \operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]} \frac{\varrho_{\mathbb{R}^n}\Big(\theta(t), f\big(t,x^*(t),\sigma\big)\Big)}{R_0(t)}. \tag{3.31}$$

Next, using again the first relation in (3.22), find a neighborhood $V(\sigma_0)\subset W(\sigma_0)$ of σ_0 such that for any $\sigma\in V(\sigma_0)$, $\sigma\neq\sigma_0$, the inequality

$$\frac{1}{1-q} \operatorname*{ess\,sup}_{t\in[t_0,T]} \frac{\varrho_{\mathbb{R}^n}\Big(\theta(t), f\big(t,x_0^*(t),\sigma\big)\Big)}{R_0(t)} < \frac{\varepsilon}{2^{\mathfrak{m}}}$$

holds true. Then provided $|\chi - x_0^*(t_i)| < 2^{-\mathfrak{m}}\varepsilon$, for any $\sigma\in V(\sigma_0)$, $\sigma\neq\sigma_0$, the mentioned solution $\bar{x}(\sigma)$ of problem (3.29), (3.30) satisfies the relation

$$\operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]} \frac{|\dot{\bar{x}}(t,\sigma)-\theta(t)|}{R_0(t)} < \varepsilon. \tag{3.32}$$

Indeed, from the above estimates, and inequalities (3.28) and (3.31) it follows that

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]}\frac{|\dot{\bar{x}}(t,\sigma)-\theta(t)|}{R_0(t)} &< \frac{1}{1-q}\operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]}\frac{\varrho_{\mathbb{R}^n}\Big(\theta(t),f\big(t,x_0^*(t),\sigma\big)\Big)}{R_0(t)}\\ &+\frac{1}{1-q}\operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]}\frac{k(t)}{R_0(t)}|\chi-x_0^*(t_i)|\\ &<\frac{\varepsilon}{2^{\mathfrak{m}}}+Q\frac{\varepsilon}{2^{\mathfrak{m}}}<\frac{\varepsilon}{2^{\mathfrak{m}}}(1+Q)<\varepsilon.\end{aligned}$$

Now, we go back to the initial Cauchy problem (3.20), (3.21). Let us show that there exists a neighborhood of σ_0 such that for any $\sigma\neq\sigma_0$ from this neighborhood, there is a solution of (3.20), (3.21) defined on the interval $[t_0,T]$ and staying, for every t, in the ball $B_{\mathbb{R}^n}\big(x_0^*(t),2^{-\mathfrak{m}}\varepsilon\big)$.

Denote $\bar{\varepsilon}\doteq 2^{-(2\mathfrak{m}+1)}\varepsilon$. According to relations (3.22), there exists a neighborhood $\Sigma(\sigma_0)\subset V(\sigma_0)$ of σ_0 such that for any $\sigma\in\Sigma(\sigma_0)$, $\sigma\neq\sigma_0$, the estimates

$$|\chi_0(\sigma)-\chi_0^*|<\bar{\varepsilon},\quad \frac{1}{1-q}\operatorname*{ess\,sup}_{t\in[t_0,T]}\frac{\varrho_{\mathbb{R}^n}\Big(\theta(t),f\big(t,x_0^*(t),\sigma\big)\Big)}{R_0(t)}<\bar{\varepsilon} \tag{3.33}$$

hold true. For every $\sigma\in\Sigma(\sigma_0)$, $\sigma\neq\sigma_0$, problem (3.20), (3.21) has a solution $\bar{x}(\sigma)$ defined on the interval $[t_0,t_1]$ and satisfying inequality (3.31). Then from (3.33), taking into account (3.25) and (3.27), we get that for a.e. $t\in[t_0,t_1]$,

$$\begin{aligned}|\bar{x}(t,\sigma)-x_0^*(t)| &\leqslant |\chi_0(\sigma)-\chi_0^*|+\int_{t_0}^t|\dot{\bar{x}}(s,\sigma)-\theta(s)|\,ds\\ &<\bar{\varepsilon}+\int_{t_0}^t\frac{|\dot{\bar{x}}(s,\sigma)-\theta(s)|}{R_0(s)}\cdot R_0(s)\,ds\\ <\bar{\varepsilon}+\int_{t_0}^t\frac{1}{1-q}&\operatorname*{ess\,sup}_{s\in[t_0,t_1]}\left(\frac{\varrho_{\mathbb{R}^n}\Big(\theta(s),f\big(s,x_0^*(s),\sigma\big)\Big)}{R_0(s)}+\frac{k(s)}{R_0(s)}\bar{\varepsilon}\right)R_0(s)ds\\ &=\bar{\varepsilon}+\frac{1}{1-q}\operatorname*{ess\,sup}_{t\in[t_0,t_1]}\frac{\varrho_{\mathbb{R}^n}\Big(\theta(t),f\big(t,x_0^*(t),\sigma\big)\Big)}{R_0(t)}\int_{t_0}^tR_0(s)ds+\\ +\bar{\varepsilon}\frac{1}{1-q}&\operatorname*{ess\,sup}_{t\in[t_0,t_1]}\nu(t)\int_{t_0}^tR_0(s)ds<\bar{\varepsilon}+\bar{\varepsilon}+\bar{\varepsilon}Q\varepsilon<2\bar{\varepsilon}+\bar{\varepsilon}=3\bar{\varepsilon}.\end{aligned}$$

Since $|\bar{x}(t_1,\sigma)-x_0^*(t_1)|<3\bar{\varepsilon}<2^{-\mathfrak{m}}\varepsilon$, the solution $\bar{x}(\sigma)$ can be extended on the interval $[t_1,t_2]$. This extension, denoted again by $\bar{x}(\sigma)$, is a solution of the Cauchy problem for inclusion (3.20) with the initial condition

$$x(t_1)=\bar{x}(t_1,\sigma)$$

and satisfies inequality (3.31) on $[t_1, t_2]$. Then for any $t \in [t_1, t_2]$, we get:

$$|\bar{x}(t,\sigma) - x_0^*(t)| \leqslant |\bar{x}(t_1,\sigma) - x_0^*(t_1)| + \int_{t_1}^{t} |\dot{\bar{x}}(t,\sigma) - \theta(t)|\,dt$$
$$< 3\bar{\varepsilon} + \int_{t_1}^{t} \frac{|\dot{\bar{x}}(s,\sigma) - \theta(s)|}{R_0(s)} \cdot R_0(s)\,dt$$
$$< 3\bar{\varepsilon} + \int_{t_1}^{t} \frac{1}{1-q} \operatorname*{ess\,sup}_{s\in[t_1,t_2]} \left(\frac{\varrho_{\mathbb{R}^n}\Big(\theta(s), f\big(s, x_0^*(s), \sigma\big)\Big)}{R_0(s)} + \frac{k(s)}{R_0(s)} 3\bar{\varepsilon} \right) R_0(s) ds$$
$$< 3\bar{\varepsilon} + \bar{\varepsilon} + 3\bar{\varepsilon} = 7\bar{\varepsilon} = (2^0 + 2^1 + 2^2)\bar{\varepsilon}.$$

From this estimate it follows that $|\bar{x}(t_2,\sigma) - x_0^*(t_2)| < 7\bar{\varepsilon} < 2^{-\mathfrak{m}}\varepsilon$, hence, the solution $\bar{x}(\sigma)$ can be extended on the interval $[t_2, t_3]$, and the extension will satisfy, for all $t \in [t_2, t_3]$, the inequality

$$|\bar{x}(t,\sigma) - x_0^*(t)| < 7\bar{\varepsilon} + \bar{\varepsilon} + 7\bar{\varepsilon} = 15\bar{\varepsilon} = \big(2^0 + 2^1 + 2^2 + 2^3\big)\bar{\varepsilon}.$$

Since $|\bar{x}(t,\sigma) - x_0^*(t)| < 15\bar{\varepsilon} < 2^{-\mathfrak{m}}\varepsilon$, the solution $\bar{x}(\sigma)$ can be extended on the interval $[t_3, t_4]$, etc. For any $t \in [t_{\mathfrak{m}-1}, t_{\mathfrak{m}}]$, we get:

$$|\bar{x}(t,\sigma) - x_0^*(t)| < \big(2^0 + 2^1 + 2^2 + ... + 2^{\mathfrak{m}-1} + 2^{\mathfrak{m}}\big)\bar{\varepsilon}$$
$$= \big(2^{\mathfrak{m}+1} - 1\big)\bar{\varepsilon} < 2^{\mathfrak{m}+1}\bar{\varepsilon} = 2^{\mathfrak{m}+1}\frac{\varepsilon}{2^{2\mathfrak{m}+1}} = \frac{\varepsilon}{2^{\mathfrak{m}}}.$$

Thus, for every $\sigma \in \Sigma(\sigma_0)$, $\sigma \neq \sigma_0$, we have a solution, the function $\bar{x}(\sigma)$, of problem (3.20), (3.21) defined on the interval $[t_0, T]$ and satisfying, for all $t \in [t_0, T]$, the inclusion $\bar{x}(t,\sigma) \in B_{\mathbb{R}^n}\big(x_0^*(t), 2^{-\mathfrak{m}}\varepsilon\big)$.

Now, let $\sigma \to \sigma_0$, then (3.24) takes place. In fact, the solution $\bar{x}(\sigma)$, $\sigma \in \Sigma(\sigma_0)$, $\sigma \neq \sigma_0$, for every $i = 0, 1, ..., \mathfrak{m} - 1$, satisfies inequality (3.32). This means that the estimate

$$\operatorname*{ess\,sup}_{t\in[t_0,T]} \frac{|\dot{\bar{x}}(t,\sigma) - \theta(t)|}{R_0(t)} = \max\left\{ \operatorname*{ess\,sup}_{t\in[t_i,t_{i+1}]} \frac{|\dot{\bar{x}}(t,\sigma) - \theta(t)|}{R_0(t)}, i = 0, 1, ..., \mathfrak{m} - 1 \right\} < \varepsilon$$

holds true, so, according to the arbitrariness of ε, we obtain (3.24).

From relation (3.24) it follows that $\dot{\bar{x}}(\sigma) \to \theta$ in $L\big([t_0, T], \mathbb{R}^n\big)$ as $\sigma \to \sigma_0$, and since $|\bar{x}(t_0,\sigma) - x_0^*(t_0)| \to 0$, we get that $\bar{x}(\sigma) \to x_0^*$ in $AC\big([t_0, T], \mathbb{R}^n\big)$ as $\sigma \to \sigma_0$. □

It should be noticed that Theorem 3.4 does not guarantee the existence of solutions for the "limit" problem $\dot{x} \in \varphi(t, x, \sigma_0)$, $x(t_0) = \chi_0^*$, but, if this problem is solvable, then we get the following statement which is a consequence of Theorem 3.4.

Corollary 3.5. *Let $\mathcal{H}(\sigma)$, $\sigma \in \Sigma$, denote the set of solutions to problem* (3.20), (3.21) *defined on the interval $[t_0, T]$. If $\mathcal{H}(\sigma_0) \neq \varnothing$ and the assumptions of Theorem* 3.4 *are complied for every $x_0^* \in \mathcal{H}(\sigma)$, then the map $\sigma \to \mathcal{H}(\sigma)$ is lower semicontinuous at σ_0.*

Note that in Theorem 3.4, conditions 1), 2), 5) should hold for all x from the ball $B_{\mathbb{R}^n}(x_0^*(t), \delta)$ of constant radius $\delta > 0$. In some problems (see Example 3.7 below) these requirements are satisfied when x changes (for every t) in the ball $B_{\mathbb{R}^n}\big(x_0^*(t), \delta\gamma(t)\big)$, where $\gamma(t) \doteq \int_{t_0}^{t} R_0(s)ds$. For $t = t_0$, this ball contains the only point $x_0^*(t_0) = \chi_0^*$. So, if $\chi_0(\sigma) \not\equiv \chi_0^*$, we cannot guarantee the continuous dependence of solutions to (3.20), (3.21) on the parameter σ. But, if the initial data does not depend on σ, i.e., $\chi_0(\sigma) \equiv \chi_0^*$, we get the following statement about continuous dependence of solutions to the Cauchy problem on the right-hand side of the inclusion. The proof is similar to that of Theorem 3.4.

Corollary 3.6. *Suppose the assumptions of Theorem* 3.4 *are fulfilled, where the map* $g : [t_0, T] \times \mathbb{R}^n \times \Sigma \multimap \mathbb{R}^n$ *is defined by the equality* $g(t, p, \sigma) \doteq f\big(t, x_0^*(t) + p\gamma(t), \sigma\big)$ *and the ball* $B_{\mathbb{R}^n}(x_0^*(t), \delta)$ *is replaced by the ball* $B_{\mathbb{R}^n}\big(x_0^*(t), \delta\gamma(t)\big)$. *Then there exists a neighborhood of* σ_0 *such that for every* $\sigma \neq \sigma_0$ *from this neighborhood, the Cauchy problem for inclusion* (3.20) *with the initial condition*

$$x(t_0) = \chi_0^*$$

has a solution $\bar{x}(\sigma)$ *defined on* $[t_0, T]$, *moreover, the convergence* $\bar{x}(\sigma) \to x_0^*$ *in* $AC\big([0, T], \mathbb{R}^n\big)$ *and convergence* (3.24) *take place as* $\sigma \to \sigma_0$.

Example 3.7. Let there be given a topological space Σ, $\sigma_0 \in \Sigma$, and a map $\psi : [0, \infty) \times \mathbb{R} \times \Sigma \to \mathbb{R}$.

Consider the Cauchy problem

$$\dot{x} \in \{\operatorname{sign} x + \psi(t, x, \sigma)\}, \ t \geqslant 0, \tag{3.34}$$

$$x(0) = 0. \tag{3.35}$$

We assume that for some $\delta \in (0, 1)$ and $T > 0$, the map $[0, T] \ni t \mapsto \psi(t, t + pt, \sigma) \in \mathbb{R}$ is measurable for all $p \in B_{\mathbb{R}}(0, \delta)$, $\sigma \in \Sigma^*$.

Put $x_0^*(t) = t$, $t \in [0, T]$, and show that, if

$$\operatorname*{ess\,sup}_{t \in [0, T]} |\psi(t, t, \sigma)| \to 0 \tag{3.36}$$

as $\sigma \to \sigma_0$, then Corollary 3.6 is applicable to problem (3.34), (3.35).

Let $R_0(t) = 1$, $t \in [0, T]$. Then we have:

$$\theta(t) = 1, \ \gamma(t) = t, \ t \in [0, T].$$

For $t \in [0, T]$, $x \in B_{\mathbb{R}}\big(x_0^*(t), \delta\gamma(t)\big) = [t - \delta\, t, t + \delta\, t]$, and $\sigma \in \Sigma^*$ define

$$\alpha(t, x, \sigma) = \mathbb{R}, \ \beta(t, x, \sigma) = \varnothing.$$

For every $t \in (0, T]$, the inequality $t - t\delta > 0$ takes place, therefore, for all $x \in B_{\mathbb{R}}(x_0^*(t), \delta\gamma(t))$, we get $\operatorname{sign} x = 1$ and

$$f(t, x, \sigma) = \{1 + \psi(t, x, \sigma)\}, \ t \in (0, T], \ \sigma \in \Sigma^*.$$

It is obvious that the map $g(\cdot, p, \sigma) : (0, T] \to \mathbb{R}$ defined by the equality $g(t, p, \sigma) = f(t, t + pt, \sigma) = \{1 + \psi(t, t + pt, \sigma)\}$, is measurable for any $p \in B_{\mathbb{R}}(0, \delta)$, $\sigma \in \Sigma^*$. Next, according to (3.36), we have that

$$\operatorname*{ess\,sup}_{t \in [t_0, T]} \frac{\varrho_{\mathbb{R}}\big(\theta(t), f(t, x_0^*(t)), \sigma\big)}{R_0(t)} = \operatorname*{ess\,sup}_{t \in (0, T]} \varrho_{\mathbb{R}}\big(1, \{1 + \psi(t, t, \sigma)\}\big) \to 0 \ \text{ as } \sigma \to \sigma_0.$$

In addition, for any $t \in (0,T]$, $x_1, x_2 \in B_{\mathbb{R}}\big(x_0^*(t), \delta\gamma(t)\big)$, and $\sigma \in \Sigma^*$, the equality

$$\operatorname{dist}_{\mathbb{R}}\big(f(t,x_1,\sigma), f(t,x_2,\sigma)\big) = 0$$

holds true.

So, all the conditions of Corollary 3.6 are complied. This means that for every σ from some neighborhood of the point σ_0, problem (3.34), (3.35) has a solutions $\bar{x}(\sigma)$ which is defined on the interval $[0,T]$ and satisfies the relations $\operatorname{ess\,sup}_{t\in[0,T]} |\dot{\bar{x}}(t,\sigma) - 1| \to 0$ and $\bar{x}(\sigma) \to x_0^*$ in $AC\big([0,T],\mathbb{R}\big)$ as $\sigma \to \sigma_0$.

At the same time, it is easy to check that, if initial condition (3.35) will depend on σ, then the continuous dependence of solutions on the parameter will fail. Indeed, even for the case of $\psi \equiv 0$, if $\chi_0(\sigma) \to 0 - 0$ as $\sigma \to \sigma_0$, then the corresponding solution $\bar{x}(t,\sigma) = -t + \chi_0(\sigma)$, being unique, does not converge to the function $x_0^*(t) = t$. Theorem 3.4 cannot be used in this situation since the map $f(t,\cdot,\sigma)$ is not even continuous on the ball $B_{\mathbb{R}}(x_0^*(t),\delta)$, and condition 5) of the theorem does not hold (see Example 3.3).

Acknowledgement. The author is grateful to Professor E.S. Zhukovskiy for useful remarks brought up in the course of work.

References

[1] Yu.G. Borisovich, B.D. Gelman, A.D. Myshkis, V.V. Obukhovskii, *Introduction to the Theory of Multivalued Maps and Differential Inclusions*, Librokom, Moscow, 2011. (in Russian)

[2] B.D. Gelman, *Multivalued contraction maps and their applications*, Vestn. Voronezh. Univ. Ser. Fiz. Mat., 2009, no. 1, 74-86. (in Russian)

[3] C.J. Himmelberg, F.S. Van Vleck, *Lipschitzian generalized differential equations*, Rend. Sem. Mat. Univ. Padova, **48**(1972), 159-169.

[4] E.S. Zhukovskiy, E.A. Panasenko, *On multi-valued maps with images in the space of closed subsets of a metric space*, Fixed Point Theory and Applications, 2013, 2013:10, 1-21.

[5] E.S. Zhukovskiy, E.A. Panasenko, *On fixed points of multi-valued maps in metric spaces and differential inclusions*, Bulletin of Udmurt University. Mathematics, Mechanics, Computer Science, 2013, no. 2, 12-26.

Received: March 27, 2014; Accepted: May 29, 2014.