Fixed Point Theory, 18(2017), No. 1, 315-328
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINT THEOREMS FOR NON-SELF MAPPINGS WITH NONLINEAR CONTRACTIVE CONDITION IN STRICTLY CONVEX MENGER PM-SPACES

RALE M. NIKOLIĆ*, SINIŠA N. JEŠIĆ** AND NATAŠA A. BABAČEV**

*Belgrade Metropolitan University
Tadeuša Košćuška 63, 11000 Belgrade, Serbia
E-mail: ralevb@open.telekom.rs

**University in Belgrade, Faculty of Electrical Engineering, Department of Mathematics
Bulevar Kralja Aleksandra 73, P.O. Box 35-54, 11120 Beograd, Serbia
E-mail: jesha@eunet.rs; natasa@etf.rs

Abstract. In this paper, existence and uniqueness of a fixed point for non-self mappings with nonlinear contractive condition in the sense of Fang [On φ-contractions in probabilistic and fuzzy metric spaces. Fuzzy Set Syst. (267) 2015, 86–99] will be proved, using the notion of strictly convex structure introduced by Ješić et al. [Ješić, S.N, Nikolić, R.M., Babačev N.A., A Common Fixed Point Theorem in Strictly Convex Menger PM-spaces, Filomat (28)(4) 2014, 735–743] for Menger PM-spaces. As a consequence of main result we will give probabilistic generalization of Assad and Kirk's result [Assad, N.A., Kirk, W.A., Fixed-point theorems for set-valued mappings of contractive type. Pacific J. Math. (43) 1972, 553–562].

Key Words and Phrases: Menger PM-spaces, strictly convex structure, fixed point, non-self mappings.
2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The Banach Contraction Mapping Principle [2] is one of the most important theorems in functional analysis. There are many generalizations of this theorem for classical metric spaces. One of the most important of them is the introduction of a nonlinear contractive principle by Boyd and Wong [3].

The notion of statistical metric spaces, as a generalization of metric spaces, with non-deterministic distance, was introduced by Menger [18] in 1942. Schweizer and Sklar [20, 21] studied the properties of spaces introduced by Menger and gave some basic results on these spaces. They studied topology, convergence of sequences, continuity of mappings, defined the completeness of these spaces, etc. The first result from the fixed point theory in probabilistic metric spaces was obtain by Sehgal and Bharucha–Reid [22] as a generalization of the classical Banach Contraction Mapping Principle.

Ćirić [4] and Jachymski [15] gave a probabilistic version of the fixed point theorem of Boyd and Wong [3]. The main result of Jachymski [15] follows.

Theorem 1.1 ([15]). *Let $(X, \mathcal{F}, T)$ be a complete Menger probabilistic space with continuous t-norm T of H-type (Hadžić type), and let $\varphi : (0, +\infty) \mapsto (0, +\infty)$ be a function satisfying conditions:*

$$0 < \varphi(t) < t \quad \text{and} \quad \lim_{n\to\infty} \varphi^n(t) = 0 \tag{1.1}$$

for every $t > 0$. If $f : X \mapsto X$ is a probabilistic φ-contraction, then f has a unique fixed point $x_* \in X$ and $\{f^n(x_0)\}$ converges to x_*, for every $x_0 \in X$.

Let us recall that a mapping $f : X \mapsto X$ is called a probabilistic φ-contraction (or a φ-contraction in probabilistic metric space) if it satisfies

$$F_{fx,fy}\big(\varphi(t)\big) \geq F_{x,y}(t)$$

for all $x, y \in X$ and every $t > 0$, where function φ is gauge function satisfying certain conditions.

Recently, Fang [9] gave a more general condition for gauge φ function than condition (1.1). Actually, he observed two classes of functions: the class Φ of all functions $\varphi : (0, +\infty) \mapsto (0, +\infty)$ satisfying the condition $\lim_{n\to\infty} \varphi^n(t) = 0$, for every $t > 0$, and the class Φ_W of all functions $\varphi : (0, +\infty) \mapsto (0, +\infty)$ satisfying the condition

$$\text{for every } t > 0 \text{ there exists } r \geq t \text{ such that } \lim_{n\to\infty} \varphi^n(r) = 0. \tag{1.2}$$

Fang [9] has showed that class Φ is a proper subclass of class Φ_W (see Example 3.1 in [9]) and by means of introducing condition (1.2) for function $\varphi : (0, +\infty) \mapsto (0, +\infty)$ Fang has improved and generalized the results of Ćirić [4], Jachymski [15], and Xiao et al. [25].

In 1970 Takahashi [24] was defined convex and normal structures for sets in metric spaces and generalized some important fixed point theorems previously proved for Banach spaces. In 1987 Hadžić [12] introduced the notion of convex structure for sets in Menger probabilistic metric spaces and proved fixed point theorem for mappings in probabilistic metric spaces with a convex structure. Ješić [16] defined convex, strictly convex and normal structure in intuitionistic fuzzy metric spaces. Recently, Ješić et al. [17] have introduced convex, strictly convex and normal structure in Menger PM-spaces.

Furthermore, in convex spaces occur cases where the involved function is not necessarily a self-mapping of a closed subset. Assad and Kirk [1] first considered non-self mappings in a metric spaces (X, d). They proved that for some non-self (single-valued) mapping $f : C \to X$, which satisfied Banach Contraction Mapping Principle $d(fx, fy) \leq \lambda d(x, y)$ for all $x, y \in C$ and $\lambda \in (0, 1)$, where X is complete metrically convex space in the sense of Menger (i.e. for every $x, y \in X, (x \neq y)$, there exists $z \in X$, such that $d(x, y) = d(x, z) + d(z, y)$), then the condition $f(\partial C) \subseteq C$ is sufficient to guarantee the existence of fixed point for mapping f, where ∂C is boundary of set C. In recent years many generalizations of mentioned theorem were proved (see e.g. [5], [6], [7], [10], [11], [13], [14] and [19]).

In this paper, using the notion of strictly convex structure for Menger probabilistic spaces the existence and uniqueness of a fixed point for non-self mappings with nonlinear contractive condition (1.2) for function $\varphi : (0, +\infty) \mapsto (0, +\infty)$, will be proved.

The obtained results hold for Menger probabilistic spaces with arbitrary t-continuous T norm. In the proof of the main result topological characterization of complete spaces with nondeterministic distances will be used. As a consequence of the main result we will give probabilistic generalization of Assad and Kirk's result [1].

2. Preliminaries

In the standard notation, let D^+ be the set of all distribution functions $F : \mathbb{R} \to [0,1]$, such that F is a nondecreasing, left-continuous mapping, which satisfies $F(0) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$. The space D^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. The maximal element for D^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1 ([21])**.** A binary operation $T : [0,1] \times [0,1] \mapsto [0,1]$ is continuous t-norm if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$, and $a,b,c,d \in [0,1]$.

Examples of t-norm are $T(a,b) = \min\{a,b\}$ and $T(a,b) = ab$.

The t-norms are defined recursively by $T^1 = T$ and

$$T^n(x_1,\dots,x_{n+1}) = T(T^{n-1}(x_1,\dots,x_n), x_{n+1})$$

for $n \ge 2$ and $x_i \in [0,1]$ for all $i \in \{1,\dots,n+1\}$.

Definition 2.2. A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X,\mathcal{F},T)$ where X is a nonempty set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x,y), the following conditions hold:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t+s) \ge T\big(F_{x,y}(t), F_{y,z}(s)\big)$, for all $x,y,z \in X$ and $s,t \ge 0$.

Remark 2.3 ([22])**.** Every metric space is a PM-space. Let (X,d) be a metric space and $T(a,b) = \min\{a,b\}$ is a continuous t-norm. Define

$$F_{x,y}(t) = \varepsilon_0\big(t - d(x,y)\big)$$

i.e

$$F_{x,y}(t) = \begin{cases} 0, & t - d(x,y) \le 0, \\ 1, & t - d(x,y) > 0, \end{cases}$$

for all $x,y \in X$ and $t > 0$. The triple $(X,\mathcal{F},T)$ is a PM-space induced by the metric d.

Definition 2.4. Let $(X,\mathcal{F},T)$ be a Menger PM-space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists positive integer N such that $F_{x_n,x}(\varepsilon) > 1-\lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1-\lambda$ whenever $n, m \geq N$.
(3) A Menger PM-space is said to be complete if every Cauchy sequence in X is convergent to a point in X.

The (ε, λ)-topology (see [21]) in a Menger PM-space $(X, \mathcal{F}, T)$ is introduced by the family of neighbourhoods $\mathcal{N}_x$ of a point $x \in X$ given by

$$\mathcal{N}_x = \big\{N_x(\varepsilon,\lambda) : \ \varepsilon > 0, \lambda \in (0,1)\big\}$$

where

$$N_x(\varepsilon,\lambda) = \big\{y \in X : \ F_{x,y}(\varepsilon) > 1-\lambda\big\}.$$

The (ε, λ)-topology is a Hausdorff topology. In this topology the function f is continuous in $x_0 \in X$ if and only if for every sequence $x_n \to x_0$ it holds that $f(x_n) \to f(x_0)$.

The following Lemma is proved by Schweizer and Sklar.

Lemma 2.5 ([21])**.** *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then the function $\mathcal{F}$ is lower semi-continuous for every fixed $t > 0$, i.e. for every fixed $t > 0$ and every two convergent sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq X$ such that $x_n \to x, y_n \to y$, for $n \to \infty$, it follows that*

$$\liminf_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t).$$

Definition 2.6. Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $A \subseteq X$. The closure of the set A is the smallest closed set containing A, denoted by $\overline{A}$.

Obviously, keeping in mind the Hausdorff topology, and the definition of converging sequences we note that the next remark holds.

Remark 2.7. $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A such that $x_n \to x$, for $n \to \infty$.

The concept of probabilistic boundedness was introduced by Egbert [8]. A version on this definition follows.

Definition 2.8. Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $A \subseteq X$. The probabilistic diameter of set A is given by

$$\delta_A(t) = \sup_{\varepsilon < t} \inf_{x,y\in A} F_{x,y}(\varepsilon).$$

The diameter of the set A is defined by

$$\delta_A = \sup_{t>0} \delta_A(t).$$

If there exists $\lambda \in (0,1)$ such that $\delta_A = 1-\lambda$ the set A will be called probabilistic semi-bounded. If $\delta_A = 1$ the set A will be called probabilistic bounded.

Lemma 2.9. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. A set $A \subseteq X$ is probabilistic bounded if and only if for each $\lambda \in (0,1)$ there exists $t > 0$ such that $F_{x,y}(t) > 1 - \lambda$ for all $x, y \in A$.*

Proof. The proof follows from the definitions of $\sup A$ and $\inf A$ of nonempty sets. □

Remark 2.10. It is not difficult to see that every metrically bounded set is also probabilistic bounded if it is considered in the induced PM-space.

Sherwood has proved the following theorem.

Theorem 2.11 ([23])**.** *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $\{F_n\}_{n\in\mathbb{N}}$ a nested sequence of nonempty, closed subsets of X such that $\delta_{F_n} \to \varepsilon_0$, for $n \to \infty$. Then there is exactly one point $x_0 \in F_n$, for every $n \in \mathbb{N}$.*

It is easy to show that the following lemma holds.

Lemma 2.12 ([23])**.** *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Let $\{F_n\}_{n\in\mathbb{N}}$ be a nested sequence of nonempty, closed subsets of X. The sequence $\{F_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero i.e. for each $\lambda \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $F_{x,y}(t) > 1 - \lambda$ for all $x, y \in F_{n_0}$ if and only if $\delta_{F_n} \to \varepsilon_0$, for $n \to \infty$.*

3. Convex structure and strictly convex structure in Menger PM-spaces

Takahashi [24] introduced the notion of metric spaces with a convex structure. This class of metric spaces includes normed linear spaces and metric spaces of the hyperbolic type.

Definition 3.1. Let (X, δ) be a metric space. We say that a metric space possesses a Takahashi's convex structure if there exists a function $W : X \times X \times [0,1] \mapsto X$ which satisfies

$$\delta\big(z, W(x,y,\theta)\big) \le \theta\delta(z,x) + (1-\theta)\delta(z,y)$$

for all $x, y, z \in X$ and arbitrary $\theta \in [0,1]$. A metric space (X, δ) with Takahashi's convex structure is called convex metric space.

Hadžić [12] introduced a generalization of the Takahashi's definition to the case of a Menger PM-space.

Definition 3.2. Let $(X, \mathcal{F}, T)$ be a Menger PM-space. A mapping $S : X \times X \times [0,1] \mapsto X$, is said to be a convex structure on X if for every $(x,y) \in X \times X$ holds $S(x,y,0) = y$, $S(x,y,1) = x$ and for all $x, y, z \in X, \theta \in (0,1)$ and $t > 0$

$$F_{S(x,y,\theta),z}(2t) \ge T\left(F_{x,z}\left(\frac{t}{\theta}\right), F_{y,z}\left(\frac{t}{1-\theta}\right)\right). \tag{3.1}$$

Example 3.3. Mapping $S : \mathbb{R} \times \mathbb{R} \times [0,1] \mapsto \mathbb{R}$, where $\mathbb{R} = (-\infty, +\infty)$, defined by

$$S(x,y,\theta) = \theta x + (1-\theta)y$$

for all $x, y \in \mathbb{R}$ and $\theta \in (0,1)$ is a convex structure on Menger PM-space $(\mathbb{R}, \mathcal{F}, T_{min})$ induced by a metric $d(x,y) = |x-y|$ on $\mathbb{R}$ where $T_{min}(a,b) = \min\{a,b\}$ is a continuous t-norm for $a, b \in [0,1]$, and

$$F_{x,y}(t) = \begin{cases} 0, & t - d(x,y) \le 0, \\ 1, & t - d(x,y) > 0. \end{cases}$$

for all $x, y \in \mathbb{R}$ and $t > 0$.

Let us prove this assertion. Firstly, we have that $S(x,y,0) = y$ and $S(x,y,1) = x$ for all $x, y \in \mathbb{R}$. Now, let us prove that inequality *(3.1)* is satisfied. If we assume that

$$T_{min}\left(F_{x,z}\left(\frac{t}{\theta}\right), F_{y,z}\left(\frac{t}{1-\theta}\right)\right) = \min\left(F_{x,z}\left(\frac{t}{\theta}\right), F_{y,z}\left(\frac{t}{1-\theta}\right)\right) = 0$$

then inequality *(3.1)* is a trivially satisfied because we get $F_{S(x,y,\theta),z}(2t) \ge 0$. Now, we will assume that $F_{x,z}\left(\frac{t}{\theta}\right) = 1$ and $F_{y,z}\left(\frac{t}{1-\theta}\right) = 1$. Then we have that $\frac{t}{\theta} > d(x,z)$ and $\frac{t}{1-\theta} > d(y,z)$ i.e. $t > \theta d(x,z)$ and $t > (1-\theta)d(y,z)$. Hence, we get

$$\begin{aligned} 2t &> \theta d(x,z) + (1-\theta)d(y,z) = \theta|x-z| + (1-\theta)|y-z| \\ &\ge \big|\theta x - \theta z + (1-\theta)y - (1-\theta)z\big| \\ &= \big|\theta x + (1-\theta)y - z\big| = d\big(\theta x + (1-\theta)y, z\big) \\ &= d\big(S(x,y,\theta), z\big), \end{aligned}$$

i.e. $2t - d\big(S(x,y,\theta),z\big) > 0$ i.e $F_{S(x,y,\theta),z}(2t) = 1$, i.e. inequality *(3.1)* holds for all $x, y, z \in \mathbb{R}$ and $t > 0$.

Remark 3.4. It is easy to see that every metric space (X,d) with a convex structure S can be consider as a Menger PM-space $(X, \mathcal{F}, T_{min})$ (the associated Menger PM-space) with the same function S.

Definition 3.5. Let $(X, \mathcal{F}, T)$ be a Menger PM-space with a convex structure $S(x,y,\theta)$. A subset $A \subseteq X$ is said to be a convex set if for every $x, y \in A$ and $\theta \in [0,1]$ it follows that $S(x,y,\theta) \in A$.

Recently, the notion of strictly convex structure was introduced by Ješić et al. [17].

Definition 3.6 ([17])**.** A convex Menger PM-space $(X, \mathcal{F}, T)$ with a convex structure $S : X \times X \times [0,1] \mapsto X$ will be called strictly convex if, for arbitrary $x, y \in X$ and $\theta \in (0,1)$ the element $z = S(x,y,\theta)$ is the unique element which satisfies

$$F_{x,y}\left(\frac{t}{1-\theta}\right) = F_{z,x}(t), \quad F_{x,y}\left(\frac{t}{\theta}\right) = F_{z,y}(t) \tag{3.2}$$

for all $t > 0$.

Lemma 3.7 ([17])**.** *Let $(X, \mathcal{F}, T)$ be a Menger PM-space with a convex structure $S(x,y,\theta)$. Suppose that for every $\theta \in (0,1)$, $t > 0$ and $x, y, z \in X$ hold*

$$F_{S(x,y,\theta),z}(t) > \min\big\{F_{z,x}(t), F_{z,y}(t)\big\}. \tag{3.3}$$

If there exists $z \in X$ such that

$$F_{S(x,y,\theta),z}(t) = \min\big\{F_{z,x}(t), F_{z,y}(t)\big\} \tag{3.4}$$

is satisfied for all $t > 0$, *then* $S(x, y, \theta) \in \{x, y\}$.

Example 3.8. Menger PM-space $(\mathbb{R}, \mathcal{F}, T_{min})$ with a convex structure

$$S(x, y, \theta) = \theta x + (1 - \theta)y$$

for all $x, y \in \mathbb{R}$ and $\theta \in (0, 1)$ from Example 3.3 is a strictly convex space in the sense of Definition 3.6 satisfying condition *(3.3)*.

Let us prove this assertion. For arbitrary $x, y \in \mathbb{R}$ and $\theta \in (0, 1)$ the element $z = S(x, y, \theta) = \theta x + (1 - \theta)y$ is the unique element which satisfies

$$\begin{aligned} F_{z,x}(t) &= \varepsilon_0\left(t - d(x, z)\right) \\ &= \varepsilon_0\big(t - d(x - \theta x, z - \theta x)\big) \\ &= \varepsilon_0\left(\frac{t}{1-\theta} - \frac{d\big((1-\theta)x, z - \theta x\big)}{1-\theta}\right) \\ &= \varepsilon_0\left(\frac{t}{1-\theta} - d\left(x, \frac{z}{1-\theta} - \frac{\theta x}{1-\theta}\right)\right) \\ &= \varepsilon_0\left(\frac{t}{1-\theta} - d(x, y)\right) \\ &= F_{x,y}\left(\frac{t}{1-\theta}\right). \end{aligned}$$

In one of previous equalities we used that

$$y = \frac{z}{1-\theta} - \frac{\theta x}{1-\theta},$$

which is obtained from $z = \theta x + (1 - \theta)y$. In the similar way it can be proved that the second equality in *(3.2)* is satisfied. Hence, we obtain that observed Menger PM-space is strictly convex in the sense of Definition 3.6 with a given convex structure $S(x, y, \theta)$.

On the other hand, we have that

$$d\big(\theta x + (1-\theta)y, z\big) < \max\big\{d(x, z), d(y, z)\big\}$$

is satisfied for all $\theta \in (0, 1)$, and it follows that

$$\begin{aligned} F_{S(x,y,\theta),z}(t) &= \varepsilon_0\Big(t - d\big(S(x, y, \theta), z\big)\Big) \\ &> \varepsilon_0\big(t - \max\{d(x, z), d(y, z)\}\big) \\ &= \min\Big\{\varepsilon_0\big(t - d(x, z)\big), \varepsilon_0\big(t - d(y, z)\big)\Big\} \\ &= \min\big\{F_{z,x}(t), F_{z,y}(t)\big\} \end{aligned}$$

holds i.e. condition *(3.3)* is satisfied.

Lemma 3.9 ([17])**.** *Let* $(X, \mathcal{F}, T)$ *be a strictly convex Menger PM-space with a convex structure* $S(x, y, \theta)$. *Then for arbitrary* $x, y \in X, x \neq y$ *there exists* $\theta \in (0, 1)$ *such that* $S(x, y, \theta) \notin \{x, y\}$.

4. Main results

Fang [9] has proved the following lemma.

Lemma 4.1 ([9]). *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and $x, y \in X$. If there exists a function $\varphi \in \Phi_W$, such that*

$$F_{x,y}\big(\varphi(t)\big) \geq F_{x,y}(t) \tag{4.1}$$

holds for every $t > 0$, then $x = y$.

Now, we can formulate and prove the main result of this paper.

Theorem 4.2. *Let $(X, \mathcal{F}, T)$ be a strictly convex, complete Menger PM-space with convex structure $S : X \times X \times [0,1] \mapsto X$ satisfying* (3.3). *Let $f : C \to X$ be a non-self mapping satisfying*

$$F_{fx,fy}\big(\varphi(t)\big) \geq F_{x,y}(t) \tag{4.2}$$

for all $x, y \in C$ and every $t > 0$, where $\varphi : (0, +\infty) \mapsto (0, +\infty)$ satisfying condition (1.2) *and C is a nonempty, closed and probabilistic bounded subset of X. Additionally, suppose that f has the property*

$$f(\partial C) \subseteq C. \tag{4.3}$$

Then f has a unique fixed point in C.

Proof. Let $x \in \partial C$ be an arbitrary point. We shall construct the sequence $\{x_n\}$ as follows. Set $x_0 = x$. Since $x \in \partial C$, by (4.3) $fx_0 \in C$. Set $x_1 = fx_0$. Define $y_2 = fx_1$. If $y_2 \in C$, set $x_2 = y_2$. If $y_2 \notin C$ let us choose $x_2 \in \partial C$ so that $x_2 = S(x_1, y_2, \theta), \theta \in (0,1)$. Continuing in this manner, we obtain than sequence $\{x_n\}$ satisfying

$$\begin{aligned} x_n &= fx_{n-1}, && \text{if} \quad fx_{n-1} \in C, \\ x_n &= S(x_{n-1}, fx_{n-1}, \theta), \theta \in (0,1) && \text{if} \quad fx_{n-1} \notin C. \end{aligned} \tag{4.4}$$

Notice that if $x_n = S(x_{n-1}, fx_{n-1}, \theta), \theta \in (0,1)$, then obviously $x_{n+1} = fx_n$ and $x_{n-1} = fx_{n-2}$, for $n = 2, 3, 4, \ldots$

Let us consider nested sequence of nonempty closed sets defined by

$$G_n = \{x_n, x_{n+1}, \ldots\} \quad \text{and} \quad F_n = \overline{G_n},\ n \in \mathbb{N}.$$

We shall prove that family $\{F_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero.

Firstly, let us prove that:

$$\delta_{G_n}\big(\varphi(t)\big) \geq \delta_{G_{n-2}}(t) \tag{4.5}$$

holds for every $t > 0$. Hence, we will observe the following three cases that are all of the possibilities:

<u>Case 1:</u> $x_{n+p} = fx_{n+p-1}$ and $x_{n+q} = fx_{n+q-1}$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$;
<u>Case 2:</u> $x_{n+p} = fx_{n+p-1}$ and $x_{n+q} = S(x_{n+q-1}, fx_{n+q-1}, \theta), \theta \in (0,1)$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$;
<u>Case 3:</u> $x_{n+p} = S(x_{n+p-1}, fx_{n+p-1}, \theta_1)$, $\theta_1 \in (0,1)$, and $x_{n+q} = S(x_{n+q-1}, fx_{n+q-1}, \theta_2)$, $\theta_2 \in (0,1)$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$.

Case 1: If $x_{n+p} = fx_{n+p-1}$ and $x_{n+q} = fx_{n+q-1}$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$, from (4.2) we have

$$\begin{aligned} F_{x_{n+p},x_{n+q}}\big(\varphi(t)\big) &= F_{fx_{n+p-1},fx_{n+q-1}}\big(\varphi(t)\big) \\ &\geq F_{x_{n+p-1},x_{n+q-1}}(t) \\ &\geq \delta_{G_{n-2}}(t). \end{aligned} \tag{4.6}$$

Case 2: If $x_{n+p} = fx_{n+p-1}$ and $x_{n+q} = S(x_{n+q-1}, fx_{n+q-1}, \theta)$, $\theta \in (0,1)$ for arbitrary $p, q \in \mathbb{N} \cup \{0\}$, then from (3.3) and (4.2) we have

$$\begin{aligned} F_{x_{n+p},x_{n+q}}\big(\varphi(t)\big) &= F_{fx_{n+p-1},S(x_{n+q-1},fx_{n+q-1},\lambda)}\big(\varphi(t)\big) \\ &> \min\Big\{F_{fx_{n+p-1},x_{n+q-1}}\big(\varphi(t)\big), F_{fx_{n+p-1},fx_{n+q-1}}\big(\varphi(t)\big)\Big\} \\ &= \min\Big\{F_{fx_{n+p-1},fx_{n+q-2}}\big(\varphi(t)\big), F_{fx_{n+p-1},fx_{n+q-1}}\big(\varphi(t)\big)\Big\} \\ &\geq \min\Big\{F_{x_{n+p-1},x_{n+q-2}}(t), F_{x_{n+p-1},x_{n+q-1}}(t)\Big\} \\ &\geq \delta_{G_{n-2}}(t). \end{aligned} \tag{4.7}$$

Case 3: If $x_{n+p} = S(x_{n+p-1}, fx_{n+p-1}, \theta_1)$, $\theta_1 \in (0,1)$, and

$$x_{n+q} = S(x_{n+q-1}, fx_{n+q-1}, \theta_2),\ \ \theta_2 \in (0,1)$$

for arbitrary $p, q \in \mathbb{N} \cup \{0\}$, then from (3.3) and (4.2) we have

$$\begin{aligned} F_{x_{n+p},x_{n+q}}\big(\varphi(t)\big) &= F_{S(x_{n+p-1},fx_{n+p-1},\lambda),S(x_{n+q-1},fx_{n+q-1},\lambda)}\big(\varphi(t)\big) \\ &> \min\Big\{F_{x_{n+p-1},x_{n+q-1}}\big(\varphi(t)\big), F_{x_{n+p-1},fx_{n+q-1}}\big(\varphi(t)\big), \\ &\qquad F_{fx_{n+p-1},x_{n+q-1}}\big(\varphi(t)\big), F_{fx_{n+p-1},fx_{n+q-1}}\big(\varphi(t)\big)\Big\} \\ &= \min\Big\{F_{fx_{n+p-2},fx_{n+q-2}}\big(\varphi(t)\big), F_{fx_{n+p-2},fx_{n+q-1}}\big(\varphi(t)\big), \\ &\qquad F_{fx_{n+p-1},fx_{n+q-2}}\big(\varphi(t)\big), F_{fx_{n+p-1},fx_{n+q-1}}\big(\varphi(t)\big)\Big\} \\ &\geq \min\Big\{F_{x_{n+p-2},x_{n+q-2}}(t), F_{x_{n+p-2},x_{n+q-1}}(t), \\ &\qquad F_{x_{n+p-1},x_{n+q-2}}(t), F_{x_{n+p-1},x_{n+q-1}}(t)\Big\} \\ &\geq \delta_{G_{n-2}}(t). \end{aligned} \tag{4.8}$$

Since the inequalities (4.6), (4.7) and (4.8) are of all the possibilities we have that

$$\delta_{G_n}\big(\varphi(t)\big) = \sup_{\varepsilon<\varphi(t)} \inf_{x,y\in G_n} F_{x,y}(\varepsilon) = \sup_{\varepsilon<\varphi(t)} \inf_{p,q\in\mathbb{N}\cup\{0\}} F_{x_{n+p},x_{n+q}}(\varepsilon) \geq \delta_{G_{n-2}}(t),$$

i.e. it follows that (4.5) holds for every $t > 0$.

Now, we shall prove that family $\{F_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero. Let $\lambda \in (0,1)$ and $t > 0$ be arbitrary. From $G_k \subseteq K$, for arbitrary $k \in \mathbb{N}$, it follows that G_k is a probabilistic bounded set. Now, from Lemma 2.9 we have that for every

$\lambda \in (0,1)$ there exist $t_0 > 0$ such that

$$F_{x,y}(t_0) > 1 - \lambda \tag{4.9}$$

for all $x, y \in G_k$. Hence, for every $\lambda \in (0,1)$ and such t_0 we get that

$$\delta_{G_k}(t_0) \geq 1 - \lambda.$$

From condition (1.2), for such t_0, there exists $s \geq t_0$, such that

$$\lim_{n\to\infty} \varphi^n(s) = 0.$$

Hence, there exists $l \in \mathbb{N}$ such that $\varphi^l(s) < t$. From the previous we can conclude that there exists an even number p, $p > l$, such that $\varphi^p(s) < t$, i.e. $\varphi^{2m}(s) < t$ where $m = \frac{p}{2}$.

Let $n = 2m + k$ and $x, y \in G_n$ be arbitrary. Applying induction in (4.5) we obtain

$$\delta_{G_n}\left(\varphi^{2m}(s)\right) \geq \delta_{G_{n-2m}}(s).$$

From the previous inequality it follows that

$$\delta_{G_n}(t) \geq \delta_{G_n}\left(\varphi^{2m}(s)\right) \geq \delta_{G_{n-2m}}(s) \geq \delta_{G_k}(t_0) \geq 1 - \lambda$$

i.e.

$$\delta_{G_n}(t) \geq 1 - \lambda.$$

Finally, since G_n and F_n have the same probabilistic diameter, we obtain that

$$\delta_{F_n}(t) \geq 1 - \lambda$$

i.e. we get that

$$F_{x,y}(t) \geq 1 - \lambda$$

for all $x, y \in F_n$, i.e. the family $\{F_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero.

Applying Theorem 2.11 and Lemma 2.12 we conclude that family $\{F_n\}_{n\in\mathbb{N}}$ has nonempty intersection, which consists of exactly one point z i.e. $z \in F_n$, for all $n \in \mathbb{N}$. Since the family $\{F_n\}_{n\in\mathbb{N}}$ has probabilistic diameter zero, then for each $\lambda \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ holds

$$F_{x_n,z}(t) > 1 - \lambda.$$

From the last inequality it follows that

$$\liminf_{n\to\infty} F_{x_n,z}(t) > 1 - \lambda$$

holds for every $\lambda \in (0,1)$.

Taking that $\lambda \to 0$ we get

$$\liminf_{n\to\infty} F_{x_n,z}(t) = 1,$$

i.e. $\lim_{n\to\infty} x_n = z$.

By the construction of sequence $\{x_n\}_{n\in\mathbb{N}}$ it follows that there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_k+1} = f x_{n_k}$. It is obvious that $\lim_{n_k\to\infty} x_{n_k+1} = z$, and $\lim_{n_k\to\infty} f x_{n_k} = z$.

From (1.2), since for arbitrary $t > 0$, there exists $r \geq t$, such that $\lim_{n\to\infty} \varphi^n(r) = 0$, it follows that there exists $l \in \mathbb{N}$ such that $\varphi^l(r) < t$. Now, from inequality (4.2) and previous we get

$$F_{fx_{n_k},fz}(t) > F_{fx_{n_k},fz}\big(\varphi^l(r)\big) \geq F_{x_{n_k},z}\big(\varphi^{l-1}(r)\big).$$

Taking lim inf in previous inequality, applying Lemma 2.5, we obtain

$$F_{\liminf\limits_{n_k\to\infty} fx_{n_k},f(z)}(t) \geq 1.$$

Hence, since previous inequality holds for arbitrary $t > 0$, we get that

$$F_{z,f(z)}(t) \geq 1$$

holds for every $t > 0$, i.e. we get that $fz = z$, i.e. z is the fixed point of f. Furthermore, since set C is closed set, we conclude that $z \in C$.

Let us prove that z is a unique fixed point. For this purpose let us suppose that there exists another fixed point, denoted by u. From the condition (4.2) follows

$$F_{fz,fu}\big(\varphi(t)\big) \geq F_{z,u}(t)$$

for every $t > 0$. Therefore we get that

$$F_{z,u}\big(\varphi(t)\big) \geq F_{z,u}(t)$$

for every $t > 0$. Finally, applying Lemma 4.1 it follows that $z = u$. This completes the proof. □

Example 4.3. Applying Theorem 4.2 we will prove that function $f : \mathbb{R} \mapsto \mathbb{R}$, where $\mathbb{R} = (-\infty, +\infty)$, defined by $f(x) = \frac{3}{5} - \frac{x^2}{2}$, has a fixed point in set $C = \Big[-\frac{1}{2}, \frac{1}{2}\Big]$. Let us show that all of conditions of Theorem 4.2 are satisfied. In accordance with Remark 2.3 we have that the triple $(\mathbb{R}, \mathcal{F}, T_{min})$ is a Menger PM-space. From Example 3.3 and Example 3.8 we have that Menger PM-space $(\mathbb{R}, \mathcal{F}, T_{min})$ is a strictly convex Menger PM space satisfying condition *(3.3)* with a convex structure

$$S(x, y, \theta) = \theta x + (1 - \theta)y$$

for all $x, y \in \mathbb{R}$ and $\theta \in (0, 1)$. Mapping $f : [-\frac{1}{2}, \frac{1}{2}] \mapsto [\frac{19}{40}, \frac{3}{5}]$ is a non-self mapping and it satisfying condition $f(\partial C) \subseteq C$ because $f(-\frac{1}{2}) = f(\frac{1}{2}) = \frac{19}{40} \in C$. It is obvious that C is a nonempty and closed set. Furthermore, C is a metrically bounded set and by Remark 2.10 it is a probabilistic bounded set, also.
Let us define function $\varphi : (0, +\infty) \mapsto (0, +\infty)$ by

$$\varphi(t) = \begin{cases} \frac{t}{1+t}, & 0 < t < 1, \\ -\frac{3}{4}t + \frac{7}{4}, & 1 \leq t \leq \frac{4}{3}, \\ t - \frac{7}{12}, & \frac{4}{3} < t < +\infty. \end{cases}$$

For function φ we have that

$$\lim_{n\to\infty} \varphi^n(t) = \lim_{n\to\infty} \frac{t}{1 + nt} = 0$$

holds for every $t \in (0,1)$, but it does not satisfy condition $\lim_{n\to+\infty}\varphi^n(t) = 0$, for every $t \geq 1$, because from $\varphi(1) = \varphi\left(\frac{19}{12}\right) = 1$ we get

$$\lim_{n\to\infty}\varphi^n(1) = \lim_{n\to\infty}\varphi^n\left(\frac{19}{12}\right) = 1 \neq 0.$$

Now, we will show that function φ satisfying condition *(1.2)* for $t \geq 1$, i.e. we will show by induction that

$$\lim_{n\to\infty}\varphi^n\left(\frac{7k}{12}\right) = 0 \tag{4.10}$$

holds for $k = 2,3,\ldots$ It is obvious that *(4.10)* holds for $k = 2$, because $\varphi\left(\frac{7}{6}\right) = \frac{7}{8} \in (0,1)$. Let us assume that *(4.10)* holds for some $k = l$. Then, for $k = l+1$, we have that $\frac{7(l+1)}{12} > \frac{4}{3}$, and it follows $\varphi\left(\frac{7(l+1)}{12}\right) = \frac{7l}{12}$. Hence, we obtain

$$\lim_{n\to\infty}\varphi^n\left(\frac{7(l+1)}{12}\right) = \lim_{n\to\infty}\varphi^{n-1}\left(\frac{7l}{12}\right) = 0$$

which shows that *(4.10)* holds for $k = l+1$, and by induction *(4.10)* holds, for every $k = 2,3,\ldots$ Finally, we can conclude that for every $t \geq 1$, exists $r = \frac{7k_0}{12} > t$ for sufficiently large $k_0 = 2,3,\ldots$, such that *(4.10)* holds. Hence, function φ satisfied condition *(1.2)*.
Notice that $\varphi(t) > \frac{t}{2}$ holds for every $t > 0$ and $|x^2 - y^2| \leq |x - y|$ holds for every $x, y \in C$, because $|x + y| \leq 1$ holds for every $x, y \in C$. Then, we get:

$$\begin{aligned}
F_{f(x),f(y)}\big(\varphi(t)\big) &= \varepsilon_0\Big(\varphi(t) - d\big(f(x), f(y)\big)\Big) = \varepsilon_0\Big(\varphi(t) - \big|f(x) - f(y)\big|\Big) \\
&= \varepsilon_0\Big(\varphi(t) - \frac{1}{2}\left|x^2 - y^2\right|\Big) \geq \varepsilon_0\Big(\frac{t}{2} - \frac{1}{2}|x - y|\Big) \\
&= \varepsilon_0(t - |x - y|) = \varepsilon_0\Big(t - d(x,y)\Big) = F_{x,y}(t),
\end{aligned}$$

i.e. nonlinear contractive condition *(4.2)* is satisfied for every $x, y \in C$.

Since, all conditions from Theorem 4.2 are satisfied we get that function f has unique fixed point $x = \frac{-5+\sqrt{55}}{5} \in C$.

Acknowledgements. This paper is supported by the Ministry of Education, Science and Technological Development of Republic of Serbia, Grant No. 174032. Authors would like to thank to the reviewer for his/her useful comments.

References

[1] N.A. Assad, W.A. Kirk, *Fixed-point theorems for set-valued mappings of contractive type*, Pacific J. Math., **43**(1972), 553-562.

[2] S. Banach, *Sur la opérations dans les ensembles abstraits et leurs applications aux équations intégrales*, Fund. Math., **3**(1922), 133-181.

[3] D.W. Boyd, J.S.W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20**(1969), 458-464.

[4] L.B. Ćirić, *Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces*, Nonlinear Anal., **72**(2010), 2009-2018.

[5] L.B. Ćirić, *Common fixed point theorems for family on non-self mappings in convex metric spaces*, Nonlinear Anal., **71**(2009), 1662-1669.

[6] L.B. Ćirić, *Quasi-contraction non-self mappings on Banach spaces*, Bull. Acad. Serbe Sci. Arts, **23**(1998), 25-31.
[7] L.B. Ćirić, *Contractive type non-self mappings on metric spaces of hyperbolic type*, J. Math. Anal. Appl., **317**(2006), 28-42.
[8] R.J. Egbert, *Products and quotients of probabilistic metric spaces*, Pacific J. Math., **24**(1968), 437-455.
[9] J.X. Fang, *On φ-contractions in probabilistic and fuzzy metric spaces*, Fuzzy Set Systems, **267**(2015), 86-99.
[10] L. Gajić, V. Rakočević, *Quasicontraction nonself-mappings on convex metric spaces and common fixed point theorems*, Fixed Point Theory Appl., **3**(2005), 365-375.
[11] L. Gajić, V. Rakočević, *Pair of non-self-mappings and common fixed points*, Appl. Math. Comput., **187**(2007), 999-1006.
[12] O. Hadžić, *Common fixed point theorems in probabilistic metric spaces with a convex structure*, Zb. Rad. Prirod.-Mat.Fak. Ser. Mat., **18**(1987), 165-178.
[13] M. Imdad, L. Khan, *Some common fixed point theorems for family of mappings in metrically convex spaces*, Nonlinear Anal., **67**(2007), 2717-2726.
[14] M. Imdad, M. Tanveer, M. Hasan, *Some common fixed point theorems in Menger PM spaces*, Fixed Point Theory Appl., Vol. 2010, doi:10.1155/2010/819269.
[15] J. Jachymski, *On probabilistic φ-contactions on Menger spaces*, Nonlinear Anal., **73**(2010), 2199-2203.
[16] S.N. Ješić, *Convex structure, normal structure and a fixed point theorem in intuitionistic fuzzy metric spaces*, Chaos, Solitons and Fractals, **41**(2008), 292-301.
[17] S.N. Ješić, R.M. Nikolić, N.A. Babačev, *A common fixed point theorem in strictly convex Menger PM-spaces*, Filomat, **28(4)**(2014), 735-743.
[18] K. Menger, *Statistical metric*, Proc. Nat. Acad. Sci. USA, **28**(1942), 535-537.
[19] B.E. Rhoades, *A fixed point theorem for some non-self mappings*, Math. Japon., **23**(1978), 457-459.
[20] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math., **10**(1960), 415-417.
[21] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland, New York, Elsevier, 1983.
[22] V.M. Sehgal, A.T. Bharucha-Reid, *Fixed points of contraction mappings in PM-spaces*, Math. System Theory, **6**(1972), 97-102.
[23] H. Sherwood, *Complete probabilistic metric spaces*, Z. Wahrsch. Verw. Gebiete, **20**(1971), 117-128.
[24] W. Takahashi, *A convexity in metric space and nonexpansive mappings*, Kodai Math. Sem. Rep., **22**(1970), 142-149.
[25] J.Z. Xiao, X.H. Zhu, X.Y. Liu, *An alternative characterization of probabilistic Menger spaces with H-type triangular norms*, Fuzzy Sets Systems, **227**(2013), 107-114.

Received: May 16, 2014; Accepted: December 13, 2015.