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FIXED POINT THEOREMS FOR SET-VALUED MAPPINGS IN b-METRIC SPACES

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Abstract. We will establish set-valued version of Suzuki's fixed point theorem when the underling space is a complete b-metric. Our method enable us to prove set-valued versions of Hardy-Rogers and Ĉirić fixed point theorems for b-metric spaces.
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1. Introduction

Fixed points and strict fixed points of multivalued operators are important concepts in the theory of set-valued dynamic systems [2, 13]. The study of fixed points for set-valued functions using Hausdorff metric was initiated by Nadler [14] in 1969, who extended Banach's fixed point theorem [4] for set-valued functions. Nadler's idea was used by some mathematicians to prove the existence of a fixed point for set-valued functions. In 2008, Suzuki [15] proved a generalization of Banach's fixed point theorem. Kikkawa and Suzuki extended Suzuki's theorem for set-valued mappings as follows:

Theorem 1.1. [11, Theorem 2] *Let (X, d) be a complete metric space and let T ba a mapping from X into the set of all nonempty closed bounded subsets of X. Assume that there exists $r \in [0, 1)$ such that $\frac{1}{1+r} d(x, Tx) \leq d(x, y)$ implies that $\mathcal{H}(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then there is some $x \in X$ such that $x \in Tx$.*

The above result was improved by Mot and Petruşel:

Theorem 1.2. [12, Theorem 6. 6] *Let (X, d) ba a complete metric space and T be a set-valued function from X into nonempty closed subsets of X. Assume that for some $a, b, c \in [0, 1)$ with $a + b + c < 1$, $\frac{1-b-c}{1+a} d(x, Tx) \leq d(x, y)$ implies that $\mathcal{H}(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty)$ for all $x, y \in X$. Then there is some $x \in X$ such that $x \in Tx$.*

In 2011, Aleomraninejad et al. [1] developed a new method to prove Suzuki's fixed point theorem for set-valued mapping. The method was extended by Yingtaweesittikul [16] for set-valued functions in general b-metric spaces.

In this paper, we use the definition Czerwik for b-metrics to improve the main result in [16] in this special case. This result will enable us to prove set-valued version of some known results in special kind of b-metric spaces.

2. Results

We recall that a b-metric d on a nonempty set X is a function $d : X \times X \to [0, \infty)$ with the following properties:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq k[d(x, y) + d(y, z)]$

for all $x, y, z \in Z$ for some $k > 1$. Clearly every metric space is a b-metric. However, the converse is not true in general. For example the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is defined by $d(x, y) = |x - y|^2$ is a b-metric on $\mathbb{R}$ for $k = 2$. However, it is not a metric. Czerwik in [7], defined a special class of b-metrics by replacing (iii) by the following:

(iv) For each $\varepsilon > 0$ and $x, y, z \in X$ if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$, then $d(x, z) < 2\varepsilon$.

It is easy to verify that " $<$ " can be replaced by " $\leq$ " in (iv). Moreover, if d satisfies (iv), then (iii) holds for $k = 2$. For the b-metric $d(x, y) = |x - y|^2$, $x, y \in \mathbb{R}$, we have $d(1, 0) = d(0, -1) = 1$ but $d(1, -1) = 4 \not< 2$. Hence the class of those b-metrics with the property (iv) is strictly smaller than the class of all b-metric spaces for $k = 2$.

In order to avoid ambiguity, if (X, d) satisfies (i)-(iii), for some $k > 1$, we say that (X, d) is a b_k-metric space. If (X, d) satisfies the properties (i), (ii) and (iv), then it is called a b-metric space. We denote by $CL(X)$ the set of all nonempty closed subsets of X. For each $x \in X$ and $A \in CL(X)$, define

$$d(x, A) = d(A, x) = \inf\{d(x, a) : a \in A\}.$$

We have the following simple observation.

Proposition 2.1. *Let (X, d)be a b-metric space and $A \in CL(X)$. Then for every $x, y \in X$, $a \in A$ and $\varepsilon > 0$, we have the following.*

(a) *$d(x, A) \geq 0$ and the equality holds only if $x \in A$.*
(b) *If $d(x, y) < \varepsilon$ and $d(y, A) < \varepsilon$, then $d(x, A) < 2\varepsilon$.*

Proof. (a) follows directly from the definition. To prove (b), let $d(y, A) < \varepsilon$, by the definition, there is some $a' \in A$ such that $d(y, a') < \varepsilon$. Since (X, d) is a b-metric, if $d(x, y) < \varepsilon$, then $d(x, a') < 2\varepsilon$. Hence $d(x, A) \leq d(x, a') < 2\varepsilon$. □

Let (X, d) be a b-metric space. For each $x_0 \in X$ and $\delta > 0$, let $N(x_0; \delta) = \{x \in X : d(x, x_0) < \delta\}$. Now if $A, B \in CL(X)$, we define the Hausdorff distance of A and B by

$$\mathcal{H}(A, B) = \inf\{r > 0 : A \subseteq \bigcup_{b \in B} N(b; r) \text{ and } B \subseteq \bigcup_{a \in A} N(a; r)\ \}.$$

Proposition 2.2. *Let (X, d) be a b-metric space and $A, B \in CL(X)$. Then we have the following.*

(a) *$\mathcal{H}$ defines a b-metric on nonempty closed bounded subsets of X.*
(b) *$d(x, B) \leq \mathcal{H}(A, B)$ for any $x \in A$.*
(c) *For each $0 < \lambda < 1$ and $x_0 \in X \setminus A$, there is some $x_1 \in A$ such that $\lambda d(x_0, A) < d(x_0, x_1)$.*

Proof. It follows from the definition that for each $A, B \in CL(X)$, $\mathcal{H}(A, B) \geq 0$ and the equality holds only if $A = B$. Moreover $\mathcal{H}(A, B) = \mathcal{H}(B, A)$. Now, let A, B and C be in $CL(X)$ and $\varepsilon > 0$. If $\mathcal{H}(A, B) < \varepsilon$ and $\mathcal{H}(B, C) < \varepsilon$, then

$$A \subseteq \bigcup_{b \in B} N(b; \varepsilon) \text{ and } B \subseteq \bigcup_{c \in C} N(c; \varepsilon).$$

Hence if $a \in A$, there is some $b \in B$ such that $d(a, b) < \varepsilon$. Moreover, there is some $c \in C$ such that $d(b, c) < \varepsilon$. Hence $d(a, c) < 2\varepsilon$. This means that $A \subseteq \bigcup_{c \in C} N(c, 2\varepsilon)$. A similar argument shows that $C \subseteq \bigcup_{a \in A} N(a, 2\varepsilon)$. Hence $\mathcal{H}(B, C) \leq 2\varepsilon$. This proves (a). Let $\mathcal{H}(A, B) < r$ and $x \in A$. Then $A \subseteq \bigcup_{b \in B} N(b; r)$. Hence there is some $b \in B$ such that $d(x, b) < r$. It follows that $d(x, B) \leq d(x, b) < r$. Thus (b) holds. (c) follows from the definition of $d(x_0, A)$. □

The following result plays an important rule in the sequel.

Lemma 2.3. *(see* [8] *or* [9]*). Suppose $d : X \times X \to [0, \infty)$ satisfies the following condition:*
For any $\varepsilon > 0$ and $x, y, z \in X$, if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$, then $d(x, z) < 2\varepsilon$.
Then the function $\rho : X \times X \to [0, \infty)$, defined by

$$\rho(x, y) = inf\{\sum_{i=1}^{n} d(x_{i-1}, x_i);\ \text{where } n \in \mathbb{N}, x_0 = x \text{ and } x_n = y\}, \quad (x, y) \in X \times X), \tag{2.1}$$

has the following properties:

(i) *$\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, for all $x, y, z \in X$.*
(ii) *$\frac{d(x,y)}{4} \leq \rho(x, y) \leq d(x, y)$ for all $x, y \in X$. Further, ρ is symmetric (i.e. $\rho(x, y) = \rho(y, x)$ if d is).*

We also need to the following observation.

Lemma 2.4. *Let X be a b-metric space and $\{x_n\}$ be a sequence in X such that for some $0 \leq r < 1$,*

$$d(x_n, x_{n+1}) \leq r d(x_{n-1}, x_n) \quad (n = 2, 3, \dots). \tag{2.2}$$

Then $\{x_n\}$ is a Cauchy sequence.

Proof. It follows from (2.2) that for each $n \geq 1$, $d(x_n, x_{n+1}) \leq r^n d(x_0, x_1)$. Given $\varepsilon > 0$, find some $n_0 \in \mathbb{N}$ such that $\frac{r^n d(x_0,x_1)}{1-r} < \varepsilon/4$. In view of Lemma 2.3, there is a metric ρ on X such that $\frac{d(x,y)}{4} \leq \rho(x, y) \leq d(x, y)$ for all $x, y \in X$. Hence for each

$m > n \geq n_0$, we have

$$\begin{aligned}(1/4)d(x_n, x_m) \leq \rho(x_n, x_m) &\leq \sum_{i=n}^{m-1} \rho(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} r^i d(x_0, x_1) \\ &\leq \frac{r^n d(x_0, x_1)}{1-r} < \varepsilon/4.\end{aligned}$$

□

Let $\mathcal{R}$ denote the class of all continuous functions $g : [0,\infty)^5 \to [0,\infty)$ with the following properties:

(i) $g(1,1,1,4,0) = g(1,1,1,0,4) = h \in (0,1)$,

(ii) g is sub-homogeneous, that is,

$$g(\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) \leq \lambda g(x_1, x_2, x_3, x_4, x_5), \quad (x_1, x_2, x_3, x_4, x_5, \lambda \geq 0).$$

(iii) If $x_i \leq y_i$ for $1 \leq i \leq 4$, then

$$g(x_1, x_2, x_3, x_4, 0) \leq g(y_1, y_2, y_3, y_4, 0) \text{ and}$$
$$g(x_1, x_2, x_3, 0, x_4) \leq g(y_1, y_2, 0, y_4).$$

Let $k \geq 1$ be fixed and let $\mathcal{R}_k$ denote the set of all continuous functions $g : [0,\infty)^5 \to [0,\infty)$ satisfying the conditions (ii), (iii) and

(iv) $g(1,1,1,2k,0) = g(1,1,1,0,2k) = h_k \in (0, \frac{1}{k})$.

We need to the following elementary results.

Lemma 2.5. *If $g \in \mathcal{R}$ and $u, v \in [0,\infty)$ are such that*

$$u \leq \max\{g(u,v,u,2(u+v),0), g(v,v,0,2(u+v)), g(v,u,v,u+v,0), g(v,u,v,0,2(u+v))\},$$

then $u \leq hv$.

Proof. See the proof of [6, Lemma 1. 3] or [16, Lemma 1. 10]. □

We recall that a point $x_0 \in X$ is said to be a fixed point of a set-valued function $T : X \to 2^X$ if $x_0 \in Tx_0$. The set of all fixed points of $T : X \to 2^X$ is denoted by $\mathcal{F}(T)$.

Lemma 2.6. *Let X be a complete b-metric space and let $T, S : X \to CL(X)$ be two set-valued functions such that for some $\lambda > 0$ and $g \in \mathcal{R}$,*

$$\lambda d(x, Tx) \leq d(x, y) \text{ or } \lambda d(x, Sx) \leq d(x, y) \text{ implies that}$$

$$\mathcal{H}(Tx, Sy) \leq g\big(d(x,y), d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx)\big) \quad (x, y \in X).$$

Then $\mathcal{F}(F) = \mathcal{F}(G)$.

Proof. See [1, Lemma 2.1] or [16, Theorem 1.9]. □

In [16], H. Yingtaweesittikul proved the following.

Theorem 2.7. [16, Theorem 2. 1] *Let (X,d) be a complete b_k-complete metric space and let $T,S : X \to CB(X)$ be two set-valued mappings. Suppose there exists $\lambda \in (0,1)$ and $g \in \mathcal{R}_k$ such that $k\lambda(1+h_k) \leq 1$ and*

$$\lambda d(x,Tx) \leq d(x,y) \text{ or } \lambda d(x,Sx) \leq d(x,y) \text{ implies that}$$

$$\mathcal{H}(Tx,Sy) \leq g\big(d(x,y), d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx)\big) \quad (x,y \in X).$$

Then $\mathcal{F}(T) = \mathcal{F}(S)$ and $\mathcal{F}(T)$ is non-empty.

Note that $h_2 = h$ but $h_2 \in (0,\frac{1}{2})$ while $h \in (0,1)$. In the next result, we improve the above theorem for special kind of b_2-metric spaces.

Theorem 2.8. *Let X be a complete b-metric space and let $T,S : X \to CL(X)$ be two set-valued functions such that for some $\lambda \in (0,1)$ and $g \in \mathcal{R}$ such that $2\lambda(1+h) < 1$ and*

$$\lambda d(x,Tx) \leq d(x,y) \text{ or } \lambda d(x,Sx) \leq d(x,y) \text{ implies that}$$

$$\mathcal{H}(Tx,Sy) \leq g\big(d(x,y), d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx)\big) \quad (x,y \in X).$$

Then $\mathcal{F}(T) = \mathcal{F}(S)$ and $\mathcal{F}(T)$ is non-empty.

Proof. Thanks to Lemma 2.6, $\mathcal{F}(T) = \mathcal{F}(S)$. Take some $1 > r > h$ and $x_0 \in X$. If $x_0 \in Tx_0$, then $x_0 \in \mathcal{F}(T)$. Otherwise, choose some $x_1 \in Tx_0$ such that $\lambda d(x_0,Tx_0) < d(x_0,x_1)$. Then

$$\begin{aligned} d(x_1,Sx_1) &\leq \mathcal{H}(Tx_0,Sx_1) \\ &\leq g\big(d(x_0,x_1), d(x_0,Tx_0), d(x_1,Sx_1), d(x_0,Sx_1), d(x_1,Tx_0)\big) \\ &\leq g\big(d(x_0,x_1), d(x_0,x_1), d(x_1,Sx_1), 2d(x_0,x_1)+2d(x_1,Sx_1), 0\big) \end{aligned}$$

By Lemma 2.5, $d(x_1,Sx_1) \leq hd(x_0,x_1) < rd(x_0,x_1)$. Let $d(x_1,Sx_1) < \mu < rd(x_0,x_1)$. By the definition, there is some $x_2 \in Sx_1$ such that $d(x_1,x_2) < \mu < rd(x_0,x_1)$. Since $\lambda d(x_1,Sx_1) < d(x_1,x_2)$, by assumption,

$$\begin{aligned} d(x_2,Tx_2) &\leq \mathcal{H}(Tx_2,Sx_1) \\ &\leq g\big(d(x_1,x_2), d(x_2,Tx_2), d(x_1,Sx_1), d(x_2,Sx_1), d(x_1,Tx_2)\big) \\ &\leq g\big(d(x_1,x_2), d(x_2,Tx_2), d(x_1,x_2), 0, 2d(x_1,x_2)+2d(x_2,Tx_2)\big). \end{aligned}$$

By applying Lemma 2.5 once again, we have $d(x_2,Tx_2) \leq hd(x_1,x_2) < rd(x_1,x_2)$. Similarly, we can find some $x_3 \in Tx_2$ such that $d(x_2,x_3) < rd(x_1,x_2)$. Using the above argument, by induction, we can obtain a sequence $\{x_n\}$ in X with the following properties:

(a) $x_{2n+1} \in T_{2n-2}$ and $x_{2n} \in Sx_{2n-1}$,
(b) $d(x_n,x_{n+1}) \leq rd(x_{n-1},x_n)$,
(c) $d(x_{2n},Tx_{2n}) \leq hd(x_{2n-1},x_{2n})$ and $d(x_{2n-1},Sx_{2n-1}) \leq hd(x_{2n-2},x_{2n-1})$

for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then by (c), x_n is a common fixed point of T and S. Otherwise, (b) and Lemma 2.4 imply that $\{x_n\}$ is a Cauchy sequence in complete b-metric space X. Let $x = \lim_{n\to\infty} x_n$. We claim that either

$\lambda d(x_{2n}, T_{2n}) \le d(x_{2n}, x)$ or $\lambda d(x_{2n+1}, T_{2n+1}) \le d(x_{2n+1}, x)$ for each $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, $\lambda d(x_{2n}, T_{2n}) > d(x_{2n}, x)$ and $\lambda d(x_{2n+1}, T_{2n+1}) > d(x_{2n+1}, x)$, then

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\le 2\big[d(x_{2n}, x) + d(x_{2n+1}, x)\big] \\ &< 2\lambda\big[d(x_{2n}, Tx_{2n}) + d(x_{2n+1}, Sx_{2n+1})\big] \\ &\le 2\lambda\big[d(x_{2n}, x_{2n+1}) + hd(x_{2n}, x_{2n+1})\big] \\ &= 2\lambda(1+h)d(x_{2n}, x_{2n+1}) < d(x_{2n}, x_{2n+1}). \end{aligned}$$

This contradiction proves our claim. Therefore, by our assumption for each $n \in \mathbb{N}$ either

$$\mathcal{H}(Tx_{2n}, Sx) \le g\big(d(x_{2n}, x), d(x_{2n}, Tx_{2n}), d(x, Sx), d(x_{2n}, Sx), d(x, Tx_{2n})\big)$$

or

$$\mathcal{H}(Tx, Sx_{2n+1}) \le$$
$$\le g\big(d(x_{2n+1}, x), d(x, Tx), d(x_{2n+1}, Sx_{2n+1}), d(x, Sx_{2n+1}), d(x_{2n+1}, Tx)\big).$$

It follows that at least one of the following two cases happens.

Case 1. There is an infinite subset $I \subseteq \mathbb{N}$ such that

$$\begin{aligned} d(x_{2n+1}, Sx) &\le \mathcal{H}(Tx_{2n}, Sx) \\ &\le g\big(d(x_{2n}, x), d(x_{2n}, Tx_{2n}), d(x, Sx), d(x_{2n}, Sx), d(x, Tx_{2n})\big) \quad (n \in I). \end{aligned}$$

In this case, for each $n \in I$, we have

$$\begin{aligned} d(x, Sx) &\le 2\big[d(x, x_{2n+1}) + d(x_{2n+1}, Sx)\big] \\ &\le 2\big[d(x, x_{2n+1}) + g\big(d(x_{2n}, x), d(x_{2n}, Tx_{2n}), d(x, Sx), d(x_{2n}, Sx), d(x, Tx_{2n})\big) \\ &\le 2\big[d(x, x_{2n+1}) + g\big(d(x_{2n}, x), d(x_{2n}, x_{2n+1}), d(x, Sx), \\ &\quad 2d(x_{2n}, x) + 2d(x, Sx), d(x, x_{2n+1})\big). \end{aligned}$$

By continuity of g, it follows that

$$d(x, Sx) \le 2g(0, 0, d(x, Sx), 2d(x, Sx), 0).$$

By Lemma 2.5, we have $d(x, Sx) = 0$. Hence $x \in \mathcal{F}(S)$.

Case 2. There is an infinite subset I of $\mathbb{N}$ such that

$$\begin{aligned} d(Tx, x_{2n+1}) &\le \mathcal{H}(Tx, Sx_{2n+1}) \\ &\le g\big(d(x_{2n+1}, x), d(x, Tx), d(x_{2n+1}, Sx_{2n+1}), d(x, Sx_{2n+1}), d(x_{2n+1}, Tx)\big) \end{aligned}$$

In this case, for each $n \in I$, we have

$$\begin{aligned} d(x, Tx) &\le 2\big[d(x, x_{2n+2}) + d(x_{2n+2}, Tx)\big] \\ &\le 2\big[d(x, x_{2n+2}) + g\big(d(x_{2n+1}, x), d(x, Tx), \\ &\qquad d(x_{2n+1}, Sx_{2n+1}), d(x, Sx_{2n+1}), d(x_{2n+1}, Tx)\big)\big] \\ &\le 2\big[d(x, x_{2n+2}) \\ &\quad + g\big(d(x_{2n+1}, x), d(x, Tx), d(x_{2n+1}, x_{2n+2}), d(x, x_{2n+2}), \\ &\qquad 2d(x_{2n+1}, x) + 2d(x, Tx)\big)\big] \end{aligned}$$

Using continuity of g, we have

$$d(x,Tx) \leq 2g\big(0, d(x,Tx), 0, 0, 2d(x,Tx)\big).$$

By Lemma 2.5, $d(x,Tx) = 0$. Hence $x \in Tx$. □

The above result enable us to prove Theorem 1.2 for b-metric spaces.

Theorem 2.9. *Let (X,d) ba a complete b-metric space and $T, S : X \to CL(X)$. Assume that for some $a, b, c \in [0,1)$ with $a+b+c < 1$, $\frac{1-b-c}{2(1+a)} d(x,Tx) \leq d(x,y)$ or $\frac{1-b-c}{2(1+a)} d(y,Sy) \leq d(x,y)$ implies that $\mathcal{H}(Tx,Sy) \leq ad(x,y) + bd(x,Tx) + cd(y,Sy)$ for all $x, y \in X$. Then $\mathcal{F}(T) = \mathcal{F}(S)$ and $\mathcal{F}(T)$ is non-empty.*

Proof. In Theorem 2.8, let $\lambda = \frac{1-b-c}{2(1+a)}$ and $g(x_1,x_2,x_3,x_4,x_5) = ax_1 + bx_2 + cx_3$ for all $x_1,x_2,x_3,x_4,x_5 \geq 0$. Since $2\lambda(1+a+b+c) < 1$, by Theorem 2.8, $\mathcal{F}(T) = \mathcal{F}(S)$ and $\mathcal{F}(T)$ is non-empty. □

By imitating the proof of Theorem 2.8, one can prove the following:

Theorem 2.10. *Let X be a complete b-metric space and let $T, S : X \to CL(X)$ be two set-valued functions such that for some $g \in \mathcal{R}$*

$$H(Tx,Sy) \leq g\big(d(x,y), d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx)\big) \quad (x, y \in X).$$

Then $\mathcal{F}(T) = \mathcal{F}(S)$ and $\mathcal{F}(T)$ is non-empty.

The next result can be considered as set-valued version of Hardy-Rogers fixed point theorem [10] in b-metric spaces.

Theorem 2.11. *Let (X,d) be a complete b-metric and $T : X \to CL(X)$ be a set-valued mapping such that for all $x, y \in X$,*

$$\mathcal{H}(Tx,Ty) \leq a\ d(x,y) + b\ d(x,Tx) + c\ d(y,Ty) + e\ d(x,Ty) + f\ d(y,Tx), \tag{2.3}$$

where $0 \leq a,b,c,e,f < 1$ and $a+b+c+2(e+f) < 1$. Then $\mathcal{F}(T) \neq \emptyset$.

Proof. By symmetry,

$$\mathcal{H}(Tx,Ty) \leq a\ d(x,y) + b\ d(y,Ty) + c\ d(x,Tx) + e\ d(y,Tx) + f\ d(x,Ty), \tag{2.4}$$

for all $x, y \in X$. Put $\alpha = \frac{b+c}{2}$ and $\beta = \frac{e+f}{2}$. Then by (2.3) and (2.4) for all $x, y \in X$,

$$\mathcal{H}(Tx,Ty) \leq a\ d(x,y) + \alpha\ [d(x,Tx) + d(y,Ty)] + \beta[\ d(x,Ty) +\ d(y,Tx)]. \tag{2.5}$$

Define $g : [0,\infty)^5 \to [0,\infty)$ by

$$g(x_1,x_2,x_3,x_4,x_5) = ax_1 + \alpha(x_2+x_3) + \beta(x_4+x_5) \quad (x_i \geq 0, 1 \leq i \leq 5).$$

Clearly g is nondecreasing with respect to each variable and sub-homogenous. Moreover, $g(1,1,1,4,0) = g(1,1,1,0,4) = a+b+c+2(e+f) < 1$. Hence $g \in \mathcal{R}$. Thanks to Theorem 2.10, $\mathcal{F}(T) \neq \emptyset$. □

A mapping $T : X \to X$ is said to be a quasi-contraction if there is some $0 \leq r < 1$ such that

$$d(Tx,Ty) \leq r \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

In 1974, Ĉirić[5] proved a fixed point theorem for quasi-contractive mappings in complete metric spaces. Aydi et al. [3] extend Ĉirić's theorem for b_k-metric spaces as follows.

Theorem 2.12. [3, Theorem 2. 2]. *Let (X, d) be a complete b_k-metric space. Suppose that T is a set-valued quasi-contractive mapping, that is*

$$\mathcal{H}(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \quad (x, y \in X).$$

If $r(k^2 + k) < 1$, then $\mathcal{F}(T) \neq \emptyset$.

Theorem 2.10 enable us to prove the following extension of Theorem 2.12 for special kind of b_2-metrics.

Theorem 2.13. *Let (X, d) be a complete b-metric and $T, S : X \to CL(X)$ be a set-valued mapping such that for all $x, y \in X$,*

$$\mathcal{H}(Tx, Sy) \leq r \max\{d(x, y),\ d(x, Tx),\ d(y, Sy),\ d(x, Sy),\ d(y, Tx)\}, \tag{2.6}$$

where $0 \leq r < \frac{1}{4}$. Then $\mathcal{F}(T) = \mathcal{F}(S)$ is nonempty.

Proof. Define $g : [0, \infty)^5 \to [0, \infty)$ by

$$g(x_1, x_2, x_3, x_4, x_5) = r \max\{x_1,\ x_2, x_3, x_4, x_5) \quad (x_i \geq 0, 1 \leq i \leq 5).$$

Then $h = r \max\{1, 1, 1, 4, 0\} = 4r < 1$. According to Theorem 2.10, $\mathcal{F}(T) = \mathcal{F}(S)$ is not empty. □

The following example shows that the above result is a genuine extension of Theorem 2.12 in special kind of b_2-metric spaces.

Example 2.14. Let $X_1 = \{x \in \mathbb{R} : 0 \leq x \leq \frac{1}{3}\}$ and $X_2 = \{a, b, c\}$ where $a = \frac{1}{2} \leq c \leq b = 1$. Let $X = X_1 \cup X_2$ and define a symmetric function $d : X \times X \to \mathbb{R}^+$ by

$d(x, x) = 0$ for all $x \in X$,
$d(x, y) = |x - y|$ if $x, y \in X_1$,
$d(x, y) = 1$ if $x \in X_1, y \in X_2$ or $x \in X_2, y \in X_1$,
$d(a, b) = \frac{1}{2}$, $d(a, c) = 1$ and $d(b, c) = 2$.

Since $d(b, c) = 2 \nleq \frac{3}{2} = d(b, a) + d(a, c)$, d is not a metric. An easy computation shows that (X, d) is complete and

$$d(x, y) \leq 2 \max\{d(x, z), d(z, y)\} \quad (x, y, z \in X).$$

Therefore (X, d) is a complete b-metric. Let $T : X \to X$ be defined by $Tx = \frac{x}{5}$ for all $x \in X$. We will show that

$$d(Tx, Ty) \leq \frac{1}{5} \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \quad (x, y \in X). \tag{2.7}$$

Clearly (2.7) holds for all $x, y \in X_1$. If $x, y \in X_2$, we have

$$d(Tx, Ty) = \frac{1}{5}|x - y| \leq \frac{1}{5} \times \frac{1}{2} \leq \frac{1}{5} d(x, y).$$

Suppose that $x \in X_1$ and $y \in X_2$, then

$$d(Tx, Ty) = \frac{1}{5}|x - y| \leq \frac{1}{5} = \frac{1}{5} d(x, y).$$

It follows from Theorem 2.13 that T has a fixed point. However, Theorem 2.12 can not be applied. In fact, in Theorem 2.12, r must be less than $\frac{1}{k^2+k}$. So that for $k=2$, $r<\frac{1}{6}$. In this example for $x_0=0$, $y_0=1$, we have $d(Tx_0,Ty_0)=\frac{1}{5}$ and

$$\max\{d(x_0,y_0),d(x_0,Tx_0),d(y_0,Ty_0),d(x_0,Ty_0),d(y_0,Tx_0)\}=\max\left\{1,0,\frac{1}{5},1,1\right\}=1.$$

But for $r<\frac{1}{6}$,

$$\begin{aligned} d(Tx_0,Tx_0) &= \frac{1}{5}\max\{d(x_0,y_0),d(x_0,Tx_0),d(y_0,Ty_0),d(x_0,Ty_0),d(y_0,Tx_0)\} \\ &\nleq r\max\{d(x_0,y_0),d(x_0,Tx_0),d(y_0,Ty_0),d(x_0,Ty_0),d(y_0,Tx_0)\}. \end{aligned}$$

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