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ON SEMILINEAR FRACTIONAL ORDER DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract. We are considering the Cauchy problem for a semilinear fractional differential inclusion in a Banach space. By using the fixed point theory for condensing multivalued maps, we prove the local and global theorems of the existence of mild solutions to this problem. We verify the compactness of the solutions set and its continuous dependence on parameters and initial data. We demonstrate also the application of the averaging principle to the investigation of the continuous dependence of the solutions set on a parameter in the case when the right-hand side of the inclusion is rapidly oscillating.

Key Words and Phrases: Fractional differential inclusion, semilinear differential inclusion, Cauchy problem, continuous dependence of solutions, averaging principle, fixed point, multivalued map, condensing map, measure of noncompactness.

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1. Introduction

Theory of differential equations of a fractional order arises from ideas of Leibnitz and Euler but only recently the interest to this subject essentially strengthened due to interesting applications in applied mathematics, physics, enginery, biology, economics and other branches of natural sciences (see, e.g., monographs [1], [3], [6], [7], [10], [13], [15], [16], [17] and references therein and papers [11], [12], [14], [18] and many others).

In the present paper we are considering the Cauchy problem for a semilinear fractional differential inclusion in a Banach space E of the following form:

$$D^q x(t) \in Ax(t) + F(t, x(t)), \ t \in [0, a], \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where D^q, $0 < q < 1$, is the Caputo fractional derivative, $F : [0, a] \times E \multimap E$ is a multivalued map with nonempty compact convex values, $A : D(A) \to E$ is a linear closed, not necessarily bounded operator in E, and $x_0 \in E$.

By using the fixed point theory for condensing multivalued maps, we prove the local and global theorems of the existence of mild solutions to problem (1.1)-(1.2). We verify the compactness of the solutions set and its continuous dependence on parameters and initial data. In the last section we demonstrate the application of the averaging principle to the investigation of the continuous dependence of the solutions set on a parameter in the case when the right-hand side of the inclusion is rapidly oscillating.

2. Preliminaries

2.1. **Fractional integral and derivative.**

Definition 2.1. (See, e.g., [15], [16]). *The fractional integral of order $\alpha \in (0, 1)$ of a function $g \in L^1([0, T]; E)$ is the function $I_0^\alpha g$ of the following form:*

$$I_0^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} g(s)\, ds,$$

where Γ is the Euler's gamma-function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha - 1} e^{-x} dx.$$

Definition 2.2. *The Caputo fractional derivative of the order $\alpha \in (N - 1, N]$ of a function $g \in C^N([0, T]; E)$ is the function $D_0^\alpha g$ of the following form:*

$$D_0^\alpha g(t) = \frac{1}{\Gamma(N - \alpha)} \int_0^t (t - s)^{N - \alpha - 1} g^{(N)}(s)\, ds.$$

2.2. **Multivalued maps.** Let $\mathcal{E}$ be a Banach space. Introduce the following notation:

- $P(\mathcal{E}) = \{A \subseteq \mathcal{E} : A \neq \varnothing\}$ denotes the collection of all non-empty subsets of $\mathcal{E}$;
- $Pv(\mathcal{E}) = \{A \in P(\mathcal{E}) : A \text{ is convex}\}$;
- $K(\mathcal{E}) = \{A \in P(\mathcal{E}) : A \text{ compact}\}$;
- $Kv(\mathcal{E}) = \{Pv(\mathcal{E}) \cap K(\mathcal{E})\}$ denotes the collection of all non-empty compact and convex subsets of $\mathcal{E}$.

Definition 2.3. *(See, e.g.,* [2], [8]*).* Let $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\beta : \mathrm{P}(\mathcal{E}) \to \mathcal{A}$ is called the measure of noncompactness (MNC) in $\mathcal{E}$ if for each $\Omega \in \mathrm{P}(\mathcal{E})$ we have:

$$\beta(\overline{\mathrm{co}}\, \Omega) = \beta(\Omega),$$

where $\overline{\mathrm{co}}\, \Omega$ denotes the closure of the convex hull of Ω.

A measure of noncompactness β is called:

1) *monotone* if for each $\Omega_0, \Omega_1 \in \mathrm{P}(\mathcal{E})$, from $\Omega_0 \subseteq \Omega_1$ follows $\beta(\Omega_0) \leq \beta(\Omega_1)$.
2) *nonsingular*, if for each $a \in E$ and each $\Omega \in \mathrm{P}(\mathcal{E})$ we have $\beta(\{a\} \cup \Omega) = \beta(\Omega)$.

If $\mathcal{A}$ is a cone in a Banach space, the MNC β is called:

3) *regular*, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in \mathrm{P}(\mathcal{E})$;
4) *real*, if $\mathcal{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering.

As the example of a real MNC obeying all above properties, we can consider the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\varepsilon > 0, \text{for which } \Omega \text{ has a finite } \varepsilon\text{-net in } \mathcal{E} \ \}.$$

Notice that the Hausdorff MNC satisfies the semi-homogeneity condition, i.e.:

$$\chi(\lambda\Omega) = |\lambda|\chi(\Omega),$$

for each $\lambda \in \mathbb{R}$ and each $\Omega \in \mathrm{P}(\mathcal{E})$.

Recall that the norm of a set $M \subset \mathcal{E}$ is defined by the formula:

$$\|M\| = \sup_{x \in M} \|x\|_{\mathcal{E}}$$

Definition 2.4. *(See, e.g.,* [4], [8]*). Let X be a metric space. A multivalued map (multimap) $\mathcal{F} : X \to P(\mathcal{E})$ is called:*

(i) *upper semicontinuous (u.s.c.)* if $\mathcal{F}^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$ is an open subset of X for each open set $V \subset \mathcal{E}$;
(ii) *closed* if its graph $\Gamma_{\mathcal{F}} = \{(x, y) : y \in \mathcal{F}(x)\}$ is a closed subset of $X \times \mathcal{E}$;
(iii) *compact* if $\mathcal{F}(X) = \cup_{x \in X} F(x)$ is a relatively compact subset of $\mathcal{E}$;
(iv) *quasicompact* if its restriction to each compact subset $A \subset X$ is compact.

Definition 2.5. *(see* [2], [8]*). A multimap $\mathcal{F} : X \subseteq \mathcal{E} \to K(\mathcal{E})$ is called condensing with respect to a MNC β (or β-condensing)* if for each bounded set $\Omega \subseteq X$ which is not relatively compact, we have:

$$\beta(F(\Omega)) \not\geq \beta(\Omega).$$

In the sequel we will need the following assertions (see [8]).

Lemma 2.6. *Let X and Y be metric spaces and $\mathcal{F} : X \to K(Y)$ a closed quasicompact multimap. Then $\mathcal{F}$ is u.s.c.*

Lemma 2.7. *Let X be a closed subset of a Banach space $\mathcal{E}$, β a monotone MNC in $\mathcal{E}$, Λ a metric space, and $G : \Lambda \times X \to K(\mathcal{E})$ a closed multimap which is β-condensing in the second argument and such that the fixed point set $Fix\ G(\lambda, \cdot) := \{x \in X : x \in G(\lambda, x)\}$ is non-empty for each $\lambda \in \Lambda$. Then the multimap $\mathcal{F} : \Lambda \to P(\mathcal{E})$, where $\mathcal{F}(\lambda) = Fix\ G(\lambda, \cdot)$ is u.s.c.*

Lemma 2.8. *Let X be a closed subset of a Banach space $\mathcal{E}$, β a monotone MNC in $\mathcal{E}$ and $F : X \to K(\mathcal{E})$ a closed multimap which is β-condensing on each bounded set. If the fixed point set* $\mathrm{Fix}\,\mathcal{F} := \{x : x \in \mathcal{F}(x)\}$ *is bounded then it is compact.*

Theorem 2.9. *Let $\mathcal{M}$ be a convex closed bounded subset of $\mathcal{E}$ and $\mathcal{F} : \mathcal{M} \to Kv(\mathcal{M})$ a β-condensing multimap, where β is a monotone nonsingular MNC in $\mathcal{E}$. Then the fixed point set* $\mathrm{Fix}\,\mathcal{F}$ *is a non-empty compact set.*

2.3. **Measurable multifunctions.** Recall some notions (see, e.g., [4], [8]). Let E be a Banach space.

Definition 2.10. For a given $p \geq 1$, a multifunction $G : [0,T] \to K(E)$ is called:

- L^p*-integrable* if it admits an L^p-Bochner integrable selection, i.e., there exists a function $g \in L^p([0,T];E)$ such that $g(t) \in G(t)$ for a.e. $t \in [0,T]$;
- L^p*-integrably bounded* if there exists a function $\xi \in L^p([0,T])$ such that
$$\|G(t)\| \leq \xi(t)$$
for a.e. $t \in [0,T]$.

The set of all L^p-integrable selections of a multifunction $G : [0,T] \to K(E)$ is denoted by $\mathcal{S}^p_G$.

Lemma 2.11. *(See* [8]*, Theorem 4.2.1). Let a sequence of functions* $\{\xi_n\} \subset L^1([0,a];E)$ *be* L^1*-integrably bounded. Suppose that*
$$\chi(\{\xi_n\}(t)) \leq \alpha(t) \quad a.e.\ t \in [0,a]$$
for all $n = 1, 2, ...,$ *where* $\alpha \in L^1_+([0,a])$*. Then for every* $\delta > 0$ *there exist a compact set* $K_\delta \subset E$*, a set* $m_\delta \subset [0,a]$ *of a Lebesgue measure* $m_\delta < \delta$*, and set of functions* $G_\delta \subset L^1([0,a];E)$ *with values in* K_δ*, such that for every* $n \geq 1$ *there exists a function* $b_n \in G_\delta$ *for which*
$$\|\xi_n(t) - b_n(t)\|_E \leq 2\alpha(t) + \delta, \qquad t \in [0,a] \setminus m_\delta.$$
Moreover, the sequence $\{b_n\}$ *may be chosen so that* $b_n \equiv 0$ *on* m_δ *and this sequence is weakly compact.*

In the sequel we will need the following notion.

Definition 2.12. *A sequence of functions* $\{\xi_n\} \subset L^p([0,a];E)$ *is called* L^p*-semicompact if it is* L^p*-integrably bounded, i.e.,*
$$\|\xi_n(t)\|_E \leq v(t) \text{ for a.e. } t \in [0,a] \text{ and for all } n = 1, 2, ...,$$
where $v \in L^p([0,a])$*, and the set* $\{\xi_n(t)\}$ *is relatively compact in* E *for a.e.* $t \in [0,a]$.

3. The local and global existence of solutions to the Cauchy problem

Let a multimap
$$F : [0,a] \times E \to Kv(E)$$
be such that:

$(F1)$ for each $x \in E$ the multifunction $F(\cdot, x) : [0,a] \to Kv(E)$ admits a strongly continuous selection;

$(F2)$ for a.e. $t \in [0,a]$ the multimap $F(t, \cdot) : E \to Kv(E)$ is u.s.c.;

$(F3)$ for each $r > 0$ there exists a function $\omega_r \in L^\infty([0,a])$ such that for each $x \in E$ with $\|x\| \leq r$ we have:
$$\|F(t,x)\| \leq \omega_r(t)$$
for a.e. $t \in [0,a]$;

$(F4)$ there exists a function $\mu \in L^{\infty}([0,a])$ such that for each bounded set $Q \subset E$ we have:

$$\chi(F(t,Q)) \leq \mu(t)\chi(Q),$$

for a.e. $t \in [0,a]$, where χ is the Hausdorff MNC in E.

On a linear operator A we pose the following condition:

(A) $A : D(A) \to E$ is a linear closed operator in E generating a C_0-semigroup$\{T(t)\}_{t\geq 0}$.

Denote $M = \sup\{\|T(t)\|\,; t \in [0;a]\}$.

For $x \in C([0,a];E)$ consider the multifunction:

$$\Phi_F : [0,a] \to Kv(E), \qquad \Phi_F(t) = F(t,x(t)).$$

From above conditions $(F1)-(F3)$ it follows (see, e.g., [8], Theorem 1.3.5) that the multifunction Φ_F is L^p-integrable for each $p \geq 1$.

To solve our problem, we will use the superposition multioperator $\mathcal{P}_F^{\infty} : C([0,a];E) \multimap L^{\infty}([0,a];E)$ defined in the following way:

$$\mathcal{P}_F^{\infty}(x) = \mathcal{S}_{\Phi_F}^{\infty}.$$

Definition 3.1. *A mild solution of the Cauchy problem* (1.1)-(1.2) *on an interval* $[0,\tau]$, $\tau \in (0,a]$ *is called a function* $x \in C([0,\tau];E)$ *which can be represented as:*

$$x(t) = \mathcal{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)\phi(s)ds, \qquad t \in [0,\tau],$$

where $\phi \in \mathcal{P}_F^{\infty}(x)$ *and*

$$\mathcal{G}(t) = \int_0^{\infty} \xi_q(\theta)T(t^q\theta)d\theta, \qquad \mathcal{T}(t) = q\int_0^{\infty} \theta\xi_q(\theta)T(t^q\theta)d\theta,$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\Psi_q(\theta^{-1/q}),$$

$$\Psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta \in \mathbb{R}^+.$$

Remark 3.2. *(See, e.g.* [18]*)* $\int_0^{\infty}\theta\xi_q(\theta)\,d\theta = \frac{1}{\Gamma(q+1)}$, $\xi_q(\theta) \geq 0$.

Lemma 3.3. *(See* [18]*, Lemma 3.4.) The operators* $\mathcal{G}$ *and* $\mathcal{T}$ *possess the following properties:*

1) *For each* $t \in [0,a]$, $\mathcal{G}(t)$ *and* $\mathcal{T}(t)$ *are linear bounded operators, more precisely, for each* $x \in E$ *we have*

$$\|\mathcal{G}(t)x\|_E \leq M\,\|x\|_E\,; \tag{3.1}$$

$$\|\mathcal{T}(t)x\|_E \leq \frac{qM}{\Gamma(1+q)}\,\|x\|_E\,; \tag{3.2}$$

2) *the operator functions* $\mathcal{G}(\cdot)$ *and* $\mathcal{T}(\cdot)$ *are strongly continuous, i.e. functions* $t \in [0,a] \to \mathcal{G}(t)x$ *and* $t \in [0,a] \to \mathcal{T}(t)x$ *are continuous for each* $x \in E$.

To search for mild solutions of problem (1.1)-(1.2) consider the map

$$S : L^{\infty}([0,a];E) \to C([0,a];E),$$

$$S(\phi)(t) = \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)\phi(s)ds$$

and the function $g_0 \in C([0,a];E)$ defined as $g_0(t) = \mathcal{G}(t)x_0$.

Consider the multioperator $G : C([0,a];E) \multimap C([0,a];E)$, given in the following way:

$$G(x) = g_0 + S \circ \mathcal{P}_F^{\infty}(x), \qquad t \in [0,a],$$

It is clear that a function $x \in C([0,a];E)$ is a mild solution of problem (1.1)-(1.2) on the interval $[0,a]$ if and only if it is a fixed point $x \in G(x)$ of the multioperator G.

Lemma 3.4. *Let a sequence $\{\eta_n\} \subset L^p([0,a];E)$, where $\frac{1}{q} < p \le \infty$, be bounded and $\eta_n \rightharpoonup \eta_0$ in $L^1([0,a];E)$. Then $S(\eta_n) \rightharpoonup S(\eta_0)$ in $C([0,a];E)$.*

Proof. For $d > 0$ consider the operator $S_d : L^1([0,a];E) \to C([0,a];E)$:

$$S_d(\eta_n) = \begin{cases} 0, \ t \le d, \\ \int_0^{t-d}(t-s)^{q-1}\mathcal{T}(t-s)\eta_n(s)ds, \ t > d \end{cases} \tag{3.3}$$

Since the integrand in the last expression is the function continuous on $[0, t-d]$, we have

$$S_d(\eta_n) \rightharpoonup S_d(\eta_0). \tag{3.4}$$

in the space $C([0,a];E)$. Let ψ be a continuous linear functional on $C([0,a];E)$, i.e., $\psi \in C^*([0,a];E)$. Then we have

$$(\psi, S(\eta_n)) = (\psi, S_d(\eta_n)) + (\psi, S(\eta_n) - S_d(\eta_n)), \ n = 0,1,2,... \tag{3.5}$$

From the definition of the operator S_d, we conclude:

$$(S(\eta_n) - S_d(\eta_n))(t) = \begin{cases} \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)\eta_n(s)ds, \ t \le d, \\ \int_{t-d}^t (t-s)^{q-1}\mathcal{T}(t-s)\eta_n(s)ds, \ t > d. \end{cases}$$

Then, by using Lemma 3.3, we obtain the following estimates:

$$\|S(\eta_n) - S_d(\eta_n)\|_{C([0,a];E)} \le$$

$$\begin{cases} \int_0^t (t-s)^{q-1}\|\mathcal{T}(t-s)\| \cdot \|\eta_n(s)\|ds, \ t \le d, \\ \int_{t-d}^t (t-s)^{q-1}\|\mathcal{T}(t-s)\| \cdot \|\eta_n(s)\|ds, \ t > d. \end{cases}$$

Then, for $p \in (\frac{1}{q}, \infty)$ the above inequalities may be continued in the following way:

$$\le \begin{cases} \left(\int_0^t (t-s)^{\frac{(q-1)p}{p-1}}ds\right)^{\frac{p-1}{p}} \left(\int_0^t \|\mathcal{T}(t-s)\|^p \cdot \|\eta_n(s)\|^p ds\right)^{\frac{1}{p}}, \ t \le d, \\ \left(\int_{t-d}^t (t-s)^{\frac{(q-1)p}{p-1}}ds\right)^{\frac{p-1}{p}} \left(\int_{t-d}^t \|\mathcal{T}(t-s)\|^p \cdot \|\eta_n(s)\|^p ds\right)^{\frac{1}{p}} \ t > d \end{cases}$$

$$\le \frac{qMd^{(q-\frac{1}{p})}}{\Gamma(1+q)}\Big[\frac{p-1}{qp-1}\Big]^{\frac{p-1}{p}}\|\eta_n\|_{L^p}.$$

For $p = \infty$ the corresponding continuation yields

$$\leq \frac{qMd^q}{\Gamma(1+q)} \|\eta_n\|_{L^\infty}.$$

(The constant M here is taken from (3.1)).

Therefore, for an arbitrary $\epsilon > 0$, we may choose such $d > 0$ that the following estimate holds true:

$$\|S(\eta_n) - S_d(\eta_n)\|_{C([0,a];E)} \leq \frac{\epsilon}{4\|\psi\|_{C^*([0,a];E)}}. \tag{3.6}$$

By virtue of (3.4), $(\psi, S_d(\eta_n)) \to (\psi, S_d(\eta_0))$, but then for a given ϵ, we may choose number n_0 such that

$$(\psi, S_d(\eta_{n_0}) - S_d(\eta_0)) < \epsilon/2 \tag{3.7}$$

Now, by using (3.5), (3.6), (3.7), we obtain:

$$\begin{gathered}(\psi, S(\eta_n) - S(\eta_0)) = (\psi, S_d(\eta_n) - S_d(\eta_0)) \\ + (\psi, S(\eta_n) - S_d(\eta_n)) + (\psi, S_d(\eta_0) - S(\eta_0)) \\ < \frac{\epsilon}{2} + 2\|\psi\|_{C^*([0,a];E)} \frac{\epsilon}{4\|\psi\|_{C^*([0,a];E)}} = \epsilon,\end{gathered}$$

concluding the proof. □

Lemma 3.5. *Let $\{f_n\}_{n=1}^\infty$ be a bounded sequence in $L^\infty([0,a];E)$ such that*

$$\chi(\{f_n(t)\}) \leq \kappa(t) \text{ a.e. } t \in [0,a],$$

where $\kappa \in L^\infty_+(0,a)$. Then

$$\chi(\{Sf_n(t)\}) \leq 2\frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \kappa(s) ds$$

Proof. Let $\|f_n\|_\infty \leq K$ for all $n = 1, 2, \ldots$ Then the sequence $\{S_d f_n\}$, by virtue of estimate (3.2), is an $\frac{qMd^qK}{\Gamma(1+q)}$-net in the space $C([0,a];E)$ of the sequence $\{Sf_n\}$. By Theorem 4.2.2 of [8] and (3.2) we have

$$\chi(\{S_d f_n(t)\}_{n=1}^\infty) \leq 2\frac{qM}{\Gamma(1+q)} \int_0^{t-d} (t-s)^{q-1} k(s) ds. \tag{3.8}$$

The result now follows from the arbitrariness of d. □

Lemma 3.6. *The operator S obeys the following properties:*

(S_1) if $\frac{1}{q} < p < \infty$, then there exists a constant $C > 0$ such that

$$\|S(\xi)(t) - S(\eta)(t)\|_E^p \leq C^p \int_0^t \|\xi(s) - \eta(s)\|_E^p ds, \qquad \xi, \eta \in L^p([0,a]);$$

(S_2) for each compact set $K \subset E$ and bounded sequence $\{\eta_n\} \subset L^\infty([0,a];E)$ such that $\{\eta_n(t)\} \subset K$ for a.e. $t \in [0,a]$, the weak convergence $\eta_n \rightharpoonup \eta_0$ in $L^1([0,a];E)$ implies the convergence $S(\eta_n) \to S(\eta_0)$ in $C([0,a];E)$.

Proof. (S_1) By using the Hölder inequality, we get:

$$\|S(\xi)(t) - S(\eta)(t)\|_E \leq \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)\, \|\xi(s) - \eta(s)\|_E \, ds$$

$$\leq \frac{qM}{\Gamma(1+q)} \left[\int_0^t (t-s)^{\frac{(q-1)p}{p-1}} ds\right]^{\frac{p-1}{p}} \left[\int_0^t \|\xi(s) - \eta(s)\|_E^p \, ds\right]^{\frac{1}{p}}$$

Then

$$\|S(\xi)(t) - S(\eta)(t)\|_E^p \leq C^p \int_0^t \|\xi(s) - \eta(s)\|_E^p \, ds,$$

where

$$C = \left[\frac{p-1}{qp-1}\right]^{\frac{p-1}{p}} \frac{qMa^{q-\frac{1}{p}}}{\Gamma(1+q)}$$

(S_2) Applying Lemma 3.3, we obtain:

$$\chi\Big(\left\{S\left(\eta_n\right)(t)\right\}\Big) \leq \int_0^t (t-s)^{q-1}\chi(\{\mathcal{T}\left(t-s\right)\eta_n\})ds = 0$$

This means that the sequence $\{S\left(\eta_n\right)(t)\}_{n=1}^{\infty} \subset E$ is relatively compact for each $t \in [0,a]$.

From the other side, if we take $t_1, t_2 \in [0,a]$ such that $0 < t_1 < t_2 \leq a$, then we have:

$$\Big\|S\left(\eta_n\left(t_2\right)\right) - S\left(\eta_n\left(t_1\right)\right)\Big\|_E$$

$$= \left\|\int_0^{t_2} \left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) \eta_n(s)ds - \int_0^{t_1} \left(t_1 - s\right)^{q-1} \mathcal{T}\left(t_1 - s\right) \eta_n(s)ds\right\|_E$$

$$= \left\|\int_{t_1}^{t_2} \left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) \eta_n(s)ds\right\|_E$$

$$+ \left\|\int_0^{t_1} \Big(\left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) - \left(t_1 - s\right)^{q-1} \mathcal{T}\left(t_1 - s\right)\Big) \eta_n(s)ds\right\|_E$$

$$\leq \left\|\int_{t_1}^{t_2} \left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) \eta_n(s)ds\right\|_E$$

$$+ \left\|\int_0^{t_1} \Big(\left(t_2 - s\right)^{q-1} - \left(t_1 - s\right)^{q-1}\Big) \mathcal{T}\left(t_1 - s\right) \eta_n(s)ds\right\|_E$$

$$+ \left\|\int_0^{t_1} \left(t_2 - s\right)^{q-1} \left(\mathcal{T}\left(t_2 - s\right) - \mathcal{T}\left(t_1 - s\right)\right) \eta_n(s)ds\right\|_E = Z_1 + Z_2 + Z_3,$$

where

$$Z_1 = \left\|\int_{t_1}^{t_2} \left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) \eta_n(s)ds\right\|_E,$$

$$Z_2 = \left\| \int_0^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \mathcal{T}(t_1 - s)\, \eta_n(s) ds \right\|_E,$$

$$Z_3 = \left\| \int_0^{t_1} (t_2 - s)^{q-1} \left(\mathcal{T}(t_2 - s) - \mathcal{T}(t_1 - s) \right) \eta_n(s) ds \right\|_E.$$

By using Lemma 3.3 and condition $(F3)$, we can for each $\epsilon_1 > 0$ choose $\delta_1 > 0$ such that $|t_2 - t_1| < \delta_1$ implies the following estimate:

$$Z_1 \leq \frac{qM \|\omega_K\|_\infty}{\Gamma(1+q)} \frac{(t_2 - t_1)^q}{q} < \epsilon_1.$$

To estimate Z_2, take constant $d > 0$, for which we have:

$$Z_2 \leq \left\| \int_0^{t_1 - d} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \mathcal{T}(t_1 - s)\, \eta_n(s) ds \right\|_E$$

$$+ \left\| \int_{t_1 - d}^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \mathcal{T}(t_1 - s)\, \eta_n(s) ds \right\|_E = I_1 + I_2,$$

where

$$I_1 = \left\| \int_0^{t_1 - d} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \mathcal{T}(t_1 - s)\, \eta_n(s) ds \right\|_E,$$

$$I_2 = \left\| \int_{t_1 - d}^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \mathcal{T}(t_1 - s)\, \eta_n(s) ds \right\|_E.$$

Consider the function $v : [d, a] \to \mathbb{R}, v(\tau) = \tau^{q-1}$. The given function is continuous on the interval $[d, a]$, hence, by the Cantor theorem, it is uniformly continuous on this interval, i.e., for each $\gamma > 0$ there exists $\delta_2 > 0$, such that $|\tau_2 - \tau_1| < \delta_2 < d, \tau_1, \tau_2 \in [d, a]$ implies

$$\left| \tau_2^{q-1} - \tau_1^{q-1} \right| < \gamma.$$

Now, taking $\tau = t - s$, we get:

$$I_1 \leq \frac{qM \|\omega_K\|_\infty \gamma (t_1 - d)}{\Gamma(1+q)} < \epsilon_2.$$

By direct integration, for I_2 we obtain:

$$I_2 \leq \frac{M \|\omega_K\|_\infty d^q (2 + 2^q)}{\Gamma(1+q)} < \epsilon_3$$

Taking into account that the family of operators $\mathcal{T}(t)$ is strongly continuous for $x \in K$, i.e., for each $\gamma_1 > 0$ there exists $\delta_3 > 0$ such that $|t_2 - t_1| < \delta_3$ implies

$$\| \mathcal{T}(t_2 - s)\, x - \mathcal{T}(t_1 - s)\, x \| < \gamma_1, \ x \in K,$$

we get the following estimate:

$$Z_3 \leq \gamma_1 a^q < \epsilon_4.$$

Therefore, for each $\epsilon > 0$ we may choose $\delta = min\{\delta_1, \delta_2, \delta_3\}$ such that

$$\left\| S\left(\eta_n\left(t_2\right)\right) - S\left(\eta_n\left(t_1\right)\right) \right\|_E \leq Z_1 + Z_2 + Z_3 < \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 < \epsilon$$

So, the sequence $\{S\left(\eta_n\right)\}$ is equicontinuous. From the Arzela-Ascoli theorem we conclude that the sequence $\{S\left(\eta_n\right)\} \subset C\left([0,a]\,;E\right)$ is relatively compact. From Lemma 3.4 we know that the weak convergence $\eta_n \rightharpoonup \eta_0$ implies $S\left(\eta_n\right) \rightharpoonup S\left(\eta_0\right)$. Since the sequence $\{S\left(\eta_n\right)\}$ is relatively compact, we conclude that $S\left(\eta_n\right) \to S\left(\eta_0\right)$ in $C\left([0,a]\,;E\right)$. □

To prove that the multioperator G is condensing, define the vector measure of noncompactness in the space $C([0,a];E)$

$$\nu : P(C([0,a];E)) \to \mathbb{R}^2_+$$

with the values in the cone $\mathbb{R}^2_+$ defined as

$$\nu(\Omega) = \max_{\mathcal{D}\in\Delta(\Omega)} \left(\varphi(\mathcal{D}), mod_C(\mathcal{D})\right),$$

where $\Delta(\Omega)$ denotes the collection of all countable subsets of Ω,

$$mod_C(\mathcal{D}) = \lim_{\delta\to 0} \sup_{x\in\mathcal{D}} \max_{|t_1-t_2|\leq\delta} \|x(t_1) - x(t_2)\|,$$

$$\varphi(\mathcal{D}) = \sup_{t\in[0,a]} e^{-pt}\chi(\mathcal{D}(t)),$$

and the constant $p > 0$ is chosen so that

$$\varrho := 2\frac{qM\,\|\mu\|_\infty}{\Gamma(1+q)} \int_0^t (t-s)^{q-1}\, e^{-p(t-s)} ds < 1.$$

Such a choice can be justified in the following way. Take $d > 0$ such that

$$2\frac{qM\,\|\mu\|_\infty}{\Gamma(1+q)} \frac{d^q}{q} < \frac{1}{2}$$

and then, choose $p > 0$ such that

$$2\frac{qM\,\|\mu\|_\infty}{\Gamma(1+q)} \frac{1}{pd^{1-q}} < \frac{1}{2}.$$

Now we have

$$2\frac{qM\,\|\mu\|_\infty}{\Gamma(1+q)} \int_0^t (t-s)^{q-1}\, e^{-p(t-s)} ds$$

$$2\frac{qM\,\|\mu\|_\infty}{\Gamma(1+q)} \left(\int_0^{t-d} (t-s)^{q-1}\, e^{-p(t-s)} ds + \int_{t-d}^t (t-s)^{q-1}\, e^{-p(t-s)} ds \right)$$

$$\leq 2\frac{qM\,\|\mu\|_\infty}{\Gamma(1+q)} \left(\frac{1}{d^{1-q}} \frac{e^{-pd}}{p} + \frac{d^q}{q} \right)$$

$$\leq 2\frac{qM\,\|\mu\|_\infty}{\Gamma(1+q)} \left(\frac{1}{pd^{1-q}} + \frac{d^q}{q} \right) < 1.$$

Lemma 3.7. *The operator G is condensing w.r.t. the measure of noncompactness ν.*

Proof. Let $\Omega \subset C([0,a];E)$ be a nonempty bounded set and

$$\nu(G(\Omega)) \geq \nu(\Omega), \tag{3.9}$$

where the inequality is taken in the sense of the order in $\mathbb{R}^2$ induced by the cone $\mathbb{R}^2_+$. Let us show that Ω is the relatively compact set.

Since the measure of noncompactness ν is nonsingular (see [8]), it is sufficient to prove the assertion for the multioperator $S \circ \mathcal{P}^{\infty}_F$.

Let the maximum of the left-hand side of the inequality be achieved on the countable set $\mathcal{D}' = \{g_n\}_{n=1}^{\infty}$. Then

$$g_n(t) = Sf_n(t), \quad f_n \in \mathcal{P}^{\infty}_F(x_n),\ n \geq 1,$$

where $\{x_n\}_{n=1}^{\infty} \subset \Omega$.

By virtue of (3.9) we have:

$$\varphi(\{g_n\}_{n=1}^{\infty}) \geq \varphi(\{x_n\}_{n=1}^{\infty}). \tag{3.10}$$

Now, we can give a upper estimate for $\varphi(\{g_n\}_{n=1}^{\infty})$.

The χ-regularity property $(F4)$ implies

$$\begin{aligned}\chi(\{f_n(s)\}_{n=1}^{\infty}) &\leq \mu(s) \cdot \chi(\{x_n(s)\}_{n=1}^{\infty})\\ &= e^{ps}\mu(s)e^{-ps} \cdot \chi(\{x_n(s)\}_{n=1}^{\infty})\\ &\leq e^{ps}\mu(s) \cdot \sup_{\xi \in [0,a]} e^{-p\xi}\chi(\{x_n(\xi)\}_{n=1}^{\infty}) = e^{ps}\mu(s) \cdot \varphi(\{x_n\}_{n=1}^{\infty}).\end{aligned}$$

Applying Lemma 3.5 and estimate (3.2), we obtain

$$\chi(\{Sf_n(t)\}_{n=1}^{\infty}) \leq 2\frac{qM\,\|\mu\|_{\infty}}{\Gamma(1+q)}\left(\int_0^t (t-s)^{q-1}\, e^{ps} ds\right) \cdot \varphi(\{x_n\}_{n=1}^{\infty}). \tag{3.11}$$

Now, from estimates (3.10), (3.11) it follows that

$$\begin{aligned}\varphi(\{x_n\}_{n=1}^{\infty}) &\leq 2\frac{qM\,\|\mu\|_{\infty}}{\Gamma(1+q)} \sup_{t \in [0,a]} \left(\int_0^t (t-s)^{q-1}\, e^{-p(t-s)} ds\right) \cdot \varphi(\{x_n\}_{n=1}^{\infty})\\ &\leq \varrho \cdot \varphi(\{x_n\}_{n=1}^{\infty}),\end{aligned}$$

implying

$$\varphi(\{x_n\}_{n=1}^{\infty}) = 0,$$

and therefore

$$\chi(\{x_n(t)\}_{n=1}^{\infty}) = 0$$

for all $t \in [0,a]$.

Now, from inequality (3.9) we have

$$mod_C(\{x_n\}_{n=1}^{\infty}) \leq mod_C\left(\{Sf_n\}_{n=1}^{\infty}\right). \tag{3.12}$$

Now prove that

$$mod_C\left(\{Sf_n\}_{n=1}^{\infty}\right) = 0.$$

To do it, let us show that the set

$$\left\{\int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f_n(s)ds : f_n(s) \in \mathcal{P}^{\infty}_F(x_n)\right\}$$

is equicontinuous. If we take $t_1, t_2 \in [0, a]$ such that $0 < t_1 < t_2 \leq a$, then for arbitrary f_n we will have

$$\left\| S\left(f_n\left(t_2\right)\right) - S\left(f_n\left(t_1\right)\right) \right\|_E$$

$$= \left\| \int_0^{t_2} \left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) f_n(s) ds - \int_0^{t_1} \left(t_1 - s\right)^{q-1} \mathcal{T}\left(t_1 - s\right) f_n(s) ds \right\|_E$$

$$= \left\| \int_{t_1}^{t_2} \left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) f_n(s) ds \right\|_E$$

$$+ \left\| \int_0^{t_1} \Big(\left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) - \left(t_1 - s\right)^{q-1} \mathcal{T}\left(t_1 - s\right) \Big) f_n(s) ds \right\|_E = Z_1 + Z_2,$$

where

$$Z_1 = \left\| \int_{t_1}^{t_2} \left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) f_n(s) ds \right\|_E,$$

$$Z_2 = \left\| \int_0^{t_1} \Big(\left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) - \left(t_1 - s\right)^{q-1} \mathcal{T}\left(t_1 - s\right) \Big) f_n(s) ds \right\|_E,$$

By using Lemma 3.3 and condition $(F3)$ we may find, for each $\epsilon_1 > 0$, such $\delta_1 > 0$, that for $|t_2 - t_1| < \delta_1$, we will have the following estimate:

$$Z_1 \leq \frac{qM \left\|\omega_K\right\|_\infty}{\Gamma\left(1+q\right)} \frac{\left(t_2 - t_1\right)^q}{q} < \epsilon_1.$$

To estimate Z_2 take any $\epsilon_2 > 0$ and choose

$$d < d_1 = \left[\frac{\epsilon_2 \Gamma(1+q)}{M \|\omega_K\|_\infty (2^q + 1)} \right]^{\frac{1}{q}}.$$

Then, if $t_1 < d$ and $t_2 - t_1 < d$, we have the following estimate

$$Z_2 \leq \int_0^{t_1} (t_2 - s)^{q-1} \|\mathcal{T}(t_2 - s)\| \cdot \|f_n(s)\| ds + \int_0^{t_1} (t_1 - s)^{q-1} \|\mathcal{T}(t_1 - s)\| \cdot \|f_n(s)\| ds$$

$$\leq \int_0^{t_2} (t_2 - s)^{q-1} \|\mathcal{T}(t_2 - s)\| \cdot \|f_n(s)\| ds + \int_0^{t_1} (t_1 - s)^{q-1} \|\mathcal{T}(t_1 - s)\| \cdot \|f_n(s)\| ds$$

$$\leq \frac{M \|\omega_K\|_\infty}{\Gamma(1+q)} \left(2^q + 1\right) d^q < \epsilon_2$$

If $t_1 > d$ we have

$$Z_2 \leq \left\| \int_0^{t_1 - d} \Big(\left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) - \left(t_1 - s\right)^{q-1} \mathcal{T}\left(t_1 - s\right) \Big) f_n(s) ds \right\|_E$$

$$+ \left\| \int_{t_1 - d}^{t_1} \Big(\left(t_2 - s\right)^{q-1} \mathcal{T}\left(t_2 - s\right) - \left(t_1 - s\right)^{q-1} \mathcal{T}\left(t_1 - s\right) \Big) f_n(s) ds \right\|_E = I_1 + I_2,$$

where

$$I_1 = \left\| \int_0^{t_1-d} \Big((t_2-s)^{q-1} \mathcal{T}(t_2-s) - (t_1-s)^{q-1} \mathcal{T}(t_1-s) \Big) f_n(s)ds \right\|_E,$$

$$I_2 = \left\| \int_{t_1-d}^{t_1} \Big((t_2-s)^{q-1} \mathcal{T}(t_2-s) - (t_1-s)^{q-1} \mathcal{T}(t_1-s) \Big) f_n(s)ds \right\|_E.$$

Choose $d < d_1$ so that

$$I_2 \leq \frac{M \left\|\omega_K\right\|_\infty d^q \left(2+2^q\right)}{\Gamma(1+q)} < \epsilon_2$$

for a given $\epsilon_2 > 0$. Since $\chi\left(\{x_n(t)\}_{n=1}^\infty\right) \equiv 0$, then, by Lemma 2.11 for each $\delta > 0$ there exist a compact set $K_\delta \subset E$, and a set $m_\delta \subseteq [0,a]$, of the Lebesgue measure $mes(m_\delta) < \delta$ such that $\{x_n(t)\}_{n=1}^\infty \subset K_\delta$ for $t \in [0,a] \setminus m_\delta$, and so, for I_1, we have the estimate

$$\begin{aligned} I_1 \leq & \left\| \int_{[0,t_1-d]\setminus m_\delta} \Big((t_2-s)^{q-1} \mathcal{T}(t_2-s) - (t_1-s)^{q-1} \mathcal{T}(t_1-s) \Big) f_n(s)ds \right\|_E \\ & + \left\| \int_{[0,t_1-d]\cap m_\delta} \Big((t_2-s)^{q-1} \mathcal{T}(t_2-s) - (t_1-s)^{q-1} \mathcal{T}(t_1-s) \Big) f_n(s)ds \right\|_E. \end{aligned}$$

Take δ so small that $mes(m_\delta) < 2\epsilon_3 d^{1-q}$ for any given $\epsilon_3 > 0$. By using condition (S_2) from Lemma 3.6 and, taking into account that $F(s,x(s)) \subset F([0,a] \times K_\delta)$ we claim that for each $\epsilon_4 > 0$, we can choose $\gamma > 0$ such that $|t_2 - t_1| < \gamma$ will imply that the first summand in the above estimate for I_1 will be less than ϵ_4.

So, for each $\epsilon > 0$ we may choose $\delta' = min\{\delta_1, \delta, \gamma\}$ such that

$$\left\| Sf_n(t_2) - Sf_n(t_1) \right\|_E \leq Z_1 + Z_2 + Z_3 \leq Z_1 + I_1 + I_2 < \epsilon$$

Since the set $\{Sf_n\}_{n=1}^\infty$ is equicontinuous, we have $mod_C\left(\{x_n\}_{n=1}^\infty\right) = 0$, hence $\nu(\Omega) = (0,0)$. So, we conclude that Ω is relatively compact set yielding that the multioperator G is condensing w.r.t. the MNC ν. □

Lemma 3.8. *The multioperator G is u.s.c.*

Proof. Since the family $\mathcal{G}(t)$ is strongly continuous, it is sufficient to prove the assertion for the multioperator $S \circ \mathcal{P}_F^\infty$.

Take a sequence $\{x_n\}_{n=1}^\infty \subset C([0,a];E)$ such that $x_n \to x$. Then for each sequence $f_n \in \mathcal{P}_F^\infty(x_n)$, $n \geq 1$ for a.e. $t \in [0,a]$, according to condition $(F4)$, the set $\{f_n(t)\}_{n=1}^\infty$, is relatively compact in E, hence the sequence $\{f_n\}_{n=1}^\infty$ is L^1-semicompact. By the Diestel criterion of weak relative compactness (see [5]), we can assume, w.l.o.g. that $f_n \overset{L^1}{\rightharpoonup} f^0$. It remains to use property (S_2) from Lemma 3.6 □

Now we can prove the local existence theorem for Cauchy problem (1.1)-(1.2).

Theorem 3.9. *Under conditions* (A), $(F1)-(F4)$ *there exists* $\tau \in (0,a]$ *such that the set of mild solutions of Cauchy problem* (1.1)-(1.2) $\Sigma_{x_0}^F[0,\tau]$ *on the interval* $[0,\tau]$ *is a nonempty subset of the space* $C([0,\tau];E)$.

Proof. Take a number $r>0$. Since the family of operators $\mathcal{G}(t)$ is equicontinuous we may choose $0<\tau_1<a$ such that

$$\|(\mathcal{G}(t)-\mathcal{G}(0))\,x_0\|_E \leq r/2 \text{ for all } t\in[0,\tau_1]. \tag{3.13}$$

Let $\overline{B}_r(\mathcal{G}(0)x_0)\subset E$ be a closed ball and $R=\|\mathcal{G}(0)x_0\|_E+r$, take $\tau_2\in(0,a]$ such that

$$\frac{M\,\|\omega_R\|_\infty\,\tau_2^q}{\Gamma(1+q)} \leq r/2, \tag{3.14}$$

where M is the constant from condition (A), and ω_R is the function from condition $(F3)$.

From Lemmas 3.7 and 3.8 we know that the multioperator G is u.s.c. and ν-condensing. Take $\tau=\min(\tau_1,\tau_2)$. Consider the ball $\overline{B}_r(x^0)\subset C([0,\tau];E)$, where x^0 is the function identically equal to $\mathcal{G}(0)x_0$.

We will show that the multioperator G transforms the ball $\overline{B}_r(x^0)$ into itself. In fact, if $x\in\overline{B}_r(x^0)$, then $\|x\|_{C([0,\tau];E)}\leq R$ for all $t\in[0,\tau]$ and from condition $(F3)$ we have

$$\|f(t)\|_E \leq \omega_R(t), \text{ a.e. } t\in[0,\tau],$$

for all $f\in\mathcal{P}_F^\infty(x)$.

Now, for $y\in G(x)$ we have

$$y(t)=\mathcal{G}(t)x_0+\int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f(s)ds,\ f\in\mathcal{P}_F^\infty(x).$$

By using (3.13)-(3.14) and Lemma 3.3, we have the following estimate:

$$\begin{aligned}\|y(t)-x_0\|_E &\leq \|(\mathcal{G}(t)-\mathcal{G}(0))\,x_0\|_E+\int_0^t (t-s)^{q-1}\,\|\mathcal{T}(t-s)\|_{L(E)}\,\|f(s)\|_E\,ds\\ &\leq \|(\mathcal{G}(t)-\mathcal{G}(0))\,x_0\|_E+\frac{M\,\|\omega_R\|_\infty\,\tau^q}{\Gamma(1+q)}\leq r/2+r/2\leq r,\end{aligned}$$

from which it follows that $y\in\overline{B}_r(x^0)$. Now we can apply Theorem 2.9 □

Let us prove the global existence result.

Theorem 3.10. *Under conditions* (A), $(F1)$, $(F2)$, $(F4)$, *and sub-linear growth condition*

$(F'3)$ *there exists a function* $\alpha\in L_+^\infty([0,a])$ *such that*

$$\|F(t,x)\|_E\leq\alpha(t)(1+\|x(t)\|_E) \textit{ for a.e. } t\in[0,a],$$

the set of all mild solutions to Cauchy problem (1.1)-(1.2) $\Sigma_{x_0}^F[0,a]$ *is a nonempty compact subset of the space* $C([0,a];E)$.

Proof. Introduce the equivalent norm in the space $C([0,a];E)$:

$$\|x\|_* = \max_{t\in[0,a]} e^{-pt}\|x(t)\|_E ,$$

where the constant $p>0$ is chosen so that for a certain $d>0$ the following inequality holds

$$\frac{qM\|\alpha\|_\infty}{\Gamma(1+q)}\left(\frac{1}{pd^{1-q}}+\frac{d^q}{q}\right)\le N<1$$

In the space $C([0,a];E)$ with the norm $\|\cdot\|_*$, consider the ball

$$\overline{B}_r(0)=\{x\in C([0,a];E)|\, \|x\|_*\le r\},$$

where $r>0$ is taken so that

$$r\ge\left(M\|x_0\|_E+\frac{M\|\alpha\|_\infty a^q}{\Gamma(1+q)}\right)(1-N)^{-1}.$$

Notice that the last inequality implies the following:

$$M\|x_0\|_E+\frac{M\|\alpha\|_\infty a^q}{\Gamma(1+q)}+Nr\le r.$$

Let us demonstrate now that that the multioperator G transforms the ball $\overline{B}_r(0)$ into itself. In fact, if we will take $x\in\overline{B}_r(0)$ and $y\in G(x)$, by using Lemma 3.3, we have, for any $f\in\mathcal{P}^\infty_F(x)$:

$$\begin{aligned}
e^{-pt}\|y(t)\|_E &\le e^{-pt}\|\mathcal{G}(t)x_0\|_E+e^{-pt}\int_0^t(t-s)^{q-1}\|\mathcal{T}(t-s)\|_{L(E)}\|f(s)\|_E\,ds\\
&\le M\|x_0\|_E+e^{-pt}\frac{Mq}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}\alpha(t)(1+\|x(s)\|_E)ds\\
&\le M\|x_0\|_E+e^{-pt}\frac{Mq\|\alpha\|_\infty}{\Gamma(1+q)}\left(\int_0^t(t-s)^{q-1}ds+\int_0^t(t-s)^{q-1}e^{ps}e^{-ps}\|x(s)\|_E\,ds\right)\\
&\le M\|x_0\|_E+e^{-pt}\frac{Mq\|\alpha\|_\infty}{\Gamma(1+q)}\frac{a^q}{q}+\|x\|_*\frac{Mq\|\alpha\|_\infty}{\Gamma(1+q)}e^{-pt}\int_0^t(t-s)^{q-1}e^{ps}ds\\
&\le M\|x_0\|_E+\frac{M\|\alpha\|_\infty a^q}{\Gamma(1+q)}\\
&+\|x\|_*\frac{Mq\|\alpha\|_\infty}{\Gamma(1+q)}e^{-pt}\left(\int_0^{t-d}(t-s)^{q-1}e^{ps}ds+\int_{t-d}^t(t-s)^{q-1}e^{ps}ds\right)\\
&\le M\|x_0\|_E+\frac{M\|\alpha\|_\infty a^q}{\Gamma(1+q)}+\|x\|_*\frac{Mq\|\alpha\|_\infty}{\Gamma(1+q)}\left(e^{-pt}\frac{1}{d^{1-q}}\frac{e^{p(t-d)}-1}{p}+\frac{d^q}{q}\right)\\
&\le M\|x_0\|_E+\frac{M\|\alpha\|_\infty a^q}{\Gamma(1+q)}+\|x\|_*\frac{Mq\|\alpha\|_\infty}{\Gamma(1+q)}\left(\frac{1}{pd^{1-q}}+\frac{d^q}{q}\right)\\
&\le M\|x_0\|_E+\frac{M\|\alpha\|_\infty a^q}{\Gamma(1+q)}+\|x\|_*N\le r.
\end{aligned}$$

So, $\|y\|_*\le r$.

From Lemmas 3.7 and 3.8 we know that the multioperator G is u.s.c. and ν-condensing. From Theorem 2.9 we obtain that the set $\Sigma^F_{x_0}[0,a]$ is nonempty.

Now we can show that the set $\Sigma_{x_0}^F[0,a]$ is a priori bounded. In fact, from the above estimate it follows that for $x \in \Sigma_{x_0}^F[0,a]$ and $f \in \mathcal{P}_F^\infty(x)$, we have for each $t \in [0,a]$:

$$\|x\|_* \leq M\,\|x_0\|_E + \frac{M\,\|\alpha\|_\infty\, a^q}{\Gamma(1+q)} + \|x\|_*\, N,$$

implying the following estimate:

$$\|x\|_* \leq \left(M\,\|x_0\|_E + \frac{M\,\|\alpha\|_\infty\, a^q}{\Gamma(1+q)}\right)(1-N)^{-1}.$$

Applying Lemma 2.8 we obtain that the set $\Sigma_{x_0}^F[0,a]$ is compact. □

4. Continuous dependence of the solutions set on parameter and initial data

Consider now the dependence of the solutions set of our problem on the parameter λ from a metric space (Λ,ρ) :

$$D^q x(t) \in Ax(t) + F(t, x(t), \lambda), \tag{4.1}$$

$$x(0) = x_0. \tag{4.2}$$

We will assume that the operator A satisfies condition (A) and the multimap $F : [0,a] \times E \times \Lambda \to Kv(E)$ obeys the following conditions:

($F1_\lambda$) for all $(x,\lambda) \in E \times \Lambda$ the multifunction $F(\cdot,x,\lambda) : [0,a] \longrightarrow Kv(E)$ admits a strongly continuous selection;

($F2_\lambda$) for a.e. $t \in [0,a]$ the multimap $F(t,\cdot,\cdot) : E \longrightarrow Kv(E)$ is u.s.c.;

($F3_\lambda$) for each $n \in \mathbb{N}$ there exists a function $\omega_n \in L^\infty([0,a])$ such that for each function $x \in \overline{B}(0,n) \subset C([0,a];E)$ we have:

$$\|F(t,x,\lambda)\| := \sup\{\|f\|_E : f \in F(t,x,\lambda)\} \leq \omega_n(t)$$

for a.e. $t \in [0,a]$;

($F4_\lambda$) there exists a function $\mu \in L^\infty([0,a])$ such that for each bounded set $Q \subset E$ we have:

$$\chi(F(t,Q,\Lambda)) \leq \mu(t)\chi(Q),$$

for a.e. $t \in [0,a]$, where χ is the Hausdorff measure of noncompactness in E.

Theorem 4.1. *Assume that conditions* (A), $(F1_\lambda) - (F4_\lambda)$ *are satisfied. Suppose that for a certain value* $\lambda_0 \in \Lambda$ *of the parameter, the solutions set* $\Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}[0,a]$ *of problem* (4.1)-(4.2) *is bounded and satisfies the following condition of extendability:*

$$\Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}[0,\tau] = \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}[0,a]|_{[0,\tau]} \text{ for each } \tau \in (0,a]. \tag{4.3}$$

Then for every $r > 0$ *there exists* $\delta > 0$ *such that for each* $\lambda \in \Lambda : \rho(\lambda,\lambda_0) < \delta$, *the set* $\sum_{x_0}^{F(\cdot,\cdot,\lambda)}[0,a]$ *is non-empty and, moreover,*

$$\Sigma_{x_0}^{F(\cdot,\cdot,\lambda)}[0,a] \subset W_r\left(\Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}[0,a]\right),$$

i.e., the multioperator $\lambda \multimap \Sigma_{x_0}^{F(\cdot,\cdot,\lambda)}[0,a]$ *is u.s.c. at the point* λ_0.

Proof. Let $C > 0$ be a constant such that :

$$\left\| \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}[0,a] \right\| \leq C, \tag{4.4}$$

and

$$M = \max_{t\in[0,a]} \|T(t)\| .$$

We will show that there exists $\delta_0 > 0$ such that for all $\lambda \in \Lambda : \rho(\lambda, \lambda_0) < \delta_0$, we have:

$$\left\| \Sigma_{x_0}^{F(\cdot,\cdot,\lambda)}(t) \right\| \leq 3C, \tag{4.5}$$

where $\Sigma_{x_0}^{F(\cdot,\cdot,\lambda)}$ is the solutions set of problem (4.1)-(4.2), i.e., $x \in \Sigma_{x_0}^{F(\cdot,\cdot,\lambda)}$ if there exists $\Delta > 0$ such that for each $\tau < \Delta$ and $y^\tau = x|_{[0,\tau]} \in C([0,\tau]; E)$, $y^\tau \in G(\lambda, y^\tau)$, where $G(\lambda, \cdot)$ is the integral multioperator in the space $C([0,\tau]; E)$ corresponding to the multimap $F(\cdot,\cdot,\lambda)$ defined for $\lambda \in \Lambda$ and $x \in C([0,\tau]; E)$ by the formula

$$G(\lambda, x) = \left\{ y : \; y(t) = \mathcal{G}(t)x_0 + z(t), z \in S \circ \mathcal{P}_{F_\lambda}^\infty(x) \right\}.$$

Suppose that (4.5) fails, then we can choose sequences $\lambda_n \in \Lambda,\ \lambda_n \to \lambda_0,\ t_n \in [0,a],\ x_n \in C([0,t_n]; E)$, such that :

$$x_n \in G(\lambda_n, x_n)(t) \text{ for } t \in [0, t_n]$$

and

$$dist\left(x_n\left(t_n\right), \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}\left(t_n\right) \right) = 2C, \tag{4.6}$$

$$dist\left(x_n\left(t_n\right), \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}\left(t\right) \right) < 2C,\ t \in [0, t_n), \tag{4.7}$$

where *dist* denotes the distance of a point from a set.

Let $t_* = \underline{lim} t_n$, from $(F3_\lambda)$ it follows that $t_* > 0$. In fact, suppose the contrary. Then there exists a subsequence of the sequence t_n (without loss of generality, we can assume that it is the sequence t_n itself), which converges to zero. Then from $(F3_\lambda)$ it follows that selections $f_n \in \mathcal{S}^\infty_{F(\cdot,x_n(\cdot),\lambda_n)}$ satisfy:

$$\|f_n(t)\| \leq \omega_{3C}(t) \text{ for a.e. } t \in [0, t_n].$$

From condition (4.3) it follows that the set $\Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}[0,a]$ is compact, so we get

$$dist\left(x_0, \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}\left(t_n\right) \right) \to 0. \tag{4.8}$$

By using (4.6), we obtain

$$0 < 2C = dist\left(x_n\left(t_n\right), \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}\left(t_n\right) \right)$$

$$\leq \|x_n(t_n) - x_0\|_E + dist\left(x_0, \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}\left(t_n\right) \right) \leq \|\left(\mathcal{G}(t_n) - I\right) x_0\|_E$$

$$+ \int_0^{t_n} (t_n - s)^{q-1} \|\mathcal{T}(t_n - s)\|_{L(E)} \cdot \|f_n(s)\|_E \, ds + dist\left(x_0, \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}\left(t_n\right) \right) \tag{4.9}$$

Notice that the first term in the right-hand side of the last inequality tends to zero while $n \to \infty$. Since the operator family $\mathcal{G}(t)$ is equicontinuous and the last term vanishes by virtue of (4.8). For the second term we have the following estimate:

$$\int_0^{t_n} (t_n - s)^{q-1} \|\mathcal{T}(t_n - s)\|_{L(E)} \cdot \|f_n(s)\|_E \, ds \le \frac{M \|\omega_{3C}\|_\infty}{\Gamma(1+q)} t_n^q \to 0.$$

Passing to the limit in (4.9) we get the contradiction.

Now, we prove that there exists $\alpha > 0$, such that all solutions x_n are defined on the interval $[0, t_* - \alpha]$ and for each x_n there exists a point $t'_n \in [0, t_* - \alpha]$ at which the following inequality holds:

$$dist\left(x_n\left(t'_n\right), \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}\left(t'_n\right)\right) \ge C/2. \tag{4.10}$$

The first part of this assertion follows from the just proved fact that $t_* > 0$. To prove the second part, notice that if at any point $t_1 \in [0, t_n)$ for some solution $x^0 \in \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}$, we have:

$$\left\|x_n(t_1) - x^0(t_1)\right\|_E < C/2,$$

then for $t_1 + \tau \in [0, t_n]$ we have the following estimate:

$$\begin{aligned} &\left\|x_n(t_1+\tau) - x^0(t_1+\tau)\right\|_E \\ &= \left\|x_n(t_1+\tau) - x_n(t_1) - x^0(t_1+\tau) + x^0(t_1) + x_n(t_1) - x^0(t_1)\right\|_E \\ &\le \|x_n(t_1+\tau) - x_n(t_1)\|_E + \left\|x^0(t_1+\tau) - x^0(t_1)\right\|_E + \left\|x_n(t_1) - x^0(t_1)\right\|_E \\ &\le \left\|x_n(t_1) - x^0(t_1)\right\|_E + \frac{2\tau^q M \|\omega_{3C}\|_\infty}{\Gamma(1+q)}. \end{aligned} \tag{4.11}$$

From the last estimate, for a small $\tau > 0$ we get:

$$\left\|x_n(t_1+\tau) - x^0(t_1+\tau)\right\|_E < C. \tag{4.12}$$

If we suppose that the desired α does not exists, then by using inequality (4.7) and choosing t_*, we obtain the following inequality:

$$\left\|x_n(t_n) - x^0(t_n)\right\|_E < C,$$

contradicting (4.6).

The family of integral multioperators $G : \Lambda \times C([0, t_* - \alpha]; E) \to Kv(C([0, t_* - \alpha]; E))$ of problem (4.1)-(4.2) is ν-condensing in the second argument and hence the sequence $\left\{x_n|_{[0,t_*-\alpha]}\right\}$ is relatively compact. It follows that for a limit point x^* of this sequence while tending t'_n to t^*, we have:

$$x^*(t) \in G(\lambda_0, x^*)(t) \text{ for } t \in [0, t_* - \alpha]$$

and

$$dist\left(x^*\left(t^*\right), \Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}\left(t^*\right)\right) \ge C/2. \tag{4.13}$$

Inequality (4.13) contradicts to condition (4.3) and to the possibility to extend the solution x^* of problem (4.1)-(4.2) to the whole interval $[0, a]$, since so extended solution does not belong to $\Sigma_{x_0}^{F(\cdot,\cdot,\lambda_0)}$.

Now, it remains only to refer to Lemma 2.7, taking the closed ball $\overline{B}_{3C}(0)$ in the space $C([0, a]; E)$ as X. $\square$

5. The averaging principle

Consider the Cauchy problem for a fractional semilinear differential inclusion in a Banach space E :

$$D^q x(t) \in Ax(t) + F(t/\epsilon, x(t)), \tag{5.1}$$

$$x(0) = x_0, \tag{5.2}$$

where ϵ is a small parameter and linear operator $A : D(A) \subseteq E \to E$ satisfies condition (A). We will assume that the multimap $F : \mathbb{R} \times E \to Kv(E)$, besides condition $(F1)$ satisfies the following additional conditions:

(F_T) the multimap F is T-periodic in the first argument, i.e., for a.e. $t \in \mathbb{R}$:

$$F(t + T, x) = F(t, x),$$

for all $x \in E$, where $T > 0$.

It is clear that this condition yields the existence of a T-periodic measurable selection for the multifunction $F(\cdot, x)$ for all $x \in E$;

$(F'2)$ the multimap F is u.s.c. in the second argument, uniformly w.r.t. the first argument, i.e., for each $x \in E$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$F(t, B_\delta(x)) \subset W_\epsilon(F(t, x)),$$

for a.e. $t \in \mathbb{R}$;

$(F'4)$ for each bounded set $\Omega \subset E$ we have:

$$\chi(F([0, T] \times \Omega)) \leq k \cdot \chi(\Omega),$$

where χ is the Hausdorff MNC in E and $k > 0$.

From condition $(F'4)$ it follows that the multimap F transforms bounded sets into bounded ones and satisfies condition $(F3)$.

Parallel to inclusion (5.1), we consider the averaged inclusion:

$$D^q x(t) \in Ax(t) + F_0(x(t)), \tag{5.3}$$

where

$$F_0(x(t)) = \frac{1}{T} \int_0^T F(s, x) ds.$$

From [8], we know the following result.

Lemma 5.1. *The multimap $F_0 : E \to Kv(E)$ is u.s.c.*

For each bounded set $\Omega \subset E$ the following estimate holds true:

$$\chi(F_0(\Omega)) \leq k \cdot \chi(\Omega), \tag{5.4}$$

where k is the constant from condition $(F'4)$. Indeed, for each $x \in \Omega$ and $f \in \mathcal{P}^\infty_F(x)$, we obtain that $f(t) \in F([0, T] \times \Omega)$ for a.e. $t \in [0, T]$. Therefore

$$\frac{1}{T} \int_0^T f(s) ds \in \overline{co} F([0, T] \times \Omega),$$

moreover, $F_0(\Omega) \in \overline{co} F([0, T] \times \Omega)$, hence, due to the definition of the MNC and condition $(F'4)$, we get (5.4).

Therefore, to study the solvability of inclusion (5.3), we can introduce the described above integral multioperator G which, as it was proved, is ν-condensing.

To prove the concluding result, we will need the following important theorem which is the "multivalued" analogue of the Krasnoselskii-Krein lemma (see [8])

Theorem 5.2. *Let F satisfy conditions $(F1)$, $(F'2)$, $(F'4)$, (F_T) and sequences $\{x_n\}_{n=1}^{\infty} \subset C([0,a];E)$ and $f_n \in \mathcal{P}^1_{\mathcal{F}_n}$, where $\mathcal{F}_n(s) = F(\frac{s}{\epsilon_n}, x_n(s))$, for a.e. $s \in [0,d]$ be given. Suppose that $\epsilon_n \to 0$, $x_n \overset{C}{\to} x^0$, and $f_n \overset{L^1}{\rightharpoonup} f^0$. Then $f^0 \in \mathcal{P}^1_{\mathcal{F}_0}$, where $\mathcal{F}_0 = F_0(x^0(s))$, for a.e. $s \in [0,a]$.*

Denote $F_\epsilon(t,x) = F(t/\epsilon, x)$, for $\epsilon > 0$.

Theorem 5.3. *Suppose that the multimap F obeys conditions $(F1)$, (F_T), $(F'2), (F'4)$ and the solutions set $\Sigma^{F_0}_{x_0}[0,a]$ satisfies the following condition of extendability:*

$$\Sigma^{F_0}_{x_0}[0,\tau] = \Sigma^{F_0}_{x_0}[0,a]|_{[0,\tau]} \text{ for each } \tau \in (0,a]. \tag{5.5}$$

Then for every $r > 0$ there exists $\epsilon_0 > 0$ such that $\Sigma^{F_\epsilon}_{x_0}[0,a] \neq \varnothing$ and

$$\Sigma^{F_\epsilon}_{x_0}[0,a] \subset W_r(\Sigma^{F_0}_{x_0}[0,a]),$$

for a.e. $\epsilon \in (0,\epsilon_0]$.

Proof. Consider the family of multioperators

$$G : [0,1] \times C([0,a];E) \to Kv(C([0,a];E))$$

$$G(\epsilon,x) = \left\{ y : \ y(t) = \mathcal{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)\phi_\epsilon(s)ds, \phi_\epsilon \in \mathcal{P}^\infty_{F_\epsilon}(x) \right\}.$$

At first, prove that G is u.s.c. at each point $(0,x)$. To do so, take sequences $\{\epsilon_n\}_{n=1}^{\infty} \subset [0,1]$, $\{x_n\}_{n=1}^{\infty} \subset C([0,a];E)$, such that $\epsilon_n \to 0$, $x_n \to x$. Then for each sequence $\phi_n \in \mathcal{P}^\infty_{F_{\epsilon_n}}(x_n)$, $n \geq 1$ for a.e. $t \in [0,a]$, the set $\{\phi_n\}_{n=1}^{\infty}$, by condition $(F'4)$, lays in a relatively compact set $F([0,T] \times \{x_n\}_{n=1}^{\infty})$, hence the sequence $\{\phi_n\}_{n=1}^{\infty}$ is L^1-semicompact. Moreover, by the Diestel criterion (see [5]) we may assume, w.l.o.g.,, that $\phi_n \overset{L^1}{\rightharpoonup} \phi^0$. By Theorem 5.2 we get $\phi^0 \in \mathcal{P}^1_{F_0}$, but the sequence ϕ_n is bounded, hence $\phi^0 \in \mathcal{P}^\infty_{F_0}$. Now it remains to use condition (S_2) from Lemma 3.6

Now, prove that the multioperator G is condensing w.r.t. the MNC ν :

$$\nu : P(C([0,a];E)) \to \mathbb{R}^2_+$$

with the values in the cone $\mathbb{R}^2_+$ defined as

$$\nu(\Omega) = \max_{\mathcal{D} \in \Delta(\Omega)} \left(\varphi(\mathcal{D}), mod_C(\mathcal{D})\right), \tag{5.6}$$

where $\Delta(\Omega)$ denotes the collection of all countable subsets of Ω,

$$\varphi(\mathcal{D}) = \sup_{t \in [0,a]} e^{-pt}\chi(\mathcal{D}(t)),$$

and the constant $p > 0$ is chosen so that

$$\varrho := 2\frac{qMk}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} e^{-p(t-s)} ds < 1,$$

where k is the constant from condition $(F'4)$.

Such a choice can be justified in the following way. Take $d > 0$ such that

$$2\frac{qMk}{\Gamma(1+q)}\frac{d^q}{q} < \frac{1}{2}$$

and then, choose $p > 0$ such that

$$2\frac{qMk}{\Gamma(1+q)}\frac{1}{pd^{1-q}} < \frac{1}{2}.$$

Now we have

$$\begin{aligned} &2\frac{qMk}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\, e^{-p(t-s)}ds \\ &= 2\frac{qMk}{\Gamma(1+q)}\left(\int_0^{t-d} (t-s)^{q-1}\, e^{-p(t-s)}ds + \int_{t-d}^t (t-s)^{q-1}\, e^{-p(t-s)}ds\right) \\ &\le 2\frac{qMk}{\Gamma(1+q)}\left(\frac{1}{d^{1-q}}\frac{e^{-pd}}{p} + \frac{d^q}{q}\right) \\ &\le 2\frac{qMk}{\Gamma(1+q)}\left(\frac{1}{pd^{1-q}} + \frac{d^q}{q}\right) < 1. \end{aligned}$$

Let $\Omega \subset C([0,a];E)$ be a nonempty bounded set and

$$\nu(G([0,1]\times\Omega)) \ge \nu(\Omega). \tag{5.7}$$

Let us show that Ω is relatively compact set.

Suppose that the maximum mentioned in formula (5.6) is achieved on the set $\mathcal{D}' = \{y_n\}_{n=1}^{\infty}$. Then there exist sequences $\{\epsilon_n\}_{n=1}^{\infty} \subset [0,1]$ $\{x_n\}_{n=1}^{\infty} \subset \Omega$, such that

$$y_n = \mathcal{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)\phi_{\epsilon_n}(s)ds, \phi_{\epsilon_n} \in \mathcal{P}^{\infty}_{F_{\epsilon_n}}(x_n)$$

Denote $\phi_{\epsilon_n} = f_n,\ n \ge 1$.

Since the MNC ν is nonsingular, it is sufficient to prove the assertion for the sequence $g_n(t) = Sf_n(t),\ f_n \in \mathcal{P}^{\infty}_{F_{\epsilon_n}}(x_n),\ n \ge 1$ instead of the sequence $\{y_n\}_{n=1}^{\infty}$.

By virtue of (5.7) we get

$$\varphi(\{g_n\}_{n=1}^{\infty}) \ge \varphi(\{x_n\}_{n=1}^{\infty}), \tag{5.8}$$

$$mod_C(\{g_n\}_{n=1}^{\infty}) \ge mod_C(\{x_n\}_{n=1}^{\infty}). \tag{5.9}$$

Now, we can give a upper estimate for $\varphi(\{g_n\}_{n=1}^{\infty})$.

The χ-regularity property $(F'4)$ implies

$$\begin{aligned} \chi(\{f_n(s)\}_{n=1}^{\infty}) &\le \chi(F([0,T])\times\{x_n\}_{n=1}^{\infty}) \\ &\le k\cdot\chi(\{x_n(s)\}_{n=1}^{\infty}) \\ &= e^{ps}ke^{-ps}\cdot\chi(\{x_n(s)\}_{n=1}^{\infty}) \\ &\le e^{ps}k\cdot\sup_{\xi\in[0,a]} e^{-p\xi}\chi(\{x_n(\xi)\}_{n=1}^{\infty}) = e^{ps}k\cdot\varphi(\{x_n\}_{n=1}^{\infty}). \end{aligned}$$

Applying Lemma 3.5 and estimate (3.2), we obtain

$$\chi(\{Sf_n(t)\}_{n=1}^{\infty}) \le 2\frac{qMk}{\Gamma(1+q)}\left(\int_0^t (t-s)^{q-1}\, e^{ps}ds\right)\cdot\varphi(\{x_n\}_{n=1}^{\infty}). \tag{5.10}$$

Now, from estimates (5.8), (5.10) it follows that

$$\varphi(\{x_n\}_{n=1}^{\infty}) \leq 2\frac{qMk}{\Gamma(1+q)} \sup_{t\in[0,a]} \left(\int_0^t (t-s)^{q-1}\, e^{-p(t-s)} ds\right) \cdot \varphi(\{x_n\}_{n=1}^{\infty})$$

$$\leq \varrho \cdot \varphi(\{x_n\}_{n=1}^{\infty}),$$

implying

$$\varphi(\{x_n\}_{n=1}^{\infty}) = 0,$$

and therefore

$$\chi(\{x_n(t)\}_{n=1}^{\infty}) = 0 \tag{5.11}$$

for all $t \in [0,a]$.

The boundedness of the multimap F on bounded sets implies the boundedness of the set $\left\{f_n :\ f_n \in \mathcal{P}^{\infty}_{F_{\epsilon_n}}(x_n)\right\}_{n=1}^{\infty}$. From condition $(F'4)$ and (5.11) it follows that the set $F([0,T] \times \{x_n(t)\}_{n=1}^{\infty})$ is relatively compact and hence the set $\{f_n(t)\}_{n=1}^{\infty}$ is also relatively compact for a.e. $t \in [0,a]$. Therefore the set $\{f_n\}_{n=1}^{\infty}$ is weakly compact in $L^1([0,a];E)$. From (5.11) it follows, by Lemma 2.11, that for each $\delta > 0$ there exist a compact set $K_\delta \subset E$, and a set $m_\delta \subseteq [0,a]$, of the Lebesgue measure $mes(m_\delta) < \delta$ such that $\{x_n(t)\}_{n=1}^{\infty} \subset K_\delta$ for $t \in [0,a] \setminus m_\delta$. Then from assertion (S_2) of Lemma 3.6 it follows that the set

$$\left\{\int_0^t (t-s)^{q-1} \mathcal{T}(t-s) f_n(s) ds |\ f_n(s) \in \mathcal{P}^{\infty}_{F_{\epsilon_n}}(\Omega)\right\}_{n=1}^{\infty}$$

is relatively compact in $C([0,a];E)$. Then, applying (5.9), we get

$$mod_C(\Omega) \leq mod_C\left(S\left([0,T]\times\Omega\right)\right) = mod_C\left(\left\{Sf_n |\ f_n(s) \in \mathcal{P}^{\infty}_{F_{\epsilon_n}}(\Omega)\right\}\right) = 0$$

where $Sf_n(t) = \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f_n(s)ds$. It means that $\nu(\Omega) = (0,0)$, and hence Ω is the relatively compact set and the multioperator G is condensing w.r.t. the MNC ν.

Therefore, we may obtain the desired result following the lines of the proof of Theorem 4.1 □

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References

[1] S. Abbas, M. Benchohra, G.M. N'Guerekata, *Topics in Fractional Differential Equations,* Developments in Mathematics, Springer, New York, 2012.

[2] R.R. Ahmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, B.N. Sadowskii, *Measures of Noncompactness and Condensing Operators,* Birkhäuser, Boston-Basel-Berlin, 1992.

[3] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus Models and Numerical Methods,* World Scientific Publishing, New York, 2012.

[4] Yu.G. Borisovich, B.D. Gelman, A.D. Myshkis, V.V. Obukhovskii, *Introduction to the Theory of Multivalued Maps and Differential Inclusions*, Librokom, Moscow, 2011. (in Russian)

[5] J. Diestel, W.M. Ruess, W. Schachermayer, *Weak compactness in* $L^1(\mu, X)$*,* Proc. Amer. Math. Soc., **118**(1993), 447-453.

[6] K. Diethelm, *The Analysis of Fractional Differential Equations,* Springer-Verlag, Berlin, 2010.

[7] R. Hilfer, *Applications of Fractional Calculus in Physics,* World Scientific, Singapore, 2000.

[8] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces,* de Gruyter Series in Nonlinear Anal. Appl., 7, Walter de Gruyter, Berlin-New-York, 2001.

[9] T.D. Ke, V. Obukhovskii, N.C. Wong, J.C. Yao, *On a class of fractional order differential inclusions with infinite delays,* Applicable Anal., **92**(2013), no. 1, 115-137.

[10] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations,* North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[11] V. Lakshmikantham, *Theory of fractional functional differential equations,* Nonlinear Anal., **69**(2008), no. 10, 3337-3343.

[12] V. Lakshmikantham, A.S. Vatsala, *Basic theory of fractional differential equations,* Nonlinear Anal., **69**(2008), no. 8, 2677-2682.

[13] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,* John Wiley, Inc., New York, 1993.

[14] V. Obukhovskii, J.C. Yao, *Some existence results for fractional functional differential equations,* Fixed Point Theory, **11**(2010), no. 1, 85-96.

[15] I. Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, 1999.

[16] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications,* Gordon and Breach Sci. Publishers, Yverdon, 1993.

[17] V.E. Tarasov, *Fractional Dynamics. Applications of Fractional Calculus to Dynamics of Particles, Fields and Media,* Nonlinear Physical Science, Springer, Heidelberg, Higher Education Press, Beijing, 2010.

[18] Z. Zhang, B. Liu, *Existence of mild solutions for fractional evolution equations,* Fixed Point Theory, **15**(2014), no. 1, 325-334.

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