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EXISTENCE RESULTS FOR A SYSTEM OF NONLINEAR INTEGRAL EQUATIONS IN BANACH ALGEBRAS UNDER WEAK TOPOLOGY

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Abstract. This paper is devoted to the study of a coupled system of nonlinear functional integral equations in suitable Banach algebras. This system is reduced to a fixed point problem for a 2×2 block operator matrix with nonlinear inputs. Hence, certain assumptions on its entries are given under a weak topology setting. These assumptions involve in particular the De Blasi measure of weak noncompactness in order to ensure the existence of solutions.
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1. Introduction

The studies of nonlinear functional integral and differential equations in Banach algebras have been discussed for a long time in the literature. This study was performed via fixed point techniques (Schauder, Darbo,...), see for example [3, 16, 19, 20, 26, 33] and the references therein. In this work, we are mainly concerned with the existence results of solutions for the following system of nonlinear integral equations occurring in some problems dealing with physics:

$$\begin{cases} x(t) = f(t, x(t)) + [a(t)y(t)] \cdot \left[\left(\int_0^{\sigma_1(t)} k(t,s) f_1(s, y(\eta(s)))ds \right) u \right] \ ; \ u \in X\backslash\{0\} \\ y(t) = \left[\left(q(t) + \int_0^{\sigma_2(t)} p\left(t, s, x(s), x(\lambda s)\right) ds \right) v \right] + g(t, y(t)) \ ; \ v \in X\backslash\{0\}, \end{cases} \tag{1.1}$$

where $\lambda \in [0,1]$, $a, q, \sigma_1, \sigma_2, \eta$ are continuous real functions on $[0,T]$, $0 < T < \infty$, and $f, g : [0,T] \times X \longrightarrow X$ as well as $f_1 : [0,T] \times X \longrightarrow \mathbb{R}$ are supposed to be weakly sequentially continuous with respect to the second variables. Here, X is a Banach algebra satisfying certain topological conditions of sequential nature. The main used tools are the fixed point theorems and the measure of weak noncompactness of De Blasi [18]. Some of the investigated fixed point questions are very natural. Note that the system (1.1) may be transformed into the following fixed point problem of the 2×2 block operator matrix

$$\begin{pmatrix} A & B \cdot B' \\ C & D \end{pmatrix} \tag{1.2}$$

with nonlinear entries defined on Banach algebras. Our assumptions are as follows: A maps a nonempty, bounded, closed, and convex subset S of a Banach algebra X into X, B, B' and D act from X into X and C from S into X.

In this direction, the authors A. Ben Amar, A. Jeribi and B. Krichen in [9] have established Schauder's and Krasnoselskii's fixed point theorems for the operator (1.2), when $B' = 1$, and X is a Banach space and have applied theirs results to a two-dimensional mixed boundary problem in $L_p \times L_p$, $p \in]1, +\infty[$.

Recently, A. Jeribi, B. Krichen and B. Mefteh in [25] have also established some new variants of fixed point theorems for the operator (1.2), when $B' = 1$. The obtained results were applied in order to prove the existence of solutions for a system of transport equations arising in biology. Due to the lack of compactness in L_1 of the operator $C(\lambda - A)^{-1}$, their analysis was carried out via arguments of weak topology and particulary the notion of the measure of weak noncompactness.

Later, the authors A. Jeribi, N. Kaddachi and B. Krichen have initiated in [27] the study of the existence of fixed point for the block operator (1.2), when X is a Banach algebra and B' is a continuous operator. An application to a system of nonlinear integral equations in $\mathcal{C}([0,1], \mathbb{R})$, the Banach algebra of all real continuous functions, was considered. For more new fixed point theorems and theirs applications on block operator matrices in a weak topology setting, the reader may consult the monograph of A. Jeribi and B. Krichen [24]. It is important to mention that all theoretical studies were based on the existence of a solution of the following equation:

$$x = Ax \cdot Bx + Cx. \tag{1.3}$$

Many authors have focused on the resolution of the equation (1.3) and have obtained a lot of valuable results in suitable Banach algebras. These studies were mainly based on the convexity of the bounded domain [16] and on the properties of the operators A, B and C (cf. completely continuous [21, 33], weakly one-set-contractive [3, 11, 28], weak continuity [6, 7, 8], weakly condensing and the potential tool of the axiomatic measures of noncompactness [3, 4, 5, 18, 30], ...). Since the weak topology is the practice setting and it is natural to investigate the fixed point problems occurring problems dealing with physics, it turns out that the above mentioned results can not be easily applied. However, because of the lack of stability of convergence for the product sequences under the weak topology, the authors in [6] have introduced a new class of Banach algebras satisfying the condition denoted by $(\mathcal{P})$:

$$(\mathcal{P})\left\{\begin{array}{l}\text{For any sequences } \{x_n\} \text{ and } \{y_n\} \text{ of X such that } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y\\ \text{then } x_n\cdot y_n \rightharpoonup x\cdot y; \text{ here } \rightharpoonup \text{ denotes weak convergence}\end{array}\right.$$

and they have established some new variants of fixed point results based on the notion of weak sequential continuity. This notion seems to be the most comfortable in use. Note that the Banach space of sequences of absolutely convergent sum l^1 satisfies the condition $(\mathcal{P})$. Moreover, if X is a Banach algebra satisfying the condition $(\mathcal{P})$ then, according to the Dobrakov's theorem, $\mathcal{C}(K, X)$ is also a Banach algebra satisfying the condition $(\mathcal{P})$, where K is a compact Hausdorff space [24].

This paper is organized as follows. In Section 2, we give some preliminary definitions and results needed in the sequel. In Sections 3 and 4, we will refine the fixed point theorems established in [27] for the block operator matrix (1.2) by using arguments of weak topology. The main result of Section 3 is Theorem 3.2. In Section 5, we give an application showing the existence of solutions of the system (1.1).

2. Notations, basic definitions and auxiliary results

Before providing the main results, let us recall some basic definitions and results needed in the remainder of the paper. Let X be a Banach algebra satisfying the condition $(\mathcal{P})$. We denote by $\mathcal{B}(X)$, the collection of all nonempty bounded subsets of X and $\mathcal{W}(X)$, the subset of $\mathcal{B}(X)$ consisting of all nonempty weakly compact subsets of X. The measure of weak noncompactness β on $\mathcal{B}(X)$ is defined by De Blasi [18] in the following way:

$$\beta(S) = \inf\big\{r > 0 : \text{ there exists } K \in \mathcal{W}(X) \text{ such that } S \subseteq K + B_r\big\}$$

where B_r is the closed ball in X centered at 0 with a radius r. Note that the function $\beta(S)$ possesses several useful properties which may be found in [1, 18] (cf. also [5], where an axiomatic approach to the notion of a measure of weak noncompactness is presented). Let $\Omega \subset X$ and $F : \Omega \longrightarrow X$. If F is bounded and $\beta(F(S)) < \beta(S)$ for any bounded subset S of Ω with $\beta(S) > 0$, then F is said to be β-condensing.

In order to recall the definition of the convex-power condensing operator, we give some notations. Let $\Omega \subset X$ be closed and convex, $F : \Omega \longrightarrow \Omega$, $x_0 \in \Omega$. For any subset S of Ω, let

$$\left\{\begin{array}{l} F^{(1,x_0)}(S) = F(S)\\ F^{(n,x_0)}(S) = F\left(\overline{co}\left\{F^{(n-1,x_0)}(S), x_0\right\}\right).\end{array}\right.$$

Definition 2.1. An operator F is said to be convex-power condensing, if F is bounded and there exists a point $x_0 \in \Omega$ and a strictly positive integer n_0 such that, for any bounded subset S of Ω with $\beta(S) > 0$, we have

$$\beta(F^{(n_0,x_0)}(S)) < \beta(S),$$

where $F^{(1,x_0)}(S) = F(S)$.

Remark 2.1. (i) By using Definition 2.1, we can see that if $\beta(F^{(n_0,x_0)}(S)) \geq \beta(S)$ then, S is relatively weakly compact set in X.

(ii) Obviously, every β-condensing operator is convex-power condensing but the reverse implication may not hold.

Definition 2.2. A mapping $T : X \longrightarrow X$ is called $\mathcal{D}$-Lipschitzian if there exists a continuous and nondecreasing function $\phi_T : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, such that

$$\|Tx - Ty\| \leq \phi_T (\|x - y\|)$$

for all $x, y \in X$ where $\phi_T(0) = 0$.
Moreover, if $\phi_T(r) < r,\ r > 0$, then T is called a nonlinear contraction on X (see [12]). In particular, if $\phi_T(r) = kr$, for some constant $0 < k < 1$, then T is a contraction.

Definition 2.3. An operator $T : X \longrightarrow X$ is said to be weakly compact, if $T(B)$ is relatively weakly compact for every nonempty bounded subset $B \subseteq X$.

Definition 2.4. An operator $T : X \longrightarrow X$ is said to be weakly sequentially continuous on X if, for every sequence $\{x_n\}$ with $x_n \rightharpoonup x$, we have $Tx_n \rightharpoonup Tx$.

Lemma 2.1. *([8], Theorem 3.2) Let $S \subset X$ be closed and convex. Suppose that $F : S \longrightarrow S$ is weakly sequentially continuous and convex-power condensing with respect to β. If $F(S)$ is bounded, then F has, at least, one fixed point in S.*

In the sequel, we will use the following lemmas which were established in [7].

Lemma 2.2. *If $K, K' \in \mathcal{W}(X)$, then $K \cdot K' = \{x \cdot y\ ;\ x \in K \text{ and } x' \in K'\} \in \mathcal{W}(X)$.*

Lemma 2.3. *If $V \in \mathcal{B}(X)$ and $K \in \mathcal{W}(X)$, then $\beta(V \cdot K) \leq \|K\|\beta(V)$.*

Lemma 2.4. *If F is Lipschitzian with a Lipschitz constant α and is weakly sequentially continuous on X, then $\beta(F(V)) \leq \alpha\beta(V)$, for all $V \in \mathcal{B}(X)$.*

3. Fixed point theorems for a 2×2 block operator matrix

At the beginning of this section, we are going to discuss a fixed point theorem for the operator matrix (1.2) involving the De Blasi's measure of weak noncompactness in a Banach algebra satisfying the condition $(\mathcal{P})$. The following definition is needed.

Definition 3.1. (see [29]) Let (X, d) be a metric space. We say that $F : X \longrightarrow X$ is a separate contraction if there exists two functions $\varphi, \psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying:
(i) ψ is strictly increasing and $\psi(0) = 0$,
(ii) $d(Fx, Fy) \leq \varphi\left(d(x, y)\right)$, and
(iii) $\psi(r) + \varphi(r) \leq r$ for $r > 0$.

Remark 3.1. Obviously, every contraction is a separate contraction.

The following result gives the sufficient conditions for the block operator matrix (1.2) which is acting on a product of Banach algebras satisfying condition $(\mathcal{P})$ to have a fixed point.

Theorem 3.1. *Let S be a bounded, closed, and convex subset of a Banach algebra X satisfying the sequential condition $(\mathcal{P})$. Assume that A, $C : S \longrightarrow X$, and B, B', $D : X \longrightarrow X$ are five operators satisfying:*
(i) A and C are weakly compact,
(ii) D is linear, bounded and there is $p \in \mathbb{N}^*$ such that D^p is a separate contraction

on X,
(iii) A, B, C and B' are weakly sequentially continuous, and
(iv) $Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx \in S$ for all $x \in S$.
Then, the block operator matrix (1.2) has, at least, one fixed point in $S \times X$.

Proof. The use of assumption (ii) and Lemma 1.2 in [29] leads to the inverse operator $(I - D^p)^{-1}$ to exist on X and that

$$(I-D)^{-1} = (I-D^p)^{-1}\sum_{k=0}^{p-1} D^k.$$

Let us define the mapping $F : S \longrightarrow X$ by the formula

$$F(x) = Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx. \tag{3.1}$$

First, let us notice that $(I-D)^{-1}$ is weakly continuous (see [13]). Next, let us show that F is weakly sequentially continuous. To do so, let $(\xi_n)_n$ be a sequence in S which converges weakly to ξ. Since $(I-D)^{-1}C(S)$ is relatively weakly compact, there exists a subsequence (ξ_{n_k}) of (ξ_n) such that $(I-D)^{-1}C(\xi_{n_k}) \rightharpoonup \gamma$. Taking into account the weak sequential continuity of the maps C and D and using the following equality:

$$(I-D)^{-1}C = C + D(I-D)^{-1}C \tag{3.2}$$

to obtain $\gamma = (I-D)^{-1}C(\xi)$. Thus,

$$(I-D)^{-1}C(\xi_{n_k}) \rightharpoonup (I-D)^{-1}C(\xi).$$

Now, we show that

$$(I-D)^{-1}C(\xi_n) \rightharpoonup (I-D)^{-1}C(\xi).$$

Suppose the contrary, then there exists a weak neighborhood V^w of $(I-D)^{-1}C(\xi)$ and a subsequence (ξ_{n_j}) of (ξ_n) such that $(I-D)^{-1}C(\xi_{n_j}) \notin V^w$ for all $j \geq 1$. Since (ξ_{n_j}) converges weakly to ξ, and arguing as before, we find a subsequence $(\xi_{n_{j_k}})$ of (ξ_{n_j}) such that $(I-D)^{-1}C(\xi_{n_{j_k}}) \rightharpoonup (I-D)^{-1}C(\xi)$. Which is absurd, since $(I-D)^{-1}C(\xi_{n_{j_k}}) \notin V^w$. As a result, $(I-D)^{-1}C$ is weakly sequentially continuous. Moreover, taking into account that X is a Banach algebra satisfying the condition $(\mathcal{P})$, and using the assumption (iii), we deduce that F is weakly sequentially continuous on S. Besides, since

$$F(S) \subseteq A(S) + B(I-D)^{-1}C(S) \cdot B'(I-D)^{-1}C(S),$$

and from the assumption (i), it follows that $F(S)$ is relatively weakly compact. Accordingly, the operator F has a fixed point x in S by using Arino, Gautier and Penot's fixed point theorem [2]. Now, the vector $y = (I-D)^{-1}Cx$ solves the problem.

Notice that the proof of Theorem 3.1 is based on the linearity of the operator D. Hence, it would be interesting to investigate the case where D is not linear.

Theorem 3.2. *Let S be a nonempty, convex, closed, and bounded subset of a Banach algebra X satisfying the sequential condition $(\mathcal{P})$. Assume that A, $C : S \longrightarrow X$, and B, B', $D : X \longrightarrow X$ are five weakly sequentially continuous operators such that:*
(i) C is weakly compact and A is β-condensing,
(ii) D is a ϕ-nonlinear contraction and $(I-D)^{-1}C(S)$ is bounded, and

(iii) $Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx \in S$ *for all* $x \in S$.
Then, the block operator matrix (1.2) *has, at least, one fixed point in* $S \times X$.

Proof. Since D is ϕ-nonlinear contraction, it follows that $(I-D)$ is injective and then, $(I-D)^{-1}$ exists on $(I-D)(X)$. In fact, the inverse operator $(I-D)^{-1}$ exists on X. Indeed, let $y \in X$ and let us define the mapping $F : X \to X$ by $F(x) = y + Dx$. Notice that F is a ϕ-nonlinear contraction. By using Theorem 1 in [12], we deduce that F has a unique fixed point x^*. Consequently, for any $y \in X$, there is a unique point x^* such that $y = (I-D)x^*$ and so $(I-D)^{-1}$ exists on X.
Now, we claim that $(I-D)^{-1}C(S)$ is relatively weakly compact. If this is not the case, then $d = \beta\left((I-D)^{-1}C(S)\right) > 0$. The use of (3.2) and also the weak compactness of $\overline{C(S)}^w$ (here, $\overline{C(S)}^w$ denotes the weak closure of $C(S)$) yields

$$\beta\left((I-D)^{-1}C(S)\right) \leq \beta\left(D(I-D)^{-1}C(S)\right). \tag{3.3}$$

Let $\varepsilon > 0$. Then, there exists a $K \in \mathcal{W}(X)$ satisfying $(I-D)^{-1}C(S) \subseteq K + B_{d+\varepsilon}$. Therefore, by using the assumption (ii) and arguing as in the proof of Theorem 2.1 in [31], we get

$$\beta\left((I-D)^{-1}C(S)\right) \leq \beta\left(D(I-D)^{-1}C(S)\right) < \beta\left((I-D)^{-1}C(S)\right)$$

which is a contradiction and the claim is approved. Consequently, $(I-D)^{-1}C$ as well as F are weakly sequentially continuous, where F is defined in (3.1). Moreover, by using Lemma 2.2 together with assumption (i), we can show that F is β-condensing. Now, by applying Theorem 3.1 in [7], we deduce that F has a fixed point x in S. Hence, the vector $y = (I-D)^{-1}Cx$ solves the problem.

Next, we will combine Theorem 3.2 and Lemma 2.4 in order to obtain the following fixed point theorem:

Corollary 3.1. *Let* S *be a nonempty, convex, closed, and bounded subset of* X. *Suppose that* A, $C : S \longrightarrow X$, *and* B, B', $D : X \longrightarrow X$ *are five weakly sequentially continuous operators satisfying:*
(i) *C is weakly compact,*
(ii) *A is a contraction with a constant k,*
(iii) *D is a ϕ-nonlinear contraction and $(I-D)^{-1}C(S)$ is bounded, and*
(iv) $Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx \in S$ *for all* $x \in S$.
Then, the block operator matrix (1.2) *has, at least, one fixed point in* $S \times X$.

By using the same arguments as in the proof of Theorem 3.2, we get the following result:

Theorem 3.3. *Let* S *be a nonempty, convex, closed, and bounded subset of* X. *Assume that* A, $C : S \longrightarrow X$, *and* B, B', $D : X \longrightarrow X$ *are five operators satisfying the following conditions:*
(i) *B and C are Lipschitzian with Lipschitz constants α and γ respectively,*
(ii) *A and B' are weakly compact,*
(iii) *D is expansive with a constant $h > \gamma + 1$ and $C(S) \subseteq (I-D)(S)$,*
(iv) *A, B and B' are weakly sequentially continuous and C is strongly continuous,*
(v) $Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx \in S$ *for all* $x \in S$.

Then, the block operator matrix (1.2) *has, at least, one fixed point in* $S \times S$ *provided that* $0 \le \alpha M < 1$, *where* $M = \|B'(S)\|$.

Proof. Since D is expansive, the inverse operator $(I - D)^{-1}$ exists on $(I - D)(X)$ (see [34]), and, for all $x, y \in (I - D)(X)$, we have

$$\|(I - D)^{-1}x - (I - D)^{-1}y\| \le \frac{1}{h - 1}\|x - y\|.$$

Then, $(I - D)^{-1}$ is continuous and, by using assumption (iv), we deduce that the operator $B(I-D)^{-1}C$ is weakly sequentially continuous on S. Therefore, the mapping $(I - D)^{-1}C$ is a contraction on S in view of assumption (iii). Thus, $(I - D)^{-1}C(S)$ is bounded. Now, the use of both assumption (ii) and Eberlein-Šmulian's theorem [17] ensures that $\overline{B'(I - D)^{-1}C(S)}^{w}$ is a weakly compact subset of X. Next, we will prove that the operator $A + B(I - D)^{-1}C \cdot B'(I - D)^{-1}C$ is β-condensing. To see this,

$$\beta\left((A + B(I - D)^{-1}C \cdot B'(I - D)^{-1}C)(S)\right) \le \beta(A(S)) + \beta\left(B(S) \cdot B'(S)\right).$$

Taking into account that $A(S)$ is relatively weakly compact, and using Lemma 2.4, we get

$$\beta\left((A + B(I - D)^{-1}C \cdot B'(I - D)^{-1}C)(S)\right) \le M\beta\left(B(I - D)^{-1}C(S)\right).$$

So, if $\beta(S) \ne 0$, we have

$$\beta\left((A + B(I - D)^{-1}C \cdot B'(I - D)^{-1}C)(S)\right) \le \frac{\alpha\gamma}{h - 1}M\beta(S) < \beta(S).$$

The result follows from Theorem 3.1 in [7].

Next, we can modify some assumptions of Theorem 3.3 in order to study the same problem.

Theorem 3.4. *Assume that S is a nonempty, convex, closed, and bounded subset of a Banach algebra X satisfying the condition* $(\mathcal{P})$. *Suppose that* $A, C : S \longrightarrow X$, B, B', $D : X \longrightarrow X$ *are five weakly sequentially continuous operators satisfying:*
(i) *B and C are Lipschitzian with Lipschitz constants α and γ respectively,*
(ii) *C is weakly compact and* $C(S) \subseteq (I - D)(S)$,
(iii) *A and D are two contractions with constants k and k' respectively, and*
(iv) $Ax + B(I - D)^{-1}Cx \cdot B'(I - D)^{-1}Cx \in S$ *for all* $x \in S$.
Then, the block operator matrix (1.2) *has, at least, one fixed point in* $S \times S$, *whenever* $0 \le k + \dfrac{\alpha\gamma}{1 - k'}M < 1$, *where* $M = \left\|\overline{B'(I - D)^{-1}C(S)}^{w}\right\| > 1$.

Proof. Notice that $(I - D)^{-1}$ exists and is continuous (see [15]). Our next task is to show that the mapping F defined in (3.1) fulfills all conditions of Lemma 2.1. We first claim that $(I - D)^{-1}C(S)$ is relatively weakly compact. If not, we have $\beta\left((I - D)^{-1}C(S)\right) > 0$. It is easy, in view of (3.2), to deduce that

$$\beta((I - D)^{-1}C(S)) \le \beta(\overline{C(S)}^{w}) + \beta(D(I - D)^{-1}C(S)) \le \beta(D(I - D)^{-1}C(S)). \quad (3.4)$$

Let $r > \beta((I-D)^{-1}C(S))$ and $K \in \mathcal{W}(X)$, such that $(I-D)^{-1}C(S) \subseteq K + B_r$. Keeping in mind that D is a k'-contraction, we infer that

$$D(I-D)^{-1}C(S) \subseteq \overline{D(K)}^w + B_{k'r}.$$

Since D is weakly sequentially continuous, it follows that $D(K)$ is relatively weakly compact. Hence,

$$\beta(D(I-D)^{-1}C(S)) \leq k'r < r.$$

By using the inequality (3.4), we deduce that

$$\beta((I-D)^{-1}C(S)) \leq \beta(D(I-D)^{-1}C(S)) \leq k'r.$$

Letting $r \to \beta((I-D)^{-1}C(S))$ we get

$$\beta((I-D)^{-1}C(S)) \leq k'\beta((I-D)^{-1}C(S)) < \beta((I-D)^{-1}C(S)),$$

which is a contradiction and the claim is approved.
An argument similar to that in the proof of Theorem 3.2 leads to the weak sequential continuity of the maps $B(I-D)^{-1}C$ and $B'(I-D)^{-1}C$, as well as F.
Therefore, the operator F is convex-power condensing. Indeed, it is easy to see that

$$F(S) \subset A(S) + B(I-D)^{-1}C(S) \cdot B'(I-D)^{-1}C(S).$$

Keeping in mind the relatively weak compactness of $B'(I-D)^{-1}C(S)$, and using the subadditivity of the De Blasi's measure of weak noncompactness, we get

$$\beta\left(F(S)\right) \leq \beta\left(A(S)\right) + \beta\left(B(I-D)^{-1}C(S) \cdot \overline{B'(I-D)^{-1}C(S)}^w\right).$$

The use of assumption (iv), as well as Lemmas 2.4 and 2.3, leads to

$$\beta\left(F(S)\right) \leq k\beta\left(S\right) + \left\|\overline{B'(I-D)^{-1}C(S)}^w\right\| \beta\left(B(I-D)^{-1}C(S)\right).$$

Since the operator $B(I-D)^{-1}C$ is Lipschitzian with a Lipschitz constant $\frac{\alpha\gamma}{1-k'}$ then,

$$\beta\left(F(S)\right) \leq \left(k + M\frac{\alpha\gamma}{1-k'}\right)\beta(S).$$

Letting $x_0 \in S$ and assuming a positive integer $n \geq 1$, then,

$$\begin{aligned}
\beta\left(F^{(n,x_0)}(S)\right) &= \beta\left(F\left(\overline{co}\left\{F^{(n-1,x_0)}(S), \{x_0\}\right\}\right)\right) \\
&\leq \left(k + M\tfrac{\alpha\gamma}{1-k'}\right)\beta\left(\overline{co}\left\{F^{(n-1,x_0)}(S), \{x_0\}\right\}\right) \\
&\leq \left(k + M\tfrac{\alpha\gamma}{1-k'}\right)^n \beta(S).
\end{aligned}$$

Since $0 < k + M\frac{\alpha\gamma}{1-k'} < 1$, it follows that F is a convex-power condensing operator. Now, we may apply Lemma 2.1 to infer that F has, at least, one fixed point x in S and consequently, the vector $y = (I-D)^{-1}Cx$ solves the problem.

Remark 3.2. It should be noted that if K is convex then, the operator D has a fixed point (see [2]).

4. Regular case

In what follows, we will study the existence of a fixed point for the block operator matrix (1.2) in the case where X is a commutative Banach algebra satisfying the condition $(\mathcal{P})$. Before stating the main result, we need the following lemma:

Lemma 4.1. *Let S be a nonempty, convex, and closed subset of X and, let A, $C : S \longrightarrow X$, and B, B', $D : X \longrightarrow X$ be five operators such that:*
(i) A, B and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions $\phi_{\mathcal{A}}, \phi_{\mathcal{B}}$ and $\phi_{\mathcal{C}}$ respectively,
(ii) D is a contraction with a constant k and $C(S) \subseteq (I-D)(S)$,
(iii) $B(I-D)^{-1}C$ is regular on $B'(S)$,
(iv) $B'(I-D)^{-1}C(S)$ is bounded with a bound M, and
(v) $Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cy \in S$, for all $x, y \in S$.
Then, $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1}$ exists on $B'(I-D)^{-1}C(S)$, whenever

$$M\phi_{\mathcal{B}} \circ \left(\frac{1}{1-k}\phi_{\mathcal{C}}\right)(r) + \phi_{\mathcal{A}}(r) < r \text{ for } r > 0.$$

Proof. From hypothesis (ii), it follows that $(I-D)^{-1}$ is well-defined on $(I-D)(X)$ and, for any $x, y \in S$, we have

$$\begin{aligned}\|B(I-D)^{-1}Cx - B(I-D)^{-1}Cy\| &\le \phi_{\mathcal{B}}\left(\|(I-D)^{-1}Cx - (I-D)^{-1}Cy\|\right)\\ &\le \phi_{\mathcal{B}} \circ \left(\tfrac{1}{1-k}\phi_{\mathcal{C}}\right)(\|x-y\|).\end{aligned}$$

Then, $B(I-D)^{-1}C$ is $\mathcal{D}$-Lipschitzian with the $\mathcal{D}$-function $\phi_{\mathcal{B}} \circ \varphi$, where $\varphi = \frac{1}{1-k}\phi_{\mathcal{C}}$. Now, let y be fixed in S and let's define a mapping

$$\begin{cases} \varphi_y : S \longrightarrow S \\ x \longrightarrow Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cy. \end{cases}$$

Notice that this operator is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function $\psi = M\phi_{\mathcal{B}} \circ \varphi + \phi_{\mathcal{A}}$. Hence, an application of the Browder's fixed point theorem [15] shows that there is a unique point $x_y \in S$ such that $\varphi_y(x_y) = x_y$. Or equivalently,

$$Ax_y + B(I-D)^{-1}Cx_y \cdot B'(I-D)^{-1}Cy = x_y$$

Consequently, in view of assumption (iii), we have

$$\left(\frac{I-A}{B(I-D)^{-1}C}\right)x_y = B'(I-D)^{-1}Cy.$$

Hence, the mapping $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1}$ is well defined on $B'(I-D)^{-1}C(S)$ and the desired result is deduced.

In the following result, we will combine Theorem 3.1 in [7] and Lemma 4.1.

Theorem 4.1. *Let S be a nonempty, convex, closed, and bounded subset of X, and let A, $C : S \longrightarrow X$ and B, B', $D : X \longrightarrow X$ be five operators satisfying:*
(i) A, B and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions $\phi_{\mathcal{A}}$, $\phi_{\mathcal{B}}$ and $\phi_{\mathcal{C}}$ respectively,
(ii) B' is Lipschitzian with a Lipschitz constant α,

(iii) D *is a contraction with a constant* $k \in \left[0, \frac{3-\sqrt{5}}{2}\right[$,
(iv) $B(I-D)^{-1}C$ *is regular on* $B'(I-D)^{-1}C(S)$,
(v) C *is strongly continuous and* $C(S) \subseteq (I-D)(S)$,
(vi) $x = Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cy$; $y \in S \Rightarrow x \in S$, *and*
(vii) $\max\left\{\left(M\phi_{\mathcal{B}} \circ \left(\frac{1}{1-k}\phi_{\mathcal{C}}\right)\right)(r) + \phi_{\mathcal{A}}(r),\ \varrho\alpha\phi_{\mathcal{C}}(r)\right\} \leq kr$, *for all* $r > 0$, *where*
$M = \|B'(I-D)^{-1}C(S)\|$ *and* $\varrho = \|B(I-D)^{-1}C(S)\| + \frac{k}{M}diam(S)$.
Then, the block operator matrix (1.2) *has a fixed point in* $S \times X$.

Proof. By using our assumptions, we infer that $B(I-D)^{-1}C$ and $B'(I-D)^{-1}C$ are two contraction operators on S and consequently, $B(I-D)^{-1}C(S)$ and $B'(I-D)^{-1}C(S)$ are two bounded subsets.
The use of Lemma 4.1 and the Browder' fixed point theorem [15] shows that the operator $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1}$ exists on $B'(I-D)^{-1}C(S)$. Define a mapping

$$\begin{cases} N : S \longrightarrow X \\ x \mapsto \left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} B'(I-D)^{-1}Cx. \end{cases} \tag{4.1}$$

Now, in view of Lemma 2.1, it suffices to prove that the operator N is weakly sequentially continuous and convex-power condensing and that $N(S)$ is bounded.

Step 1: N is weakly sequentially continuous.
We claim that $\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1}$ is a continuous operator on $B'(I-D)^{-1}C(S)$. To see this, let $\{x_n\}_{n=0}^{\infty}$ be any sequence in $B'(I-D)^{-1}C(S)$ converging to a point x, and let

$$\begin{cases} y_n = \left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} x_n \\ y = \left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} x. \end{cases}$$

Or, equivalently

$$\begin{cases} y_n = Ay_n + B(I-D)^{-1}Cy_n \cdot x_n \\ y = Ay + B(I-D)^{-1}Cy \cdot x. \end{cases}$$

Then

$$\|y_n - y\| \leq \left(M\phi_{\mathcal{B}} \circ \left(\frac{1}{1-k}\phi_{\mathcal{C}}\right) + \phi_{\mathcal{A}}\right)(\|y_n - y\|) + \|B(I-D)^{-1}Cy\|\|x_n - x\|.$$

Moreover, taking into account that the operator $B(I-D)^{-1}C$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\phi_{\mathcal{B}} \circ \left(\frac{1}{1-k}\phi_{\mathcal{C}}\right)$ and using the growth of the functions $\phi_{\mathcal{B}}$ and $\phi_{\mathcal{C}}$, we get for any $y \in S$

$$\|B(I-D)^{-1}Cy\| < \|B(I-D)^{-1}Ca\| + \frac{k}{M}\|y - a\|$$

for some fixed point $a \in S$. Consequently,

$$\limsup_n \|y_n - y\| \leq \left(M\phi_{\mathcal{B}} \circ \left(\frac{1}{1-k}\phi_{\mathcal{C}}\right) + \phi_{\mathcal{A}}\right)(\limsup_n \|y_n - y\|).$$

If $\limsup_n \|y_n - y\| \neq 0$, then we get a contradiction and the claim is proved.
Now, keeping in mind the continuity of $(I-D)^{-1}$ and B', and using assumption (v), we deduce that N is weakly sequentially continuous.

Step 2: N is convex-power condensing.
Let $x_1, x_2 \in S$ and $y_1, y_2 \in X$, such that $y_1 = Nx_1$ and $y_2 = Nx_2$. Then

$$\begin{cases} y_1 = Ay_1 + B(I-D)^{-1}Cy_1 \cdot B'(I-D)^{-1}Cx_1 \\ \\ y_2 = Ay_2 + B(I-D)^{-1}Cy_2 \cdot B'(I-D)^{-1}Cx_2 \end{cases}$$

and, by using assumption (vii), we have

$$\begin{array}{rcl} \|y_1 - y_2\| & \leq & \left(\phi_{\mathcal{A}} + M\phi_{\mathcal{B}} \circ \left(\frac{1}{1-k}\phi_{\mathcal{C}}\right)\right)(\|y_1 - y_2\|) + \frac{\varrho\alpha}{1-k}\phi_{\mathcal{C}}(\|x_1 - x_2\|) \\ \\ & \leq & k\|y_1 - y_2\| + \frac{k}{1-k}\|x_1 - x_2\|. \end{array}$$

This implies that

$$\|Nx_1 - Nx_2\| \leq \frac{k}{(1-k)^2}\|x_1 - x_2\|.$$

The use of Lemma 2.4 and Step 1 leads to the following

$$\beta(N(S)) \leq \frac{k}{(1-k)^2}\beta(S) < \beta(S).$$

Hence, if $\beta(S) > 0$, then N is β-condensing and so is convex-power condensing. The result follows from Lemma 2.1.

Remark 4.1. The assumption (vi) of Theorem 4.1 was introduced by Burton [14] instead of assuming that $Ax + Tx \cdot T'y \in S$, for all $x, y \in S$.

Now, we may combine Theorem 2.5 in [10] and Lemma 4.1 in order to obtain the following fixed point theorem.

Theorem 4.2. *Let S be a nonempty, weakly compact, and convex subset of X, and let A, $C : S \longrightarrow X$ and B, B', $D : X \longrightarrow X$ be five operators satisfying:*
(i) A, B and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions $\phi_{\mathcal{A}}, \phi_{\mathcal{B}}$ and $\phi_{\mathcal{C}}$ respectively,
(ii) B' is continuous on S,
(iii) T is regular on $B'(S)$, where $T = B(I-D)^{-1}C$,
(iv) D is a contraction with a constant k,
(v) C is strongly continuous and $C(S) \subseteq (I-D)(S)$, and
(vi) $x = Ax + TxT'y$; $y \in S \Rightarrow x \in S$, where $T' = B'(I-D)^{-1}C$.
Then, the block operator matrix (1.2) has a fixed point, whenever

$$M\phi_B \circ \left(\frac{1}{1-k}\phi_C\right)(r) + \phi_A(r) < r \text{ for } r > 0,$$

where $M = \|T'(S)\|$.

Proof. Similarly to the proof of Theorem 4.1, the operator N already defined in (4.1) is weakly sequentially continuous. Moreover, taking into account that S is weakly compact, and using the Eberlein-Šmulian's theorem [17] we deduce that $N(S)$ is relatively weakly compact. Hence, and from Theorem 2.5 in [10], we deduce that the equation $Nx = x$ has, at least, one solution in S. Consequently, the use of vector $y = (I - D)^{-1}Cx$ solves the problem.

5. Existence theory

In this section, we illustrate the applicability of Theorem 3.2 and Theorem 4.1 by two examples of Banach algebras in order to study the existence of solutions for the system (1.1).

Example 5.1. Let X be a Banach algebra satisfying the condition $(\mathcal{P})$.
We will seek the solutions of the system (1.1) in the space $C(J, X)$ of all continuous functions on $J = [0, T], 0 < T < \infty$ endowed with the norm $\|.\|_\infty$.
Clearly, $C(J, X)$ becomes a Banach algebra satisfying the condition $(\mathcal{P})$ (see [10]).
Let us now introduce the following assumptions:

(H_0) The functions a and k are such that:
  (a) $a : J \longrightarrow X$ is continuous, and
  (b) $k : J \times J \to \mathbb{R}$ is nonnegative and continuous function.
(H_1) $\sigma_1, \sigma_2, \eta : J \longrightarrow J$ are continuous,
(H_2) $q : J \longrightarrow \mathbb{R}$ is continuous,
(H_3) The function $p : J \times J \times X \times X \longrightarrow \mathbb{R}$ is weakly sequentially continuous such that, for an arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \longrightarrow p(t, s, x, y)$ is continuous,
(H_4) The mapping $f : J \times X \longrightarrow X$ is such that:
  (a) f is weakly sequentially continuous, and
  (b) f is a contraction operator with a constant k',
(H_5) The function $f_1 : J \times X \longrightarrow \mathbb{R}$ is such that:
  (a) f_1 is weakly sequentially continuous with respect to the second variable, and
  (b) $\|f_1(., x(.))\| \leq \lambda r$, if $\|x\|_\infty \leq r$, for $r > 0$.
(H_6) The function $g : J \times X \longrightarrow X$ is such that:
  (a) g is weakly sequentially continuous with respect to the second variable,
  (b) g is a Φ-nonlinear contraction with respect to the second variable, and
  (c) $\Phi(r) < (1 - \lambda)r$, for all $r > 0$.
Our existence result for problem (1.1) is in the following result:

Theorem 5.1. *Suppose that the assumptions (H_0)-(H_6) are satisfied. Moreover, assume that there exists a real number $r_0 > 0$ such that*

$$\begin{cases} |p\,(t, s, x(s), x(\lambda s))| \leq r_0, \text{ for } x \in C(J, X) \text{ such that } \|x\|_\infty \leq r_0, \text{ and} \\ \|f(t, x(t))\| \leq k'\|x(t)\|, \quad \text{for } t \in J \text{ and } x \in C(J, X) \text{ such that } \|x\|_\infty \leq r_0. \\ \|g(., x(.))\| \leq \lambda\|x\|_\infty, \text{ for } x \in C(J, X) \text{ such that } \|x\|_\infty \leq r_0. \\ \|a\|_\infty \leq \dfrac{(1 - k')r_0}{\delta^2 KT\lambda\|u\|_\infty}, \text{ with } u \in X\backslash\{0\}, \end{cases} \tag{5.1}$$

where $K = \sup_{t,s\in J} k(t,s)$ *and* $\lambda\delta = (\|q\|_\infty + Tr_0)\|v\|_\infty + r_0$. *Then, the system* (1.1) *has, at least, one solution in* $C(J,X)\times C(J,X)$.

Proof. Let S be the closed ball B_{r_0} on $C(J,X)$ centered at origin of radius $r_0 > 0$, where r_0 satisfies the inequalities in (5.1). We recall that the problem (1.1) can be written in the following form

$$\begin{cases} x(t) = Ax(t) + By(t)\cdot B'y(t) \\ y(t) = Cx(t) + Dy(t) \end{cases}$$

where

$$\begin{cases} (Ax)(t) = f(t,x(t)),\ t\in J, \\ (Bx)(t) = a(t)x(t),\ t\in J, \\ (Cx)(t) = \left(q(t) + \displaystyle\int_0^{\sigma_2(t)} p\left(t,s,x(s),x(\lambda s)\right)ds\right)v;\ t\in J, 0<\lambda<1 \text{ and } v\in X\backslash\{0\}. \\ (Dx)(t) = g\left(t,x(t)\right),\ t\in J, \text{ and} \\ (B'x)(t) = \left(\displaystyle\int_0^{\sigma_1(t)} k(t,s)f_1(s,x(\eta(s)))ds\right)u;\ t\in J \text{ and } u\in X\backslash\{0\}. \end{cases} \tag{5.2}$$

In order to apply Theorem 3.2, we have to verify the following steps.
Claim 1: $(I-D)^{-1}C(S)$ is bounded. Indeed, since D is a Φ-nonlinear contraction, then the inverse operator $(I-D)^{-1}$ is well-defined on $C(J,X)$ (See [12]). Let $(x,y)\in S\times C(J,X)$ be such that $y=(I-D)^{-1}Cx$. Then, for all $t\in J$, we have

$$y(t) = \left(q(t) + \int_0^{\sigma_2(t)} p\left(t,s,x(s),x(\lambda s)\right)ds\right)v + g\left(t,y(t)\right).$$

Since $y\in C(J,X)$ then, there is $t^*\in J$ such that

$$\begin{aligned} \|y\|_\infty &= \|y(t^*)\| \\ &\le \left|q(t^*) + \int_0^{\sigma_2(t^*)} p\left(t^*,s,x(s),x(\lambda s)\right)ds\right| \|v\| \\ &+ \|g(t^*,y(t^*)) - g(t^*,x(t^*))\| + \|g(t^*,x(t^*))\| \\ &\le (\|q\|_\infty + Tr_0)\|v\| + \Phi\left(\|x(t^*)-y(t^*)\|\right) + \|g(t^*,x(t^*))\| \\ &< (\|q\|_\infty + Tr_0)\|v\| + (1-\lambda)\|y(t^*)\| + \|x\|_\infty \\ &\le (\|q\|_\infty + Tr_0)\|v\| + r_0 + (1-\lambda)\|y\|_\infty. \end{aligned}$$

Consequently, $\|y\|_\infty < \delta$ where $\delta = \frac{1}{\lambda}\left[(\|q\|_\infty + Tr_0)\|v\| + r_0\right]$. Hence, $(I-D)^{-1}C(S)$ is bounded with a bound δ which end the first claim
It should be noted that the operators defined in (5.2) are well-defined. Indeed, the maps Ax, By and Dy are continuous on J in view of assumptions (H_0), $(H_4)(b)$ and

$(H_6)(b)$, for all $(x, y) \in S \times C(J, X)$. Now, we claim that the two maps Cx and $B'y$ are continuous on J for all $(x, y) \in S \times (I - D)^{-1}C(S)$. To see this, let $\{t_n\}$ be any sequence in J converging to a point t in J. Then,

$$\begin{aligned}\|(B'y)(t_n) - (B'y)(t)\| &\leq \left[\int_0^{\sigma_1(t_n)} |k(t_n, s) - k(t, s)|\, |f_1(s, y(\eta(s)))|\, ds\right] \|u\| \\ &+ \left|\int_{\sigma_1(t_n)}^{\sigma_1(t)} k(t, s) f_1(s, y(\eta(s)))ds\right| \|u\|.\end{aligned}$$

Moreover, taking into account that $(I - D)^{-1}C(S)$ is bounded with a bound δ, and using the assumption $(H_5)(b)$, we get

$$\begin{aligned}\|(B'y)(t_n) - (B'y)(t)\| &\leq \left[\int_0^T |k(t_n, s) - k(t, s)|\, \lambda\delta ds + \left|\int_{\sigma_1(t_n)}^{\sigma_1(t)} K\lambda\delta ds\right|\right] \|u\| \\ &\leq \left[\int_0^T |k_{t_n}(s) - k_t(s)|\, ds + K\, |\sigma_1(t_n) - \sigma_1(t)|\right] \lambda\delta\|u\|.\end{aligned}$$

The continuity of k and σ_1 on $[0, T]$ implies that the function $B'y$ is continuous.
Now, the use of the first inequality in (5.1) and the dominated convergence theorem shows that the operator C is well defined.

Claim 2: In the proof of Theorem 3.2, we need to show that B and B' are weakly sequentially continuous on $(I - D)^{-1}C(S)$ and A and C are weakly sequentially continuous on S.
We begin to show the property for the operator B. Let $\{x_n\}_{n=0}^{\infty}$ be a weakly convergent sequence of $(I - D)^{-1}C(S)$ to a point x. Since $(I - D)^{-1}C(S)$ is bounded, we can apply the Dobrakov's theorem [22] in order to get

$$x_n(t) \rightharpoonup x(t) \ \text{ in } \ X.$$

Using the condition $(\mathcal{P})$, we obtain

$$(Bx_n)(t) \rightharpoonup (Bx)(t) \ \text{ in } \ X.$$

Since $\{Bx_n\}_{n=0}^{\infty}$ is bounded with a bound $\|a\|_\infty\delta$ then, we can again apply the Dobrakov's theorem to obtain $Bx_n \rightharpoonup Bx$. Consequently, B is weakly sequentially continuous.
Now, the use of assumption (H_5) and the Dobrakov's theorem allows us we obtain

$$f_1(t, x_n(t)) \rightharpoonup f_1(t, x(t)) \ \text{ in } \mathbb{R}.$$

Moreover, the use of the dominated convergence theorem leads to

$$\lim_{n\to\infty}\int_0^{\sigma_1(t)} k(t, s) f_1(s, x_n(\eta(s)))ds = \int_0^{\sigma_1(t)} k(t, s) f_1(s, x(\eta(s)))ds.$$

Then, $B'x_n \rightharpoonup B'x$ and so B' is weakly sequentially continuous on $(I-D)^{-1}C(S)$. Therefore, since g is weakly sequentially continuous with respect to the second variable and since $g(\cdot, x_n)$ is bounded, then the operator D defined in (5.2) is also weakly sequentially continuous. Moreover, taking into account that S is bounded and using the Dobrakov's theorem [22] we show that A is a weakly sequentially continuous operator on S.
Now, we show that C is weakly sequentially continuous on S. To see this, let $\{x_n\}_{n=0}^{\infty}$ be any sequence in S weakly converging to a point $x \in S$. Then by using the Dobrakov's theorem, we get for all $t \in J,\ x_n(t) \rightharpoonup x(t)$. Then, by assumption (H_3) and the dominated convergence theorem, we obtain

$$(Cx_n)(t) \to (Cx)(t) \ in \ X.$$

Thus, $Cx_n \rightharpoonup Cx$. As a result, C is weakly sequentially continuous on S.
Claim 3: Let $\{x_n\}$ be any sequence in S, we have $(Cx_n)(t) = r_n(t) \cdot v$, where

$$r_n(t) = q(t) + \int_0^{\sigma_2(t)} p\left(t, s, x_n(s), x_n(\lambda s)\right) ds.$$

Since $|r_n(t)| \leq \|q\|_\infty + Tr_0$ in view of the first inequality in (5.1), it follows that there is a renamed subsequence such that $r_n(t) \to r(t)$, which implies that

$$(Cx_n)(t) \rightharpoonup (Cx)(t) \ in \ X.$$

As a result, $C(S)(t)$ is sequentially relatively weakly compact. Next, we will show that $C(S)$ is a weakly equi-continuous set. If we take $\varepsilon > 0,\ x \in S,\ x^* \in X^*$ and $t, t' \in J$ such that $t \leq t',\ t' - t \leq \varepsilon$, and using the first inequality in (5.1), we obtain

$$|x^*\left((Cx)(t) - (Cx)(t')\right)| \quad \leq \quad [w(q,\varepsilon) + w(p,\varepsilon) + r_0 w(\sigma,\varepsilon)]\, \|x^*(v)\|,$$

where

$$\begin{cases} w(q,\varepsilon) = \sup\left\{|q(t) - q(t')| \ : \ t, t' \in J \ ; \ |t - t'| \leq \varepsilon\right\}, \\ \\ w(p,\varepsilon) = \sup\left\{|p_t(s,x,y) - p_{t'}(s,x,y)| \ ; \ |t - t'| \leq \varepsilon \ ; \ x, y \in S\right\}, \text{ and} \\ \\ w(\sigma,\varepsilon) = \sup\left\{|\sigma(t) - \sigma(t')| \ : \ t, t' \in J \ ; \ |t - t'| \leq \varepsilon\right\}. \end{cases}$$

Taking into account the assumption (H_3), and in view of the uniform continuity of the functions p, q and σ on the set J, it follows that $w(q,\varepsilon) \to 0$, $w(p,\varepsilon) \to 0$ and $w(\sigma,\varepsilon) \to 0$ as $\varepsilon \to 0$. An application of the Arzelà-Ascoli's theorem [32], we conclude that $C(S)$ is sequentially relatively weakly compact in X. Again, an application of Eberlein-Šmulian's theorem [17] shows that $C(S)$ is relatively weakly compact. As a result, C is weakly compact.
Now the use of the assumption (H_4) and Lemma 2.4 shows the operator A is condensing.
Claim 4: It should be noted that, for all $x \in (I-D)^{-1}C(S)$, there exists a unique $z \in C(J,X)$ such that $z = x$, with $\|z\| \leq \delta$.
Let $x,\ y \in S \times C(J,X)$, such that

$$y = Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx.$$

Then, for all $t \in J$, we have

$$\begin{aligned}\|y(t)\| &\leq \|f(t,x(t))\| + \|a(t)z(t)\| \left\| \left(\int_0^{\sigma_1(t)} k(t,s) f_1(s, z(\eta(s)))ds \right) u \right\| \\ &< k'\|x(t)\| + \|a\|_\infty K \|z(t)\| \|h_\delta\|_{L^1} \|u\|_\infty,\end{aligned}$$

where $K = \sup\limits_{t,s\in J} k(t,s)$.

Since $y \in C(J,X)$, there is $t^* \in J$ such that $\|y\|_\infty = \|y(t^*)\|$ and consequently, $\|y\|_\infty \leq r_0$ in view of the last inequality in (5.1). As a result,

$$Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cx \in S, \quad \text{for all } \ x \in S.$$

To end the proof, we apply Theorem 3.2, we get that the block operator matrix (1.2) has, at least, a fixed point in $\mathcal{B}_r$, for all $r \geq r_0$, equivalently the problem (1.1) has a solution in $S \times C(J,X)$.

Example 5.2. Let $C([0,1],\mathbb{R})$ be the Banach algebra of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the sup-norm $\|.\|_\infty$ defined by $\|f\|_\infty = \sup\limits_{t\in[0,1]} |f(t)|$, for each $f \in C(J,\mathbb{R})$.

We will use Theorem 4.1 to examine the existence of solutions of the following problem (in short *FIE*)

$$\begin{cases} x(t) = \displaystyle\int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\eta_1(s)))ds + y(t) \cdot \left[\int_0^{\sigma_2(t)} k_2(t,s) f_2(s, y(\eta_2(s)))ds \right] \\ y(t) = \dfrac{1}{1+b(t)|x(t)|} - g(t, \dfrac{1}{1+b(t)|x(t)|}) + g(t, y(t)) \end{cases} \tag{5.3}$$

for all $t \in J$, where the functions $\sigma_1, \sigma_2, \eta_1, \eta_2, k_1, k_2, b, f_1, f_2, g$ are given, whereas $x = x(t)$ and $y = y(t)$ are unknown functions.

Let us recall the following definition which will play a crucial role below.

Definition 5.1. A mapping $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to satisfy L^1-Carathéodory condition or simply is called L^1-Carathéodory if:

(a) $t \longrightarrow f(t,x)$ is measurable for each $x \in \mathbb{R}$,

(b) $x \longrightarrow f(t,x)$ is almost everywhere continuous for $t \in [0,1]$, and

(c) for each real number $r > 0$, there exists a function $h_r \in L^1([0,1],\mathbb{R})$ such that $|f(t,x)| \leq h_r(t)$; $t \in J$ for all $x \in \mathbb{R}$ with $|x| \leq r$.

Let us now introduce the following assumptions:

(H_7) The functions $\sigma_i, \eta_i : J \longrightarrow J$ are continuous for $i = 1,2$.

(H_8) The function $b : [0,1] \longrightarrow \mathbb{R}$ is continuous and nonnegative.

(H_9) The functions $k_i : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ are continuous and nonnegative for $i = 1,2$.

(H_9) The function $f_1 : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is generalized Lipschitz with Lipschitz function l_1.

(H_{10}) The function $f_2 : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is L^1-Carathéodory and it is generalized

Lipschitz with a Lipschitz function l_2.
(H_{11}) The function $g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that:
(a) The partial $x \longrightarrow g(t,x)$ is a contraction with a constant k, for $t \in [0,1]$.
(b) The partial $t \longrightarrow g(t,x)$ is a continuous mapping on J, for all $x \in C([0,1],\mathbb{R})$.
(H_{12}) There exists $r > 2$ such that:
(a) $|f_1(t,x(t))| \leq |x(t)|$, for $x \in C([0,1],\mathbb{R})$ such that $\|x\|_\infty \leq r$.
(b) $K_2\|h_r\|_{L^1} \leq (1-K_1)r$, where $K_1 = \sup\limits_{t,s\in[0,1]} k_1(t,s) < 1$ and $K_2 = \sup\limits_{t,s\in[0,1]} k_2(t,s)$.
(c) $\dfrac{\|b\|_\infty(1+k)}{1-k} K_2\|h_r\|_{L^1} + K_1\|l_1\|_{L^1} < k$.
(d) $0 \leq \dfrac{rK_2\|l_2\|_{L^1}}{(1-k)^2} < 1$.

Theorem 5.2. *Under assumptions* (H_7)-(H_{12})*, the problem* FIE (5.3) *has, at least, one solution in* $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$.

Proof. Observe that the above problem (5.3) may be written in the following form

$$\begin{cases} x(t) \in Ax(t) + By(t) \cdot B'y(t) \\ y(t) = Cx(t) + Dy(t) \end{cases}$$

where A, B, C, D and B' on $C(J,\mathbb{R})$ defined by:

$$\begin{cases} (Ax)(t) = \displaystyle\int_0^{\sigma_1(t)} k_1(t,s) f_1(s, x(\eta_1(s)))ds \ ; \ t \in J \\ (Bx)(t) = x(t) \ ; \ t \in J \\ (Cx)(t) = \dfrac{1}{1+b(t)|x(t)|} - g(t, \dfrac{1}{1+b(t)|x(t)|}) \ ; \ t \in J \\ (Dx)(t) = g\left(t, x(t)\right) \ ; \ t \in J \\ (B'x)(t) = \displaystyle\int_0^{\sigma_2(t)} k_2(t,s) f_2(s, x(\eta_2(s)))ds \ ; \ t \in J. \end{cases} \tag{5.4}$$

We will show that A, B, C, D and B' satisfy all conditions of Theorem 4.1.
Since the two maps f_2 and k_2 are continuous and since f_2 is L^1-Carathéodory, it follows from the dominated convergence theorem that B' is continuous on S (See Granas [23]), where

$$S := \{y \in C([0,1],\mathbb{R}) \text{ such that } \|y\| \leq r\}.$$

Therefore, it is easy to see that, by composition the operators defined in (5.4) are well defined.
Claim 1. A, B, C and B' are Lipschitzian. To see this, for all $x, y \in S$ we have

$$\begin{aligned} \|Ax - Ay\| &\leq \sup_{t\in J} \int_0^{\sigma_1(t)} k_1(t,s) \left|f_1(s, x(\eta_1(s))) - f_1(s, y(\eta_1(s)))\right| ds \\ &\leq \sup_{t\in J} \int_0^{\sigma_1(t)} K_1 l_1(s) \left|x(\eta_1(s)) - y(\eta_1(s))\right| ds \\ &\leq K_1\|x - y\| \int_0^1 l_1(s)ds. \end{aligned}$$

This shows that A is Lipschitzian with a Lipschitz constant $K_1\|l_1\|_{L^1}$.
By using the same argument, we conclude that B' is Lipschitzian with a Lipschitz constant $K_2\|l_2\|_{L^1}$. Again, from assumption $(H_{11})(a)$, it follows that the operator C is Lipschitzian with a Lipschitz constant $(1+k)\|b\|_\infty$ and the first claim is approved.
Claim 2. Let $x \in S$ and $t \in J$ then, one can easily verify that

$$(Cx)(t) = (I-D)\left(\frac{1}{1+b|x|}\right)(t).$$

Hence, there exists a point $y \in S$ such that $(Cx)(t) = (I-D)(y)(t)$. Consequently, $C(S) \subseteq (I-D)(S)$ and from our assumptions we infer that the inverse operator $(I-D)^{-1}$ exists on $C(S)$ and that $(I-D)^{-1}C$ is Lipschitzian on S, with a Lipschitz constant $\frac{1+k}{1-k}\|b\|_\infty$.
Claim 3. C is a strongly continuous mapping on S. Indeed, let $\{x_n\}_{n=0}^\infty$ be any sequence in S weakly converging to a point x. Then, $x \in S$ since S is weakly closed in $C([0,1],\mathbb{R})$ and by using the Dobrakov's theorem [22] we have for all $t \in [0,1]$

$$x_n(t) \rightharpoonup x(t) \text{ in } \mathbb{R}.$$

Since C is Lipschitzian then, $Cx_n(t) \to Cx(t)$ and consequently $Cx_n \to Cx$. This shows that C is a strongly continuous operator on S.
Claim 4. Since $(I-D)^{-1}Cx = \dfrac{1}{1+b|x|}$ for all $x \in S$, then the operator $B(I-D)^{-1}C$ is regular on $S \supseteq B'(I-D)^{-1}C(S)$. Therefore, the operator inverse $\left(\frac{I}{T}\right)^{-1}$ exists on $B'(S)$. Indeed, let $x, y \in B'(S)$ such that

$$x(t)(1+b(t)|x(t)|) = y(t), \ t \in \ J.$$

This implies that

$$|x(t)|(1+b(t)|x(t)|) = |y(t)|.$$

For each $t \in [0,1]$ such that $b(t) = 0$, we have $x = y$. Then, for each $t \in [0,1]$ such that $b(t) > 0$, we obtain

$$\left(\sqrt{b(t)}|x(t)| + \frac{1}{2\sqrt{b(t)}}\right)^2 = \frac{1}{4b(t)} + |y(t)|$$

which further implies the following equation

$$\sqrt{b(t)}|x(t)| = \frac{-1}{2\sqrt{b(t)}} + \sqrt{\frac{1}{4b(t)} + |y(t)|}.$$

Hence,

$$b(t)|x(t)| = \frac{-1}{2} + \sqrt{\frac{1}{4} + b(t)|y(t)|}$$

and consequently,

$$x(t) = \frac{y(t)}{1+b(t)|x(t)|} = \frac{y(t)}{\dfrac{1}{2} + \sqrt{\dfrac{1}{4} + b(t)|y(t)|}}.$$

Consider G the function defined by the expression

$$\begin{cases} G : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R}) \\ x \longrightarrow G(x) = \dfrac{x}{\dfrac{1}{2} + \sqrt{\dfrac{1}{4} + b|x|}}. \end{cases}$$

It is easy to verify that, for all $x \in C(J,\mathbb{R})$, we have

$$\left(\left(\frac{I}{B(I-D)^{-1}C}\right) \circ G\right)(x) = \left(G \circ \left(\frac{I}{B(I-D)^{-1}C}\right)\right)(x) = x.$$

We conclude that

$$\left(\frac{I}{B(I-D)^{-1}C}\right)^{-1} x = \frac{x}{\dfrac{1}{2} + \sqrt{\dfrac{1}{4} + b|x|}}.$$

Moreover, taking into account that $K_1\|l_1\|_{L^1} < 1$ and $A(S) \subset S$, and using the fixed point theorem of Boyd and Wong [12], we deduce that $(I - A)^{-1}$ exists on $(I - A)(S)$. Consequently, by using [20], we have

$$\left(\frac{I-A}{B(I-D)^{-1}C}\right)^{-1} = \left(\frac{I}{B(I-D)^{-1}C}\right)^{-1} (I-A)^{-1}.$$

So, the operator $\left(\dfrac{I-A}{B(I-D)^{-1}C}\right)^{-1}$ exists on $B'(I-D)^{-1}C(S)$.

Claim 5. By using the assumption (H_{10}), we have

$$\begin{aligned} M_1 &= \sup_{x\in S} \|B'(I-D)^{-1}Cx\| \\ &\leq \sup_{x\in S} \|B'x\| \\ &\leq \sup_{x\in S} \left\{ \sup_{t\in[0,1]} \left| \int_0^{\sigma_2(t)} k_2(t,s) f_2(s, x(\eta_2(s)))ds \right| \right\} \\ &\leq K_2\|h_r\|_{L^1}. \end{aligned}$$

Consequently, in view of (H_{12}), we have

$$\frac{(1+k)\|b\|_\infty}{1-k} M + K_1\|l_1\|_{L^1} < k.$$

Now, since

$$M_2 = \sup_{x\in S} \|B(I-D)^{-1}Cx\| \leq \sup_{x\in S} \|Bx\| \leq r,$$

we have

$$0 \leq \frac{M_2 K_2\|l_2\|_{L^1}}{(1-k)^2} < 1.$$

Next, let us fix an arbitrary $x \in C(J,\mathbb{R})$ and $y \in S$, such that

$$x = Ax + B(I-D)^{-1}Cx \cdot B'(I-D)^{-1}Cy$$

or, equivalently

$$\text{for all } \ t \in [0,1], \ x(t) = Ax(t) + B(I-D)^{-1}Cx(t) \cdot B'(I-D)^{-1}Cy(t).$$

Then,

$$\begin{aligned}
|x(t)| &\leq |Ax(t)| + |Tx(t)||T'y(t)| \\
&\leq \int_0^{\sigma_1(t)} |k_1(t,s) f_1(s, x(\eta_1(s)))|\, ds + \\
&\quad \frac{1}{1+b(t)|x(t)|} \left| \int_0^{\sigma_2(t)} k_2(t,s) f_2\left(s, \frac{1}{1+b(\eta_2(s))|x(\eta_2(s))|}\right) ds \right| \\
&\leq \int_0^{\sigma_1(t)} k_1(t,s)\, |f_1(s, x(\eta_1(s)))|\, ds + \\
&\quad \frac{1}{1+b(t)|x(t)|} \int_0^1 k_2(t,s) \left| f_2\left(s, \frac{1}{1+b(\eta_2(s))|x(\eta_2(s))|}\right) \right| ds \\
&\leq \int_0^1 K_1\, |x(\eta_1(s))|\, ds + \int_0^1 K_2 h_r(s) ds \leq K_1 \|x\|_\infty + K_2 \|h_r\|_{L^1} \leq r.
\end{aligned}$$

To end the proof, we apply Theorem 5.2, we get that (5.3) has, at least, one solution in $S \times S$.

References

[1] J. Appell, E. De Pascale, *Some parameters associated with the Hausdorff measure of noncompactness in spaces of measurable functions,* Boll. Un. Mat. Ital. B (6), **3**(1984), no. 2, 497-515. (in Italian)

[2] O. Arino, S. Gautier, J.-P. Penot, *A fixed point theorem for sequentially continuous mappings with application to ordinary differential equations,* Funkcial. Ekvac., **27**(1984), no. 3, 273-279.

[3] J. Banas, M. Lecko, *Fixed points of the product of operators in Banach algebras,* Panamer. Math. J., **12**(2002), no. 2, 101-109.

[4] J. Banas, L. Olszowy, *On a class of measures of non-compactness in Banach algebras and their application to nonlinear integral equations,* Z. Anal. Anwend., **28**(2009), no. 4, 475-498.

[5] J. Banas, J. Rivero, *On measures of weak noncompactness,* Ann. Mat. Pura Appl., **151**(1988), no. 1, 213-224.

[6] A. Ben Amar, S. Chouayekh, A. Jeribi, *New fixed point theorems in Banach algebras under weak topology features and applications to nonlinear integral equations,* J. Funct. Anal., **259**(2010), no. 9, 2215-2237.

[7] A. Ben Amar, S. Chouayekh, A. Jeribi, *Fixed point theory in a new class of Banach algebras and application,* Afr. Mat. **24**(2013), no. 4, 705-724.

[8] A. Ben Amar, S. Chouayekh, A. Jeribi, *Fixed point results in Banach algebras for weakly convex-power condensing operators and applications,* Preprint.

[9] A. Ben Amar, A. Jeribi, B. Krichen, *Fixed point theorems for block operator matrix and an application to a structured problem under boundary conditions of Rotenberg's model type,* Math. Slovaca, **64**(2014), no. 1, 155-174.

[10] A. Ben Amar, A. Jeribi, M. Mnif, *Some fixed point theorems and application to biological model,* Numer. Funct. Anal. Optim., **29**(2008), no. 1-2, 1-23.

[11] A. Ben Amar, M. Mnif, *Leray-Schauder alternatives for weakly sequentially continuous mappings and application to transport equation,* Math. Methods Appl. Sci., **33**(2010), no. 1, 80-90.

[12] D.W. Boyd, J.S.W. Wong, *On nonlinear contractions,* Proc. Amer. Math. Soc., **20**(1969), 458-464.

[13] H. Brezis, *Analyse fonctionnelle. Théorie et applications,* Masson, Paris, 1983.

[14] T.A. Burton, *A fixed-point theorem of Krasnoselskii,* Appl. Math. Lett., **11**(1998), no. 1, 85-88.

[15] F.E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces,* Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 2, Chicago, Ill., 1968). Amer. Math. Soc., Providence, R.I., 1976, 1-308.

[16] J. Caballero, B. Lopez, K. Sadarangani, *Existence of nondecreasing and continuous solutions of an integral equation with linear modification of the argument,* Acta Math. Sin. (Engl. Ser.), **23**(2007), 1719-1728.

[17] J.B. Conway, *A Course in Functional Analysis,* Springer-Verlag, Berlin 1990.

[18] F.S. De Blasi, *On a property of the unit sphere in a Banach space,* Bull. Math. Soc. Sci. Math. R.S. Roumanie (N.S.), **21(69)**(1977), no. 3-4, 259-262.

[19] B.C. Dhage, *On a fixed point theorem in Banach algebras with applications,* Appl. Math. Lett., **18**(2005), no. 3, 273-280.

[20] B.C. Dhage, *On some nonlinear alternatives of Leray-Schauder type and functional integral equations*, Arch. Math., Brno, **42**(2006), no. 1, 11-23.

[21] B.C. Dhage, D. O'Regan, *A fixed point theorem in Banach algebras with applications to functional integral equations,* Funct. Differ. Equ., **7**(2000), no. 3-4, 259-267.

[22] I. Dobrakov, *On representation of linear operators on* $C_0(T, X)$, Czechoslovak Math. J., **21(96)**(1971), 13-30.

[23] A. Granas, R.B. Guenther, J.W. Lee, *Some general existence principles in the Carathéodory theory of nonlinear differential systems,* J. Math. Pures Appl., **70**(1991), no. 2, 153-196.

[24] A. Jeribi, B. Krichen, *Nonlinear Functional Analysis in Banach Spaces and Banach Algebras: Fixed Point Theory Under Weak Topology for Nonlinear Operators and Block Operator Matrices with Applications,* (Monographs and Research Notes in Mathematics Series), CRC Press/ Taylor and Francis, 2015.

[25] A. Jeribi, B. Krichen, B. Mefteh, *Existence solutions of a two-dimensional boundary value problem for a system of nonlinear equations arising in growing cell populations,* J. Biological Dynamics, **7**(2013), no. 1, 218-232.

[26] A. Jeribi, B. Krichen, B. Mefteh, *Existence of solutions of a nonlinear Hammerstein integral equation*, Numer. Funct. Anal. Optim., **35**(2014), no. 10, 1328-1339.

[27] N. Kaddachi, A. Jeribi, B. Krichen, *Fixed Point Theorems of Block Operator Matrices On Banach Algebras and an Application to Functional Integral Equations*, Math. Methods Appl. Sci., **36**(2013), no. 6, 659-673.

[28] R.W. Legget, *On certain nonlinear integral equations,* J. Math. Anal. Appl., **57**(1977), no. 2, 462-468.

[29] Y. Liu, Z. Li, *Schaefer type theorem and periodic solutions of evolution equations,* J. Math. Anal. Appl., **316**(2006), no. 1, 237-255.

[30] A. Majorana, S.A. Marano, *Continuous solutions of a nonlinear integral equation on an unbounded domain,* J. Integral Eq. Appl., **6**(1994), no. 1, 119-128.

[31] D. O'Regan, M.A. Taoudi, *Fixed point theorems for the sum of two weakly sequentially continuous mappings,* Nonlinear Anal., **73**(2010), no. 2, 283-289.

[32] I. Vrabie, C_0*-Semigroups and Applications,* Elsevier, New-york, 2003.

[33] D.R. Smart, *Fixed Point Theorems,* Cambridge University Press, Cambridge, 1980.

[34] T. Xiang, R. Yuan, *A class of expansive-type Krasnosel'skii fixed point theorems,* Nonlinear Anal., **71**(2009), no. 7, 3229-3239.

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