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# EXISTENCE, LOCALIZATION AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR THE DIRICHLET BVP WITH $\phi$-LAPLACIAN

DIANA-RALUCA HERLEA

Babeş-Bolyai University, Faculty of Mathematics and Computer Science
1, Kogălniceanu Street, 400084 Cluj-Napoca, Romania
E-mail: dherlea@math.ubbcluj.ro

**Abstract.** The aim of this paper is to discuss the existence, localization and multiplicity of positive solutions for the Dirichlet boundary value problem with $\phi$-Laplacian. Our approach is based on Krasnosel'skiĭ's fixed point theorem in cones and on a weak Harnack type inequality. As concerns the systems, the localization is established by the vector version of Krasnosel'skiĭ's theorem, where the compression-expansion conditions are expressed on components.
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## 1. Introduction

In this paper, we focus on the existence, localization and multiplicity of positive solutions for the following Dirichlet boundary value problem

$$\begin{cases} (\phi\,(u'))' + f\,(t,u) = 0, & 0 < t < 1 \\ u\,(0) = u\,(1) = 0, \end{cases} \tag{1.1}$$

where $\phi$ is a homeomorphism from $(-a,a)$ to $\mathbb{R}$, $0 < a \leq \infty$.

According to the related literature [3]-[6], there are two basic models in this context:

(1) The $p$-Laplacian operator, where $a = \infty$,

$$\phi(u) = |u|^{p-2}u, \text{ with } p > 1.$$

(2) The curvature operator in Minkowski space, where $a = 1$,

$$\phi(u) = \frac{u}{\sqrt{1-u^2}}.$$

The problem (1.1) can be considered as a particular case, for $n = 1$, of the corresponding problem for an $n$-dimensional system,

$$\begin{cases} (\phi_i\,(u_i'))' + f_i\,(t, u_1, u_2, ..., u_n) = 0, & 0 < t < 1 \\ u_i\,(0) = u_i\,(1) = 0 \quad (i = 1, 2, ..., n)\,. \end{cases} \tag{1.2}$$

A somewhat similar approach of the case $a = \infty$ was used in [1], [2]. However, the multiplicity of solutions is discussed in [2] by a different method, and the case of systems was not considered at all in [1], [2].

First we shall concentrate on the problem (1.1) for a single equation, and then we shall extend the results to the general case (1.2) of systems.

The equation from problem (1.1) with different boundary conditions has been studied in a large number of papers using fixed point methods, degree theory, upper and lower solution techniques and variational methods. We refer to the papers [3]-[7], [10], [12], [15], and the bibliographies therein.

We are interested not only on the existence of positive solutions to the problems (1.1) and (1.2), but also on their localization and multiplicity. We shall succeed this by using the technique based on Krasnosel'skiĭ's fixed point theorem in cones [11].

**Theorem 1.1.** **(Krasnosel'skiĭ)** *Let $(X, |\cdot|)$ be a normed linear space; $K \subset X$ a cone; $r, R \in \mathbb{R}_+$, $0 < r < R$, $K_{r,R} = \{u \in K : r \leq |u| \leq R\}$, and let $N : K_{r,R} \to K$ be a compact map. Assume that one of the following conditions is satisfied:*

(a) *$N(u) \nless u$ if $|u| = r$, and $N(u) \ngtr u$ if $|u| = R$;*

(b) *$N(u) \ngtr u$ if $|u| = r$, and $N(u) \nless u$ if $|u| = R$.*

*Then $N$ has a fixed point $u$ in $K$ with $r \leq |u| \leq R$.*

Here for two elements $u, v \in X$, the strict ordering $u < v$ means $v - u \in K \setminus \{0\}$.

In applications, the technique based on Krasnosel'skiĭ's theorem requires the construction of a suitable cone of positive functions. In the case of most boundary value problems this is done using the associated Green functions and their properties. Alternatively, for other problems for which Green functions are not known, one can use weak Harnack type inequalities associated to the differential operators and the boundary conditions, as shown in [15] and [16]. In our case, such an inequality will arise as a consequence of the concavity of the positive solutions.

In the case of systems, we shall allow the homeomorphisms $\phi_i$ have different domains and we shall be interested to localize each component $u_i$ of a solution $u = (u_1, u_2, ..., u_n)$. In this respect, we shall use the following vector version of Krasnosel'skiĭ 's theorem given in [13], [14], and applied to different types of problems in [9], [10], [13].

**Theorem 1.2.** ([13]) *Let $(X, |.|)$ be a normed linear space; $K_1, K_2, ..., K_n \subset X$ cones; $K := K_1 \times K_2 \times ... \times K_n$; $r, R \in \mathbb{R}^n_+$, $r = (r_1, r_2, ..., r_n)$, $R = (R_1, R_2, ..., R_n)$ with $0 < r_i < R_i$ for all $i$, $K_{r,R} = \{u \in K : r_i \leq |u_i| \leq R_i,\ i = 1, 2, ..., n\}$ and let $N : K_{r,R} \to K$, $N = (N_1, N_2, ..., N_n)$ be a compact map. Assume that for each $i = 1, 2, ..., n$, one of the following conditions is satisfied in $K_{r,R}$ :*

(a) *$N_i(u) \nless u_i$ if $|u_i| = r_i$, and $N_i(u) \ngtr u_i$ if $|u_i| = R_i$;*

(b) *$N_i(u) \ngtr u_i$ if $|u_i| = r_i$, and $N_i(u) \nless u_i$ if $|u_i| = R_i$.*

*Then $N$ has a fixed point $u = (u_1, u_2, ..., u_n)$ in $K$ with $r_i \leq |u_i| \leq R_i$ for $i = 1, 2, ..., n$.*

Note that in the previous theorem, the same symbol $<$ is used to denote the strict ordering induced by any of the cones $K_1, K_2, ..., K_n$.

It deserves to be underlined that the compression condition (a) has to be satisfied by some indices $i$, and the expansion condition (b) by the others. In applications, this fact allows the nonlinear terms of the system to have different behaviors both in components and in variables.

## 2. Positive solutions of $\phi$-Laplace equations

In this section, we prove existence of positive solutions for the problem (1.1). We make the following assumptions: $\phi : (-a, a) \to \mathbb{R}$, $0 < a \leq \infty$ is an increasing homeomorphism such that $\phi(0) = 0$; $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

By a positive solution of the problem (1.1) we understand a function $u \in C^1[0,1] \cap C([0,1]; \mathbb{R}_+)$, with $u(0) = u(1) = 0$, such that $u'(t) \in (-a, a)$ for every $t \in [0,1]$, $\phi \circ u'$ is continuously differentiable on $[0,1]$, and the equation in (1.1) is satisfied on $[0,1]$.

In order to obtain the equivalent integral equation to the problem (1.1), let us first consider the problem:

$$\begin{cases} (\phi(u'))' + h(t) = 0, & 0 < t < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{2.1}$$

where $h \in C[0,1]$.
Integration of the differential equation from (2.1) gives

$$\phi(u'(t)) = \phi(u'(0)) - \int_0^t h(s)\, ds.$$

Then

$$u'(t) = \phi^{-1}\left(\phi(u'(0)) - \int_0^t h(s)\, ds\right).$$

Integrating from 0 to $t$ and taking into account that $u(0) = 0$, we have

$$u(t) = \int_0^t \phi^{-1}\left(\phi(u'(0)) - \int_0^\tau h(s)\, ds\right) d\tau. \tag{2.2}$$

If we denote $b := \phi(u'(0))$ and we substitute into (2.2), we obtain

$$u(t) = \int_0^t \phi^{-1}\left(b - \int_0^\tau h(s)\, ds\right) d\tau. \tag{2.3}$$

For $t = 1$, (2.3) becomes

$$\int_0^1 \phi^{-1}\left(b - \int_0^\tau h(s)\, ds\right) d\tau = 0. \tag{2.4}$$

According to Lemma 2 from [3], there exists a unique $b = b(h)$ satisfying (2.4). In addition, the mapping $b : C[0,1] \to \mathbb{R}$ is continuous and takes bounded sets into bounded sets.

Taking this into account, for all $t \in [0,1]$ we may define the integral operator $S : L^1[0,1] \to C^1[0,1]$ by

$$(Sh)(t) = \int_0^t \phi^{-1}\left(b(h) - \int_0^\tau h(s)\,ds\right) d\tau, \tag{2.5}$$

which has the following properties:
(a) For each $h \geq 0$, $Sh \geq 0$;
(b) If $h_1 \geq h_2 \geq 0$ then $Sh_1 \geq Sh_2$.
Indeed, property (a) will arise as a consequence of the concavity of $u = Sh$, but property (b) requires the following comparison result, which can be deduced from the general result in [8]. However, in our case a direct simple proof can be done.

**Lemma 2.1.** *Assume that $h_1, h_2 \in C[0,1]$, with $h_i(t) = -\left(\phi\left(u_i'\right)\right)'$, where $u_i(0) = u_i(1) = 0$, $i = 1,2$. Under the assumptions on $\phi$, if $h_1 \geq h_2 \geq 0$ then $u_1(t) \geq u_2(t)$ for each $t \in [0,1]$.*

*Proof.* Suppose for a contradiction that $u_1 \ngeq u_2$. Then there exists an interval $[t_0, t_1]$, with $0 \leq t_0 < t_1 \leq 1$ where $u_1(t) < u_2(t)$, for all $t \in (t_0, t_1)$ and $u_1(t_0) = u_2(t_0)$.

From $h_1 \geq h_2$, one has that $\left(\phi\left(u_2'\right)\right)' - \left(\phi\left(u_1'\right)\right)' \geq 0$. Then $\phi\left(u_2'\right) - \phi\left(u_1'\right)$ is increasing. On the other hand, the concavity of $u_1$ and $u_2$ implies $u_2'(t_0) \geq u_1'(t_0)$, which shows that $\phi\left(u_2'\right) - \phi\left(u_1'\right) \geq 0$ in $t_0$ and then on the entire interval $[t_0, t_1]$. Thus $u_2' - u_1' \geq 0$ and then we have that $u_2 - u_1$ is increasing on $[t_0, t_1]$ and equal to zero in $t_0$ and $t_1$. This implies that $u_1 \equiv u_2$ on $[t_0, t_1]$ which is a contradiction.

Now, returning to our problem (1.1), we have its equivalence to the integral equation

$$u = S \circ N_f(u), \tag{2.6}$$

where $N_f(u) = f(\cdot, u)$.

Next, we may define the integral operator $T : C([0,1]; \mathbb{R}_+) \to C([0,1]; \mathbb{R}_+)$ by

$$T(u)(t) = \int_0^t \phi^{-1}\left(b - \int_0^\tau f(s, u(s))\,ds\right) d\tau, \tag{2.7}$$

where $b = b(f(\cdot, u(\cdot)))$. Thus, finding positive solutions to (1.1) is equivalent to the fixed point problem for the operator $T$ on $C([0,1]; \mathbb{R}_+)$. Note that standard argument based on Ascoli-Arzela's theorem, guarantee that $T$ is completely continuous. Let $|.|_\infty$ denote the max norm on $C\,[0,1]$.

In order to apply Krasnosel'skiĭ's fixed point theorem in cones we need a weak Harnack type inequality for the differential operator $Lu := -(\phi(u'))'$.

**Lemma 2.2.** *For each $t_0, t_1 \in (0,1)$ with $t_0 < t_1$, and any $u \in C^1[0,1] \cap C\left([0,1]; \mathbb{R}_+\right)$ with $u\,(0) = u\,(1) = 0$, $u'\,(t) \in (-a, a)$ for every $t \in [0,1]$, $\phi \circ u' \in W^{1,1}(0,1)$ and $(\phi(u'))' \leq 0$ a.e. on $[0,1]$, one has*

$$u(t) \geq \gamma(t)|u|_\infty, \quad \text{for all } t \in [0,1], \tag{2.8}$$

*where* $\gamma(t) = \begin{cases} \min\{t_0, 1 - t_1\}, & \text{for all } t \in [t_0, t_1] \\ 0, & \text{otherwise.} \end{cases}$

*Proof.* Since $\phi$ is increasing and $\phi(u')$ is nonincreasing on $[0,1]$, the function $u'$ is nonincreasing on $[0,1]$. Therefore, $u$ is positive and concave on $[0,1]$. If $\min\limits_{t\in[t_0,t_1]} u(t) = 0$, then the concavity of $u$ implies $u = 0$ on $[0,1]$, and so (2.8) holds. If $\min\limits_{t\in[t_0,t_1]} u(t) > 0$, then we may assume without loss of generality that $\min\limits_{t\in[t_0,t_1]} u(t) = 1$ (otherwise, multiply (2.8) by a suitable positive constant). Then $u(t_0) = 1$ or $u(t_1) = 1$.

Assume that $u(t_0) = 1$. Since $u$ is concave, $|u|_\infty$ is reached on $[t_0, 1]$. On the other hand the graph of $u$ for $t \in [t_0, 1]$ is under the line $u = \dfrac{t}{t_0}$, containing the points $(0,0)$ and $(t_0, 1)$. So we have that $|u|_\infty \leq \dfrac{1}{t_0}$. Hence $t_0\,|u|_\infty \leq 1$. Finally, since $1 \leq u(t)$ for $t \in [t_0, t_1]$, we obtain

$$u(t) \geq t_0|u|_\infty, \text{ for all } t \in [t_0, t_1].$$

Similarly, if $u(t_1) = 1$, from the concavity of $u$, $|u|_\infty$ is reached on $[0, t_1]$. On the other hand its graph for $t \in [0, t_1]$ is under the line $u = \dfrac{t-1}{t_1-1}$, containing the points $(1,0)$ and $(t_1, 1)$ and so we have that $|u|_\infty \leq \dfrac{1}{1-t_1}$. Therefore $(1-t_1)\,|u|_\infty \leq 1$ and so we obtain

$$u(t) \geq (1-t_1)|u|_\infty, \text{ for all } t \in [t_0, t_1].$$

Notice that a graphic representation would make more clear the above reasoning.

For our first result we make the following assumptions:

(A1) $\phi : (-a, a) \to \mathbb{R}$, $0 < a \leq \infty$ is an increasing homeomorphism such that $\phi(0) = 0$;

(A2) $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $f(t,.)$ is nondecreasing on $\mathbb{R}_+$ for each $t \in [0,1]$.

**Theorem 2.3.** *Let* (A1) *and* (A2) *hold and assume that there exist* $\alpha, \beta > 0$ *with* $\alpha \neq \beta$ *such that*

$$|Sf(\cdot, \gamma(\cdot)\alpha)|_\infty > \alpha, \tag{2.9}$$

$$|Sf(\cdot, \beta)|_\infty < \beta. \tag{2.10}$$

*Then* (1.1) *has at least one positive solution* $u$ *with* $r \leq |u|_\infty \leq R$, *where* $r = \min\{\alpha, \beta\}$, $R = \max\{\alpha, \beta\}$.

*Proof.* We shall apply Krasnosel'skiĭ's fixed point theorem in cones. In our case, $X = C[0,1]$, the cone $K$ is the following one

$$K = \{u \in C([0,1]; \mathbb{R}_+) : u(0) = u(1) = 0 \text{ and } u(t) \geq \gamma(t)|u|_\infty, \text{ for all } t \in [0,1]\},$$

and $T$ is the operator given by (2.7).

Notice that if $u, v \in C([0,1]; \mathbb{R}_+)$ and $v < u$, that is $u - v \in K \setminus \{0\}$, then $(u-v)(1) \geq \gamma(1)\,|u-v|_\infty > 0$. Hence

$$|u|_\infty \geq u(1) > v(1). \tag{2.11}$$

First we remark that $T(K) \subset K$. Indeed, if $u \in K$ and $v := T(u)$, then $-(\phi(v'))' = f(t,u)$. We have $f(t, u(t)) \geq 0$ for every $t \in [0,1]$, so $(\phi(v'))' \leq 0$

on $[0,1]$. Then Lemma 2.2 guarantees that $v(t) \geq \gamma(t)\,|v|_\infty$ for $t \in [0,1]$, that is $v \in K$ as desired.

Next we prove that

$$u \ngtr T(u) \quad \text{for every } \ u \in K \ \text{ with } \ |u|_\infty = \alpha. \tag{2.12}$$

To this end, assume the contrary, i.e. $u > T(u)$ for some $u \in K$ with $|u|_\infty = \alpha$. Then using the definition of $K$, and the monotonicity of $f$ and $\phi$, we have that $f(\cdot,u) \geq f(\cdot,\gamma(\cdot)\alpha)$ and so $Sf(\cdot,u) \geq Sf(\cdot,\gamma(\cdot)\alpha)$. Hence

$$|Sf(\cdot,u)|_\infty \geq |Sf(\cdot,\gamma(\cdot)\alpha)|_\infty. \tag{2.13}$$

Now, using (2.11) and (2.13), we deduce

$$\alpha = |u|_\infty \geq |T(u)|_\infty = |Sf(\cdot,u)|_\infty \geq |Sf(\cdot,\gamma(\cdot)\alpha)|_\infty,$$

which contradicts (2.9). Thus (2.12) holds.

The next step is to prove that

$$u \nless T(u) \quad \text{for every } \ u \in K \ \text{ with } \ |u|_\infty = \beta. \tag{2.14}$$

Assume the contrary, i.e. $u < T(u)$ for some $u \in K$ with $|u|_\infty = \beta$. Then we would obtain

$$\beta = |u|_\infty \leq |T(u)|_\infty = |Sf(\cdot,u)|_\infty \leq |Sf(\cdot,\beta)|_\infty,$$

which contradicts (2.10). Thus (2.14) holds.

Now Krasnosel'skiĭ's theorem applies and yields the result.

**Remark 2.4.** *The existence and localization result, Theorem 2.3, immediately yields multiplicity results for the problem (1.1), in case that several (finitely many or infinitely many) couples of distinct numbers $\alpha,\beta$ satisfying (2.9), (2.10) exist such any two of the corresponding intervals $(\alpha,\beta)$ are disjoint.*

The next theorems are about the existence of at least one pair $\alpha,\beta$ satisfying the conditions (2.9), (2.10), and the existence of a sequence of positive solutions of the problem (1.1), respectively. Their proofs are as in [10]. However, for reader's convenience we reproduce them.

**Theorem 2.5.** *Let* (A1) *and* (A2) *hold and assume that one of the following conditions is satisfied:*

(i) $\limsup\limits_{\lambda\to\infty} \dfrac{|Sf(\cdot,\gamma(\cdot)\lambda)|_\infty}{\lambda} > 1$ *and* $\liminf\limits_{\lambda\to 0} \dfrac{|Sf(\cdot,\lambda)|_\infty}{\lambda} < 1$;

(ii) $\limsup\limits_{\lambda\to 0} \dfrac{|Sf(\cdot,\gamma(\cdot)\lambda)|_\infty}{\lambda} > 1$ *and* $\liminf\limits_{\lambda\to\infty} \dfrac{|Sf(\cdot,\lambda)|_\infty}{\lambda} < 1$.

*Then* (1.1) *has at least one positive solution.*

*Proof.* In order to apply Theorem 2.3, we look for two numbers $\alpha,\beta > 0$, $\alpha \neq \beta$ with

$$|Sf(\cdot,\gamma(\cdot)\alpha)|_\infty > \alpha \quad \text{and} \quad |Sf(\cdot,\beta)|_\infty < \beta.$$

In case (i), one can chose $\alpha$ large enough and $\beta$ small enough; while in case (ii), $\alpha$ is chosen small enough and $\beta$ is chosen large enough.

**Theorem 2.6.** *Let* (A1) *and* (A2) *hold. If the condition*

(iii) $\limsup_{\lambda\to\infty} \frac{|Sf(\cdot,\gamma(\cdot)\lambda)|_\infty}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \frac{|Sf(\cdot,\lambda)|_\infty}{\lambda} < 1$

*holds, then* (1.1) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $|u_n|_\infty \to \infty$ *as* $n\to\infty$.

*If the condition*

(iv) $\limsup_{\lambda\to 0} \frac{|Sf(\cdot,\gamma(\cdot)\lambda)|_\infty}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \frac{|Sf(\cdot,\lambda)|_\infty}{\lambda} < 1$

*holds, then* (1.1) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $u_n \to 0$ *as* $n\to\infty$.

*Proof.* Clearly (iii) guarantees the existence of two sequences $(\alpha_n)_{n\geq 1}, (\beta_n)_{n\geq 1}$ such that

$$0<\alpha_n<\beta_n<\alpha_{n+1} \text{ for every } n\geq 1, \text{ and } \alpha_n\to\infty \text{ as } n\to\infty. \tag{2.15}$$

For each $n$, Theorem 2.3 yields a positive solution $u_n$ with $\alpha_n \leq |u_n|_\infty \leq \beta_n$. The condition (2.15) implies that these solutions are distinct and that $|u_n|_\infty \to \infty$ as $n\to\infty$. A similar reasoning can be done in case (iv).

Notice that the conditions (iii) and (iv) show that $f$ is oscillating towards $\infty$ or zero, respectively.

## 3. Positive solutions of $\phi$-Laplace systems

In this section we extend the above results to the general case (1.2). We shall allow the homeomorphisms $\phi_i$ have different domains, namely $\phi_i : (-a_i, a_i) \to \mathbb{R}$, $0 < a_i \leq \infty$, and we shall assume that $\phi_i$ are increasing with $\phi_i(0) = 0$, and that $f_i : [0,1]\times\mathbb{R}^n_+ \to \mathbb{R}_+$ are continuous functions $(i = 1, 2, ..., n)$. Under these assumptions, problem (1.2) is equivalent to the integral system

$$u_i(t) = \int_0^t \phi_i^{-1}\left(b_i - \int_0^\tau f_i(s,u(s))\,ds\right)d\tau, \quad i=1,2,...,n,$$

where $u = (u_1, u_2, ..., u_n)$ and $b_i = b_i(f_i(\cdot, u(\cdot)))$.

According to Lemma 2.2, for each $i$ a weak Harnack type inequality holds for the differential operator $L_i v := -(\phi_i(v'))'$ and the boundary conditions $v(0) = v(1) = 0$. Based on this we define the cones

$$K_i = \{u_i \in C([0,1];\mathbb{R}_+) : u_i(0) = u_i(1) = 0 \text{ and } u_i(t) \geq \gamma_i(t)|u_i|_\infty, \text{ for all } t\in[0,1]\}, \tag{3.1}$$

for $i = 1, 2, ..., n$. We note that the functions $\gamma_i$ are given by Lemma 2.2 for possibly different subintervals $[t_0, t_1]$. Now we consider the product cone

$$K := K_1 \times K_2 \times ... \times K_n$$

in $C([0,1],\mathbb{R}^n)$.

Let $T : C([0,1];\mathbb{R}^n_+) \to C([0,1];\mathbb{R}^n_+)$, $T = (T_1, T_2, ..., T_n)$ be defined by

$$T_i(u)(t) = \int_0^t \phi_i^{-1}\left(b_i - \int_0^\tau f_i(s,u(s))\,ds\right)d\tau \qquad (i=1,2,...,n).$$

If $u_j \in K_j$ for each $j$, then $f_i(s, u(s)) \geq 0$ and from Lemma 2.2, one has $T_i(u) \in K_i$. Thus the cone $K$ is invariant by $T$.

The following result is a generalization of Theorem 2.3 and guarantees the existence of positive solutions to the problem (1.2) and their component-wise localization. For any index $i \in \{1, 2, ..., n\}$, we shall say that the homeomorphism $\phi_i : (-a_i, a_i) \to \mathbb{R}$ satisfies (A1) if $\phi_i$ is increasing and $\phi_i(0) = 0$, and that the continuous function $f_i : [0, 1] \times \mathbb{R}_+^n \to \mathbb{R}_+$ satisfies (A2) if for each $t \in [0, 1]$, $f_i(t, x_1, ..., x_n)$ is nondecreasing on $\mathbb{R}_+$ with respect to any variable $x_j$, $j = 1, 2, ..., n$.

**Theorem 3.1.** *Let $\phi_i$, $f_i$ satisfy* (A1) *and* (A2) *for $i = 1, 2, ..., n$. Assume that there exist $\alpha_i$, $\beta_i > 0$ with $\alpha_i \neq \beta_i$ such that*

$$|Sf_i(\cdot, \gamma_1(\cdot)\alpha_1, ..., \gamma_n(\cdot)\alpha_n)|_\infty > \alpha_i, \tag{3.2}$$

$$|Sf_i(\cdot, \beta)|_\infty < \beta_i, \tag{3.3}$$

*for $i = 1, 2, ..., n$, where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$. Then* (1.2) *has at least one positive solution $u = (u_1, u_2, ..., u_n)$ with $r_i \leq |u_i|_\infty \leq R_i$, where $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$, $i = 1, 2, ..., n$.*

*Proof.* The result is a consequence of the vector version of Krasnosel'skiĭ's fixed point theorem in cones.

We shall say that for a given index $i$, the condition (i) holds if for every $\lambda_1, \lambda_2, ..., \lambda_{i-1} > 0$,

$$\limsup_{\lambda_i \to \infty} \frac{|Sf_i(\cdot, \gamma_1(\cdot)\lambda_1, ..., \gamma_n(\cdot)\lambda_n)|_\infty}{\lambda_i} > 1 \quad \text{and} \quad \liminf_{\lambda_i \to 0} \frac{|Sf_i(\cdot, \lambda)|_\infty}{\lambda_i} < 1,$$

uniformly with respect to $\lambda_{i+1}, \lambda_{i+2}, ..., \lambda_n \in (0, \infty)$, where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$. We shall understand the condition (ii) in a similar manner. Analogously, we say that (iii) holds for some index $i$, if for every $\lambda_1, \lambda_2, ..., \lambda_{i-1} > 0$,

$$\limsup_{\lambda_i \to \infty} \frac{|Sf_i(\cdot, \gamma_1(\cdot)\lambda_1, ..., \gamma_n(\cdot)\lambda_n)|_\infty}{\lambda_i} > 1 \quad \text{and} \quad \liminf_{\lambda_i \to \infty} \frac{|Sf_i(\cdot, \lambda)|_\infty}{\lambda_i} < 1,$$

uniformly with respect to $\lambda_{i+1}, \lambda_{i+2}, ..., \lambda_n \in (0, \infty)$, where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$. The condition (iv) is understood in a similar manner.

Finally, we note that Theorem 3.2 from [10] can be applied to our problem (1.2) in order to guarantee the existence of multiple solutions.

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