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ABOUT THE ATTRACTORS OF INFINITE ITERATED FUNCTION SYSTEMS

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Abstract. The aim of this paper is to establish a sufficient condition for the attractor of an infinite iterated function system to become a dendrite. We consider the family of the associated graphs of an attractor and prove that, in some conditions, the attractor is a dendrite if all the associated graphs are infinite trees.

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1. Introduction

Iterated function systems were introduced in their present form by J. Hutchinson ([8]) and popularized by M. Barnsley ([1]). Infinite iterated function systems were introduced in ([15]) and some results have been obtained by N. A. Secelean for the case when the attractor is compact ([13]). Also R. Miculescu and A. Mihail ([11]) studied the shift space associated to the attractors of infinite iterated function systems, which are nonempty closed and bounded subsets of complete metric spaces. Topological properties of the attractors of infinite iterated function systems have also been studied in ([2], [3], [5], [10], [11]). Generalizations of infinite iterated function systems can be found in ([4], [9]). In fact, in ([9]), K. Leśniak presented a multivalued approach of infinite iterated function systems. For a nonempty set X we will denote by $\mathcal{P}(X)$ the set of nonempty subsets of X, by $\mathcal{K}(X)$ the set of nonempty compact subsets of X and by $\mathcal{B}(X)$ the set of nonempty bounded closed subsets of X.

Definition 1.1. *Let (X,d) be a metric space. The* ***Hausdorff-Pompeiu semidistance*** *is the application $h:\mathcal{P}(X)\times\mathcal{P}(X)\to[0,+\infty]$ defined by*

$$h(A,B)=\max\{d(A,B),d(B,A)\}=\inf\{r\in[0,\infty]\mid A\subset \mathbf{B}(B,r)\text{and}\ \ B\subset\mathbf{B}(A,r)\},$$

where $d(A,B)=\sup\limits_{x\in A}d(x,B)=\sup\limits_{x\in A}\left(\inf\limits_{y\in B}d(x,y)\right)$.

Theorem 1.1. ([1], [6], [14]) *Let (X,d) be a metric space and $h:\mathcal{P}(X)\times\mathcal{P}(X)\to[0,\infty]$ the Hausdorff-Pompeiu semidistance. Then*:

1) $(\mathcal{B}(X), h)$ *and* $(\mathcal{K}(X), h)$ *are metric spaces with* $(\mathcal{K}(X), h)$ *closed in* $(\mathcal{B}(X), h)$.
2) *If* (X, d) *is complete, then* $(\mathcal{B}(X), h)$ *and* $(\mathcal{K}(X), h)$ *are complete metric spaces.*

In this paper by $\mathcal{K}(X)$ and $\mathcal{B}(X)$ we will refer to $(\mathcal{K}(X), h)$ and $(\mathcal{B}(X), h)$, respectively.

Definition 1.2. *Let* (X, d) *be a metric space. For a function* $f : X \to X$ *let us denote by* $Lip(f) \in [0, +\infty]$ *the* ***Lipschitz constant*** *associated to* f, *which is*:

$$Lip(f) = \sup_{x,y \in X;\, x \neq y} \frac{d(f(x), f(y))}{d(x, y)}.$$

We say that f *is a* ***Lipschitz function*** *if* $Lip(f) < +\infty$ *and a* ***contraction*** if $Lip(f) < 1$.

Proposition 1.1. ([14]) *Let* (X, d) *be a metric space. Then:*
1) *If* H *and* K *are two nonempty subsets of* X, *then* $h(H, K) = h(\overline{H}, \overline{K})$,
2) *If* $(H_i)_{i\in I}$ *and* $(K_i)_{i\in I}$ *are two families of nonempty subsets of* X, *then:*

$$h\left(\bigcup_{i\in I} H_i, \bigcup_{i\in I} K_i\right) = h\left(\overline{\bigcup_{i\in I} H_i}, \overline{\bigcup_{i\in I} K_i}\right) \leq \sup_{i\in I} h(H_i, K_i).$$

3) *If* H *and* K *are two nonempty subsets of* X *and* $f : X \to X$ *is a Lipschitz function, then* $h(f(K), f(H)) \leq Lip(f) \cdot h(K, H)$.

Definition 1.3. *A family* $(f_i)_{i\in I}$ *of continuous functions* $f_i : X \to X$ *for every* $i \in I$, *is said to be* ***bounded*** *if for every bounded set* $A \subset X$ *the set* $\bigcup_{i\in I} f_i(A)$ *is bounded.*

Definition 1.4. a) *An* ***infinite iterated function system*** *consists of a bounded family of continuous functions* $(f_i)_{i\in I}$ *on* X *and it is denote by* $\mathcal{S} = (X, (f_i)_{i\in I})$.
b) *For an infinite iterated function system* $\mathcal{S} = (X, (f_i)_{i\in I})$, *the* ***fractal operator*** *is the function defined by* $F_{\mathcal{S}} : \mathcal{B}(X) \to \mathcal{B}(X)$, $F_{\mathcal{S}}(B) = \overline{\bigcup_{i\in I} f_i(B)}$ *for every* $B \in \mathcal{B}(X)$.

Remark 1.1. Let $\mathcal{S} = (X, (f_i)_{i\in I})$ be an infinite iterated function system. If the functions f_i are contractions for every $i \in I$ with $c = \sup_{i\in I} Lip(f_i) < 1$, then the function $F_{\mathcal{S}}$ is a contraction that satisfies $Lip(F_{\mathcal{S}}) \leq \sup_{i\in I} Lip(f_i) < 1$.

Using Banach's contraction theorem one can prove the following:

Theorem 1.2. ([15]) *Let* (X, d) *be a complete metric space and* $\mathcal{S} = (X, (f_i)_{i\in I})$ *an infinite iterated function system such that* $c = \sup_{i\in I} Lip(f_i) < 1$. *Then there exists a unique set* $A \in \mathcal{B}(X)$, *which is called the attractor of* $\mathcal{S}$, *such that* $F_{\mathcal{S}}(A) = A$.

We remind the following well-known definitions.

Definition 1.5. a) *A metric space* (X, d) *is called* ***connected*** *if there are no nonempty sets* $B, C \subset X$ *such that* $X = B \cup C$ *and* $\overline{B} \cap C = \overline{C} \cap B = \varnothing$.
b) *A metric space* (X, d) *is called arcwise connected if for every* $x, y \in X$ *there exists a continuous function* $\varphi : [0, 1] \to X$ *such that* $\varphi(0) = x$ *and* $\varphi(1) = y$.

c) *A metric space* (X, d) *is called* ***locally connected*** *if for every* $x \in X$ *and every neighbourhood* V *of* x*, there exists a connected neighbourhood* U *such that* $x \in U \subset V$.

Definition 1.6. *Let* (X, τ) *be a nonempty topological space. Then*:

a) *Two paths* φ *and* ψ *in* X, $\varphi : [a, b] \to X$ *and* $\psi : [\alpha, \beta] \to X$ *are called equivalent if there exists* $h : [\alpha, \beta] \to [a, b]$ *a homeomorphism strictly increasing such that* $\varphi \circ h = \psi$.

b) *The relation from a) is indeed an equivalence on* X *because the following conditions are fulfilled*: *reflexivity* (*if* $h = Id$ *then* $\varphi \circ h = \varphi$); *simetry* (*if* $\varphi \circ h = \psi$ *then* $\varphi = \psi \circ h^{-1}$); *tranzitivity* (*if* $\varphi \circ h = \psi$ *and* $\psi \circ h_1 = \chi$ *then* $\varphi \circ (h \circ h_1) = (\varphi \circ h) \circ h_1 = \psi \circ h_1 = \chi$). *In these conditions, an equivalence class of paths is called* ***curve***.

c) *If* X *is a compact, connected and locally connected space, then* X *is called* ***dendrite*** *if for every* $x, y \in X$ *there exists an injective path* $\varphi : [0, 1] \to X$ *such that* $\varphi(0) = x$, $\varphi(1) = y$ *and every injective path* $\psi : [0, 1] \to X$ with $\psi(0) = x$ *and* $\psi(1) = y$ *is equivalent to* φ (*i.e. there exists a unique injective curve joining* x *to* y).

Definition 1.7. *Let* (X, d) *be a metric space and* $(A_i)_{i\in I}$ *a family of nonempty subsets of* X. *The family* $(A_i)_{i\in I}$ *is said to be* ***connected*** *if for every* $i, j \in I$ *there exists* $(i_k)_{k=\overline{1,n}} \subset I$ *such that* $i_0 = i$, $i_n = j$ *and* $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ *for every* $k \in \{1, \ldots, n-1\}$. *If the family is not connected, it is said to be* ***disconnected***.

Next we shortly present the shift space of an infinite iterated function system. For more details one can see ([11]). We start with some set notations: $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}^* = \mathbb{N} - \{0\}$, $\mathbb{N}^*_n = \{1, 2, \ldots, n\}$. For two nonempty sets A and B, B^A denotes the set of functions from A to B. By $\Lambda = \Lambda(B)$ we will understand the set $B^{\mathbb{N}^*}$ and by $\Lambda_n = \Lambda_n(B)$ we will understand the set $B^{\mathbb{N}^*_n}$. The elements of $\Lambda = \Lambda(B) = B^{\mathbb{N}^*}$ will be written as infinite words $\omega = \omega_1\omega_2 \ldots \omega_m\omega_{m+1} \ldots$, where $\omega_m \in B$ and the elements of $\Lambda_n = \Lambda_n(B) = B^{\mathbb{N}^*_n}$ will be written as finite words $\omega = \omega_1\omega_2 \ldots \omega_n$. By λ we will understand the empty word. Let us remark that $\Lambda_0(B) = \{\lambda\}$. By $\Lambda^* = \Lambda^*(B)$ we will understand the set of all finite words $\Lambda^* = \Lambda^*(B) = \bigcup\limits_{n\geq 0} \Lambda_n(B)$. We denote by $|\omega|$ the length of the word ω. An element of $\Lambda = \Lambda(B)$ is said to have length $+\infty$.

If $\omega = \omega_1\omega_2 \ldots \omega_m\omega_{m+1} \ldots$ or if $\omega = \omega_1\omega_2 \ldots \omega_n$ and $n \geq m$, then $[\omega]_m = \omega_1\omega_2 \ldots \omega_m$. More generally, $[\omega]^l_m = \omega_{l+1}\omega_{l+2} \ldots \omega_m$ and, therefore, we have $[\omega]_m = [\omega]_l[\omega]^l_m$ for $\omega \in \Lambda_n(B)$ and $n \geq m > l \geq 1$ or for $\omega \in \Lambda(B)$ and $m > l \geq 1$. For two words $\alpha, \beta \in \Lambda^*(B) \cup \Lambda(B)$, $\alpha \prec \beta$ means $|\alpha| \leq |\beta|$ and $[\beta]_{|\alpha|} = \alpha$. For $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$, by $\alpha\beta$ we will understand the joining of the words α and β namely $\alpha\beta = \alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m$ and respectively $\alpha\beta = \alpha_1\alpha_2 \ldots \alpha_n\beta_1\beta_2 \ldots \beta_m\beta_{m+1} \ldots$.

Let I be a nonempty set. On $\Lambda = \Lambda(I) = (I)^{\mathbb{N}^*}$ we can consider the metric $d_s(\alpha, \beta) = \sum\limits_{k=1}^{\infty} \dfrac{1 - \delta^{\beta_k}_{\alpha_k}}{3^k}$, where $\delta^y_x = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$, $\alpha = \alpha_1\alpha_2 \ldots$ and $\beta = \beta_1\beta_2 \ldots$.

Definition 1.8. *The metric space* $(\Lambda(I), d_s)$ *is called the* ***shift space*** *associated with an infinite iterated function system.*

The relation between the attractor of an infinite iterated function system and the shift space is studied in ([11]).

2. Main results

In this section we investigate some topological properties of the attractors of infinite iterated function system. We establish sufficient conditions under which the attractor of an infinite iterated function system becomes a dendrite. We will start with some general properties of the dendrites. We remind the notion of infinite graph and establish some relation between the attractor and the graph of intersections associated with it.

Definition 2.1. a) *By an* ***infinite graph*** (I, G), *we understand an infinite set* I *of* ***vertices*** *and a subset* G *of* $I \times I = \{(i, j) \mid i, j \in I\}$. *An element of* G *will be called an* ***edge***.

b) *A graph* (I, G) *is called* ***symmetric*** *if for every* $(i, j) \in G$ *we have* $(j, i) \in G$.

From now on, throughout this paper we will consider only symmetric infinite graphs and when there is no confusion upon the set of vertices, we will denote them only by G.

c) *Let* (I, G) *be a graph and* $x, y \in I$ *arbitrarily chosen. Un* ***path*** *from* x *to* y *is a family of vertices* $(v_0, v_1, \ldots, v_k)$ *with* $v_0 = x$, $v_k = y$ *and for every* $i \in \{1, \ldots, k\}$ *we have* $(v_{i-1}, v_i) \in G$. *By the* ***length of path*** *we will understand the number of edges which form the path.*

d) *Let* (I, G) *be a graph. A path* $(v_0, v_1, \ldots, v_k)$ *is called a cycle if:* $k \geq 3$, $v_0 = v_k$ *and* $v_0, v_1, \ldots, v_{k-1}$ *are different.*

e) *A graph* (I, G) *is called* ***connected*** *if for every* $x, y \in I$, $x \neq y$, *there exists a path from* x to y.

f) *A graph* (I, G) *is called an* ***infinite tree*** *if it is connected and has no cycles.*

Definition 2.2. *Let* X *be a nonempty set and* $(A_i)_{i \in I}$ *a family of nonempty subsets of* X. *Then*:

a) *The graph* (I, G), *where* $G = \{(i, j) \mid i, j \in I,\ i \neq j \text{ such that } A_i \cap A_j \neq \emptyset\}$ *is called the* ***graph of the intersections*** *associated to the family of sets* $(A_i)_{i \in I}$. *We remark that the graph of intersections is symmetric.*

b) *The family* $(A_i)_{i \in I}$ *is said to be a tree of sets if for every* $i, j \in I$, $i \neq j$, *there exists a unique sequence* $(i_k)_{k=\overline{1,n}} \subset I$ *with* $i_1, i_2, \ldots, i_n$ *different such that* $i_1 = i$, $i_n = j$, *and* $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ *for every* $k \in \{1, 2, \ldots, n-1\}$.

Remark 2.1. a) The family $(A_i)_{i \in I}$ is a tree of sets if and only if the graph of the intersections associated to the family $(A_i)_{i \in I}$ is a tree.

b) If the family of sets $(A_i)_{i \in I}$ is a tree of sets, then the intersection of three different sets of the family is empty.

c) The family $(A_i)_{i \in I}$ is connected if and only if the graph of intersections associated to it is connected.

Notation 2.1. Let a, b be real numbers such that $a < b$. Then $\mathcal{D}([a, b])$ denotes the set of the divisions of the interval $[a, b]$. For a division of the interval $[a, b]$, $\Delta = (a = y_0 < y_1 < \ldots < y_n = b)$, the norm of Δ will be $||\Delta|| = \max\limits_{k=\overline{0,n-1}} |y_k - y_{k+1}|$.

We prove now some preliminary lemmas.

Lemma 2.1. *Let* (X,d) *be a metric space and* $\varphi,\varphi^{'}:[0,1]\to X$ *continuous and injective functions such that there exist two sequences of divisions of the interval* $[0,1]$, $(\Delta_l)_{l\in\mathbb{N}},(\Delta_l^{'})_{l\in\mathbb{N}}\in\mathcal{D}([0,1])$ *satisfying the following properties:*

a) $\Delta_l=(0=y_0^l<y_1^l<\ldots<y_{n_l}^l=1)$ *and* $\Delta_l^{'}=(0=z_0^l<z_1^l<\ldots<z_{n_l}^l=1)$ *have the same number of elements for all* $l\in\mathbb{N}$,

b) $||\Delta_l||\xrightarrow{l\to\infty}0$ *and* $||\Delta_l^{'}||\xrightarrow{l\to\infty}0$,

c) $\max\limits_{k=\overline{0,n_l}} d(\varphi(y_k^l),\varphi^{'}(z_k^l))\xrightarrow{l\to\infty}0$.

Then there exists a unique continuous, bijective and increasing function $u:[0,1]\to[0,1]$ *such that* $\varphi^{'}\circ u=\varphi$ *(i.e.* φ *and* $\varphi^{'}$ *are equivalent).*

Proof. Let $t\in[0,1]$. Then there exists a sequence $(k_l(t))_{l\in\mathbb{N}}$ of natural numbers such that $y_{k_l(t)}^l\le t\le y_{k_l(t)+1}^l$. It is easy to see that $d(\varphi(y_{k_l(t)}^l),\varphi(t))\xrightarrow{l\to\infty}0$. Therefore from point c) $d(\varphi'(z_{k_l(t)}^l),\varphi(t))\xrightarrow{l\to\infty}0$. If $u\in[0,1]$ is a limit point of the sequence $(z_{k_l(t)}^l)_{l\in\mathbb{N}}$ then, from the continuity of φ', we have that $\varphi'(u)=\varphi(t)$. Since $[0,1]$ is a compact set and $(z_{k_l(t)}^l)_{l\in\mathbb{N}}\subset[0,1]$ it follows that the sequence $(z_{k_l(t)}^l)_{l\in\mathbb{N}}$ has at least one limit point.

If $u'\in[0,1]$ is such that $\varphi'(u')=\varphi(t)$, we should have $\varphi'(u')=\varphi(t)=\varphi'(u)$. Since φ' is injective, it follows that $u=u'$. Since the sequence $(z_{k_l(t)}^l)_{l\in\mathbb{N}}$ has a unique limit point, $(z_{k_l(t)}^l)_{l\in\mathbb{N}}\subset[0,1]$ and $[0,1]$ is a compact set it follows that $z_{k_l(t)}^l\xrightarrow{l\to\infty}u$. Thus, we have proved that for every $t\in[0,1]$ there exists a unique $u(t)\in[0,1]$ such that $\varphi(t)=\varphi'(u(t))$. This means that there exists a unique function $u:[0,1]\to[0,1]$ such that $\varphi^{'}\circ u=\varphi$. Since φ is injective it results that u is also injective.

We will prove that u is strictly increasing. Let $t_1,t_2\in[0,1]$ be such that $t_1<t_2$. Then there exist the sequences $(k_l(t_1))_{l\in\mathbb{N}}\subset\mathbb{N}$ and $(k_l(t_2))_{l\in\mathbb{N}}\subset\mathbb{N}$ such that $y_{k_l(t_1)}^l\le t_1\le y_{k_l(t_1)+1}^l$ and $y_{k_l(t_2)}^l\le t_2\le y_{k_l(t_2)+1}^l$.

It follows that $y_{k_l(t_1)}^l\le y_{k_l(t_2)+1}^l$ and so $z_{k_l(t_1)}^l\le z_{k_l(t_2)+1}^l$. Since $z_{k_l(t_1)}^l\xrightarrow{l\to\infty}u(t_1)$, $z_{k_l(t_2)}^l\xrightarrow{l\to\infty}u(t_2)$ and $|z_{k_l(t_2)}^l-z_{k_l(t_2)+1}^l|\le||\Delta_l^{'}||\xrightarrow{l\to\infty}0$ it follows that $u(t_1)\le u(t_2)$.

Interchanging $\varphi^{'}$ with φ, there exists a unique function $v:[0,1]\to[0,1]$ such that $\varphi^{'}=\varphi\circ v$. Since φ and φ' are injective functions, it follows that v is the inverse of u. Therefore u is bijective. Also u is a continuous function, since every bijective and increasing function between two closed intervals is continuous.

Lemma 2.2. *Let* (X,d) *be a complete metric space such that* $X=\bigcup\limits_{i=1}^{n}A_i$ *with* A_i *compact sets satisfying card*$(A_i\cap A_j)\in\{0,1\}$ *for every* $i,j\in\{1,\ldots,n\}$ *different. We suppose that the graph of intersections associated to the family* $(A_i)_{i=\overline{1,n}}$*, is a tree. Then for any continuous and injective function* $\varphi:[0,1]\longrightarrow X$ *such that there exists* $l\in\{1,\ldots,n\}$ *for which* $\varphi(0)\in A_l$ *and* $\varphi(1)\in A_l$*, we have that* $\varphi([0,1])\subset A_l$.

Proof. Let $\varphi:[0,1]\to X$ be an injective path such that $\varphi(0)\in A_l$ and $\varphi(1)\in A_l$, where $l\in\{1,\ldots,n\}$. We remark first that A_l has at least two elements since φ is injective. If one of the sets, namely A_j, has one element it follows that $A_j \subset \bigcup_{\substack{i=1;\\ i\neq j}}^{n} A_i$, since the family of sets $(A_i)_{i=\overline{1,n}}$ is connected and so $A_j \cap \left(\bigcup_{\substack{i=1;\\ i\neq j}}^{n} A_i\right) \neq \emptyset$. Therefore we can suppose that the sets A_i have at least two elements for all $i \in \{1,\ldots,n\}$.

Let us suppose that there exists $t \in (0,1)$ such that $\varphi(t) \notin A_l$. It results that $t \in A_j$ for $j \in \{1,\ldots,n\}\backslash\{l\}$. Then there exists a unique sequence $(i_k)_{k=\overline{1,m}} \subset I$ such that $i_1 = l$, $i_m = j$, $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$, $k \in \{1,\ldots,m-1\}$ and $i_1,\ldots,i_m$ different. Let $a \in X$ be such that $\{a\} = A_{i_1} \cap A_{i_2}$. Then there exists $t_1 \in (0,t)$ such that $\varphi(t_1) = a$. Indeed, if we suppose that $a \notin \varphi([0,t])$, then $\varphi([0,t]) \subset \bigcup_{i=1}^{n} A_i\backslash\{a\}$. We consider the sets $\widetilde{A}_i = A_i \setminus \{a\}$ for $i \in \{1,\ldots,n\}$. Since the family of sets $(A_i)_{i=\overline{1,n}}$ is a tree, it results from Remark 2.1 that the family of sets $(\widetilde{A}_i)_{i=\overline{1,n}}$ is disconnected and the sets $\widetilde{A}_l$ and $\widetilde{A}_j$ belong to different connected components of the family of sets $(\widetilde{A}_i)_{i=\overline{1,n}}$. Let $(\widetilde{A}_i)_{i\in J}$ be the connected component which contains the set $\widetilde{A}_j$. We consider $B = \bigcup_{i\in J} \widetilde{A}_i$ and $C = \bigcup_{i=1;i\notin J}^{n} \widetilde{A}_i$. Then $\varphi(0) \in \widetilde{A}_l \subset C$, $\varphi(1) \in \widetilde{A}_j \subset B$, $\varphi([0,t]) \subset B \cup C$ and $\overline{B} \cap C = B \cap \overline{C} = \emptyset$. This contradicts the fact that $\varphi([0,t])$ is a connected set. In a similar way there exists $t_2 \in (t,1)$ such that $\varphi(t_2) = a$. Hence $\varphi(t_1) = \varphi(t_2) = a$ and $t_1 < t_2$ which contradicts the injectivity of φ.

Lemma 2.3. *Let (X,d) be a complete metric space such that $X = \bigcup_{i\in I} A_i$ where A_i are compact sets for every $i \in \{1,\ldots,n\}$. We suppose that the graph of intersections associated to the family $(A_i)_{i\in I}$ is a tree. Let $x,y \in X$, $x \neq y$ and a chain of sets $\{A_{i_j}\}_{j=\overline{0,m}}$ such that $i_0,i_1,\ldots,i_m$ are different, $x \in A_{i_0}$, $x \notin A_{i_1}$, $y \in A_{i_m}$, $y \notin A_{i_{m-1}}$ and $A_{i_j} \cap A_{i_{j+1}} \neq \emptyset$ for every $j \in \{1,\ldots,m-1\}$, $m \in \mathbb{N}$. Then for any continuous and injective function $\varphi : [0,1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$ there exists a division $\Delta = (0 = y_0 < y_1 < \ldots < y_m = 1)$ of the interval $[0,1]$ such that $\varphi(y_j) \in A_{i_j} \cap A_{i_{j+1}}$ for every $j \in \{0,\ldots,m-1\}$.*

Proof. We suppose that for every $t \in (0,1)$ we have $\varphi(t) \notin A_{i_j} \cap A_{i_{j+1}}$ for some $j \in \{0,\ldots,m-1\}$. Then there exists A_v such that $A_v \cap A_{i_j} \neq \varnothing$, $A_v \cap A_{i_{j+1}} \neq \varnothing$ and $A_v \cap \mathrm{Im}\varphi \neq \varnothing$. Then $A_{i_0},\ldots,A_{i_j}, A_v, A_{i_j},\ldots,A_{i_m}$ is a new chain of sets that joining x to y. But this contradicts the fact that the family $(A_i)_{i=\overline{1,n}}$ is a tree of sets.Thus there exists $y_j \in (0,1)$ such that $\varphi(y_j) \in A_{i_j} \cap A_{i_{j+1}}$ for every $j \in \{0,\ldots,m-1\}$. Let us suppose now that $y_1 < y_0$.

From Lemma 2.2 applied to the interval $[0,y_0]$, it results that $\varphi([0,y_0]) \subset A_{i_0}$. But since $\varphi(y_1) \in A_{i_1} \cap A_{i_2}$, we obtain that $\varphi(y_1) \in A_{i_0} \cap A_{i_1} \cap A_{i_2}$. Thus $A_{i_0} \cap A_{i_1} \cap A_{i_2} \neq \varnothing$ which is a contradiction with Remark 2.1. Hence $y_1 > y_0$ and inductively $y_{i+1} > y_i$ for every $i \in \{0,\ldots,m-1\}$ and thus $\Delta = (0 < y_0 < y_1 < \ldots < y_{m-1} < 1)$ is a division of the interval $[0,1]$ such that $\varphi(y_j) \in A_{i_j} \cap A_{i_{j+1}}$ for every $j \in \{0,\ldots,m-1\}$.

Notation 2.2. a) Let (X,d) be a complete metric space, $\mathcal{S}=(X,(f_i)_{i\in I})$ an infinite iterated function system on X and A the attractor of $\mathcal{S}$. For $\omega=\omega_1\omega_2\ldots\omega_m\in\Lambda_m(I)$, f_ω denotes $f_{\omega_1}\circ f_{\omega_2}\circ\ldots\circ f_{\omega_m}$ and H_ω denotes $f_\omega(H)$ for a subset $H\subset X$. By H_λ we will understand the set H. In particular $A_\omega=f_\omega(A)$.

b) Let (X,d) be a complete metric space and $\mathcal{S}=(X,(f_{i\in I})_{i\in I})$ an infinite iterated function system on X. For every $m\in\mathbb{N}^*$, we denote by $\mathcal{S}^m$ the infinite iterated function system $\mathcal{S}^m=(X,(f_\omega)_{\omega\in\Lambda_m(I)})$ and we remark that $A(\mathcal{S})=A(\mathcal{S}^m)$.

c) By G^m we will denote the graph of intersections associated to the family of sets $(A_\omega)_{\omega\in\Lambda_m(I)}$ for every $m\in\mathbb{N}^*$.

The following result gives a characterization of the arcwise connected attractors of infinite iterated function systems and it is proven in ([5]).

Theorem 2.1. ([5]) *Let (X,d) be a complete metric space and $\mathcal{S}=(X,(f_k)_{k\in I})$ an infinite iterated function system such that $c=\sup_{k\in I}Lip(f_k)<1$. We denote by A the attractor of $\mathcal{S}$ and by A_k the set $f_k(A)$ for every $k\in I$. If $A=\bigcup_{k\in I}A_k$ and the family of sets $(A_k)_{k\in I}$ is connected, then A is arcwise connected.*

Now we can give the main result which is a sufficient condition for the attractor of an infinite iterated function system to become a dendrite.

Theorem 2.2. *Let (X,d) be a complete metric space and $\mathcal{S}=(X,(f_i)_{i\in I})$ an infinite iterated function system with $c=\sup_{i\in I}Lip(f_i)<1$. We denote by A the attractor of $\mathcal{S}$ and by G^m the graph of intersections associated with the family of sets $(A_\omega)_{\omega\in\Lambda_m(I)}$ for every $m\in\mathbb{N}^*$. If A is compact, locally connected, $A=\bigcup_{i\in I}A_i$ and the associated graphs G^m are infinite trees for every $m\in\mathbb{N}^*$, then A is a dendrite.*

Proof. Since G is an infinite tree, it results that G is connected. Thus, from the hypothesis and Theorem 2.1, it follows that A is arcwise connected. We will prove that A is a dendrite. Let $x,y\in A$, $x\neq y$. We suppose that there exist two continuous and injective functions $\varphi,\varphi^{'}:[0,1]\to A$ such that $\varphi(0)=\varphi^{'}(0)=x$ and $\varphi(1)=\varphi^{'}(1)=y$. To prove that A is a dendrite it is enough to prove that φ and $\varphi^{'}$ are equivalent. We intend to use Lemma 2.1 to prove the equivalence. For that, we will construct two sequences $(\Delta_l)_{l\geq 0}$ and $(\Delta_l')_{l\geq 0}$ of divisions of the unit interval $[0,1]$ such that:

a) $\Delta_l=(0=y_0^l<y_1^l<\ldots<y_{n_l}^l=1)$ and $\Delta_l^{'}=(0=z_0^l<z_1^l<\ldots<z_{n_l}^l=1)$ have the same number of elements for all $l\in\mathbb{N}$,

b) $||\Delta_l||\xrightarrow{l\to\infty}0$ and $||\Delta_l^{'}||\xrightarrow{l\to\infty}0$,

c) $\max_{k=\overline{0,n_l}}d(\varphi(y_k^l),\varphi^{'}(z_k^l))\xrightarrow{l\to\infty}0$.

Let $l\in\mathbb{N}$ be fixed. If there exists an $\alpha\in\Lambda_l(I)$ such that $x,y\in A_\alpha$ then we take $\Delta_0=(y_0^0=0<y_1^0=1)$ and $\Delta_0'=(z_0^0=0<z_1^0=1)$. We have $\varphi(y_0^0)=\varphi(0)=\varphi^{'}(0)=\varphi^{'}(z_0^0)=x$ and $\varphi(y_1^0)=\varphi(1)=\varphi^{'}(1)=\varphi^{'}(z_1^0)=y$. If there does not exist an $\alpha\in\Lambda_l(I)$ such that $x,y\in A_\alpha$ then there exists $\alpha_x,\alpha_y\in\Lambda_l(I)$ such that $x\in A_{\alpha_x}$, $y\in A_{\alpha_y}$ and $\alpha_x\neq\alpha_y$. Since G^l is a tree, the sets A_{α_x} and A_{α_y} are joined by a unique chain of sets $\{A_{\omega_j}\}_{j=\overline{1,m_l}}$ such that $\alpha_x=\omega_1$, $\alpha_y=\omega_{m_l}$, $\omega_j\in\Lambda_l(I)$, $A_{\omega_j}\cap A_{\omega_{j+1}}\neq\emptyset$

for $j \in \{1, \ldots, m_l - 1\}$ and $i_1, i_2, \ldots, i_{m_l} \in I$ different. We can suppose that $x \notin A_{\omega_2}$ and $y \notin A_{\omega_{m_l-1}}$ by replacing α_x with ω_2 if $x \in A_{\omega_2}$ and α_y with ω_{m-1} if $y \in A_{\omega_{m_l-1}}$.

Now, from Lemma 2.3, there exist $\Delta_l = (0 = y_0^l < y_1^l < \ldots < y_{m_l}^l = 1)$ and $\Delta_l' = (0 = z_0^l < z_1^l < \ldots < z_{m_l}^l = 1)$ divisions of the interval $[0,1]$ such that $\varphi(y_j^l), \varphi^{'}(z_{j+1}^l) \in A_{\omega_j} \cap A_{\omega_{j+1}}$. It results that

$$\max_{k=\overline{0,m_l}} d(\varphi(y_k^l), \varphi^{'}(z_k^l)) \leq \max_{k=\overline{0,m_l}} d(A_{\omega_k}) \leq c^l d(A),$$

where $c = \sup\limits_{k \in I} Lip(f_k) < 1$. Therefore $\max\limits_{k=\overline{0,n_l}} d(\varphi(y_k^l), \varphi^{'}(z_k^l)) \xrightarrow{l\to\infty} 0$. We remark now that $d_l = \max\limits_{k=\overline{0,m_l-1}} d(\varphi(y_k^l), \varphi(y_{k+1}^l)) \leq \max\limits_{k=\overline{0,m_l-1}} d(A_{\omega_k}) \leq c^l d(A)$. Therefore $d_l = \max\limits_{k=\overline{0,m_l-1}} d(\varphi(y_k^l), \varphi(y_{k+1}^l)) \xrightarrow{l\to\infty} 0$. Let now $\delta_\mu = \inf\limits_{x,y \in [0,1]; |x-y| \geq \mu} d(\varphi(x), \varphi(y))$ for every $\mu \in [0,1)$. It is obvious that $\delta_\mu \leq \delta_\nu$ if $\mu \leq \nu$. Since φ is injective and $[0,1]$ is a compact set, we have $\delta_\mu > 0$ for every $\mu > 0$. We suppose by contradiction that the sequence $(||\Delta_l||)_{l \geq 0}$ is not convergent to 0. Then there exist $\varepsilon > 0$ and a subsequence of divisions $(||\Delta_{l_k}||)_{k \geq 0}$ such that $||\Delta_{l_k}|| \geq \varepsilon$. Then

$$d_{l_k} = \max_{j=\overline{0,n_l-1}} d(\varphi(y_{j+1}^{l_k}), \varphi(y_j^{l_k})) \geq \delta_{||\Delta_{l_k}||} \geq \delta_\varepsilon > 0.$$

This contradicts the fact that $d_l \xrightarrow{l\to\infty} 0$. It follows that $||\Delta_l|| \xrightarrow{l\to\infty} 0$. In a similar way one can prove that $||\Delta_l'|| \overset{l\to\infty}{\longrightarrow} 0$.

Hence, A is compact, arcwise connected, locally connected and we have proved that every two points of A can be joined by a unique continuous and injective curve. Thus, according to Definition 1.5, A is a dendrite.

Example 2.1. (Attractors of infinite-von Koch type) (for more details see [14]). We consider the set $X = [0,1] \times [0,1] \subset \mathbb{R}^2$ and the contractions $f_n : X \to X$ given by:

$$f_n(x,y) = \begin{cases} \frac{1}{2^{p+1}}\left(\frac{x}{3} + 2^{p+1} - 2, \frac{y}{3}\right), & \text{if} \quad n = 4p+1 \text{ and } p \geq 0 \\ \frac{1}{2^{p+1}}\left(\frac{x}{6} - \frac{y\sqrt{3}}{6} + 2^{p+1} - \frac{5}{3}, \frac{x\sqrt{3}}{6} + \frac{y}{6}\right), & \text{if} \quad n = 4p+2 \text{ and } p \geq 0 \\ \frac{1}{2^{p+1}}\left(\frac{x}{6} + \frac{y\sqrt{3}}{6} + 2^{p+1} - \frac{3}{2}, -\frac{x\sqrt{3}}{6} + \frac{y}{6} + \frac{\sqrt{3}}{6}\right), & \text{if} \quad n = 4p+3 \text{ and } p \geq 0 \\ \frac{1}{2^p}\left(\frac{x}{3} + 2^p - \frac{4}{3}, \frac{y}{3}\right), & \text{if} \quad n = 4p \text{ and } p \geq 1 \end{cases}$$

The attractor A of the countable iterated function system $\mathcal{S} = (X, \{f_n\}_{n \in \mathbb{N}^*})$ is an infinite-von Koch curve obtained by smaller copies of Koch's curve. We remark that A is compact, arcwise connected, locally connected and, moreover, $A = \bigcup\limits_{n \geq 1} f_n(A)$. Also, the graphs of intersections associated to A have the following edges:

$G^1 : \{(1,2),(2,3),(3,4),(4,5),\ldots\}$.

$G^2 : \{(11,12),(12,13),(13,14),(14,21),(21,22),(22,23),(23,24),(24,31),(31,32),(32,33),(33,34),(34,41),(41,42),(42,43),(43,44),(44,55),(55,56),(56,57),\ldots\}$.

$G^3 : \{(111,112),(112,113),(113,114),(114,121),(121,122),(12,123),(123,124),(124,131),(131,132),(132,133),(133,134),(134,141),(141,142),(142,143),(143,144),(144,211),(211,212),\ldots\},\ldots$

Thus G^m is an infinite tree for every $m \in \mathbb{N}$ and from Theorem 2.2 it results that A is a dendrite.

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