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SELF-ADAPTIVE PROJECTION ALGORITHMS FOR SOLVING THE SPLIT EQUALITY PROBLEMS

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Abstract. In this paper, we first introduce a self-adaptive projection algorithm by adopting Armijo-like searches to solve the spit equality problem (SEP), then we propose a relaxed self-adaptive projection algorithm by using projections onto half-spaces instead of those onto the original convex sets, which is much more practical. Weak convergence results for both algorithms are analyzed.
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1. Introduction

Recently, Moudafi [15] introduced a new convex feasibility problem, which was named the split equality problem (SEP) by Byrne and Moudafi [7]. Its interest covers many situations, for instance in domain decomposition for PDE's, game theory and intensity-modulated radiation therapy (IMRT). In domain decomposition for PDE's, this equals to the variational form of a PDE's in a domain that can be decomposed into two non overlapping subdomains with a common interface (see e.g. [2]). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables (see e.g. [3]). In (IMRT), this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see [8] for further details). Attouch [4] pointed more applications of the SEP in optimal control theory, surface energy and potential games, whose variational form can be seen as a SEP. Algorithms for solving the split equality problem receive great attention; see for instance [15, 7, 2, 3, 10, 16, 11].

Let H_1, H_2, H_3 be real Hilbert spaces, let $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators.

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The SEP can mathematically be formulated as the problem of finding x, y with the property

$$x \in C,\ y \in Q,\ \text{such that}\ \ Ax = By, \tag{1.1}$$

which allows asymmetric and partial relations between the variables x and y. If $H_2 = H_3$ and $B = I$, then the split equality problem (1.1) reduces to the split feasibility problem (originally introduced in Censor and Elfving [9]) which is to find $x \in C$ with $Ax \in Q$.

For solving the SEP (1.1), Moudafi [15] introduced the following alternating CQ algorithm (ACQA for short)

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.2}$$

where $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, λ_A and λ_B are the spectral radius of A^*A and B^*B, respectively. By studying the projected Landweber algorithm of the SEP (1.1) in a product space, Byrne and Moudafi [7] obtained the following algorithm (PSA for short):

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \tag{1.3}$$

where γ_k, the stepsize at the iteration k, is chosen in the interval $(\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon)$. It is easy to see that the alternating CQ algorithm (1.2) is sequential (like Gauss-Seidel iteration iteration) while the algorithm (1.3) is simultaneous (like Jacob iteration).

Observe that in the algorithms (1.2) and (1.3), the determination of the stepsize γ_n depends on the operator (matrix) norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of A^*A and B^*B). This means that in order to implement the alternating CQ algorithm (1.2), one has first to compute (or, at least, estimate) operator norms of A and B, which is in general not an easy work in practice.

Consider this, Qu and Xiu [17] modified the CQ algorithm by adopting the Armijo-like searches to get the step-size for solving the split feasibility problem. Based on their algorithm, Zhao and Yang [20] presented self-adaptive projection algorithms for the multiple-sets split feasibility problem.

Inspired by them, in this paper, we introduce a self-adaptive projection algorithm for solving the SEP (1.1). The advantage of our choice of the stepsizes lies in the fact that no prior information about the operator norms of A and B is required, and still convergence is guaranteed. Since the projections on closed convex sets C and Q is generally difficult to compute, we also practise a relaxed version of the self-adaptive projection algorithm, where the closed convex sets C and Q are both level sets of convex functions,

The rest of this paper is organized as follows. In the next section, some useful facts and tools are given. The weak convergence theorem of the proposed self-adaptive projection algorithm is obtained in section 3. In section 4, we consider a relaxed self-adaptive projection algorithm where the sets C and Q are level sets of convex functions.

2. Preliminaries

In this section, we review some definitions and lemmas which will be used in this paper.

Let H be a Hilbert space and I be the identity operator on H. If $f : H \to \mathbb{R}$ is a differentiable functional, then denote by ∇f the gradient of f. If $f : H \to \mathbb{R}$ is a subdifferentiable functional, then denote by ∂f the subdifferential of f. Given a sequence (x_k, y_k) in $H_1 \times H_2$, $\omega_w(x_k, y_k)$ stands for the set of cluster points in the weak topology. $'x_k \to x'$ (resp., $'x_k \rightharpoonup x'$) means the strong (resp., weak) convergence of (x_k) to x.

The projection is an important tool for our work in this paper. Let Ω be a closed convex subset of real Hilbert space H. Recall that the (nearest point or metric) projection from H onto Ω, denoted P_Ω, is defined in such a way that, for each $x \in H$, $P_\Omega x$ is the unique point in Ω such that

$$\|x - P_\Omega x\| = \min\{\|x - z\| : z \in \Omega\}.$$

The following two lemmas are useful characterizations of projections.

Lemma 2.1. *Given $x \in H$ and $z \in \Omega$. Then $z = P_\Omega x$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \quad \textit{for all} \quad y \in \Omega.$$

Lemma 2.2. *For any $x, y \in H$ and $z \in \Omega$, it holds*

(i) $\|P_\Omega(x) - P_\Omega(y)\|^2 \leq \langle P_\Omega(x) - P_\Omega(y), x - y\rangle$;

(ii) $\|P_\Omega(x) - z\|^2 \leq \|x - z\|^2 - \|P_\Omega(x) - x\|^2$.

Throughout this paper, assume the split equality problem (1.1) is consistent and denote by Γ the solution of (1.1), i.e.

$$\Gamma = \{x \in C, y \in Q : Ax = By\},$$

then Γ is closed, convex and nonempty. The split equality problem (1.1) can be written as the following minimization problem:

$$\min_{x \in H_1, y \in H_2} \iota_C(x) + \iota_Q(y) + \frac{1}{2}\|Ax - By\|^2,$$

where $\iota_C(x)$ is a indicator function of the set C defined by

$$\iota_C(x) = \begin{cases} 0, & x \in C \\ +\infty, & \text{otherwise.} \end{cases}$$

Observe that $\nabla_x\{\frac{1}{2}\|Ax-By\|^2\} = A^*(Ax-By)$, $\nabla_y\{\frac{1}{2}\|Ax-By\|^2\} = -B^*(Ax-By)$ and $\partial\iota_C(x) = N_C(x)$, $\partial\iota_Q(y) = N_Q(y)$, where N_C, N_Q are the normal cone to the convex sets C and Q, respectively. By writing down the optimality conditions we obtain

$$\begin{cases} 0 \in \nabla_x\{\frac{1}{2}\|Ax - By\|^2\} + \partial\iota_C(x) = A^*(Ax - By) + N_C(x), \\ 0 \in \nabla_y\{\frac{1}{2}\|Ax - By\|^2\} + \partial\iota_Q(y) = -B^*(Ax - By) + N_Q(y), \end{cases}$$

which implies, for $\gamma > 0, \beta > 0$,

$$\begin{cases} x - \gamma A^*(Ax - By) \in x + \gamma N_C(x), \\ y + \beta B^*(Ax - By) \in y + \beta N_Q(y), \end{cases}$$

which in turn leads to the fixed point formulation

$$\begin{cases} x = (I + \gamma N_C)^{-1}(x - \gamma A^*(Ax - By)), \\ y = (I + \beta N_Q)^{-1}(y + \beta B^*(Ax - By)). \end{cases}$$

Since $(I + \gamma N_C)^{-1} = P_C$ and $(I + \beta N_Q)^{-1} = P_Q$, we have

$$\begin{cases} x = P_C(x - \gamma A^*(Ax - By)), \\ y = P_Q(y + \beta B^*(Ax - By)). \end{cases} \tag{2.1}$$

We will see that solutions of the fixed point equations (2.1) are exactly the solutions of the SEP (1.1).

Proposition 2.1. *Given $x^* \in H_1$ and $y^* \in H_2$. Then (x^*, y^*) solves the SEP (1.1) if and only if (x^*, y^*) solves the fixed point equations (2.1).*

Proof. We have already proved that if (x^*, y^*) solves SEP (1.1), then it also solves the fixed point equations (2.1). Conversely, assume that (x^*, y^*) solves the fixed point equations (2.1), it is obvious that $x^* \in C$ and $y^* \in Q$. (2.1) and Lemma 2.1 imply that

$$\begin{cases} \langle x^* - \gamma A^*(Ax^* - By^*) - x^*, u - x^* \rangle \leq 0, & u \in C, \\ \langle y^* + \beta B^*(Ax^* - By^*) - y^*, v - y^* \rangle \leq 0, & v \in Q. \end{cases}$$

That is,

$$\begin{cases} \langle A^*(Ax^* - By^*), x^* - u \rangle \leq 0, & u \in C, \\ \langle B^*(Ax^* - By^*), v - y^* \rangle \leq 0, & v \in Q. \end{cases}$$

Hence,

$$\begin{cases} \langle Ax^* - By^*, Ax^* - Au \rangle \leq 0, & u \in C, \\ \langle Ax^* - By^*, Bv - By^* \rangle \leq 0, & v \in Q. \end{cases} \tag{2.2}$$

Adding up two inequalities in (2.2) gives

$$\langle Ax^* - By^*, Bv - Au + Ax^* - By^* \rangle \leq 0, \quad u \in C, v \in Q.$$

Letting $(u, v) \in \Gamma$, i.e. $Au = Bv$, we obtain $Ax^* = By^*$, that is $(x^*, y^*) \in \Gamma$. □

Let F denote a mapping on H. For any $x \in H$ and $\alpha > 0$, define:

$$x(\alpha) = P_\Omega(x - \alpha F(x)), \quad e(x, \alpha) = x - x(\alpha).$$

From the nondecreasing property of $\|e(x, \alpha)\|$ on $\alpha > 0$ by Toint [18] (see Lemma 2(1)) and the nonincreasing property of $\|e(x, \alpha)\|/\alpha$ on $\alpha > 0$ by Gafni and Bertsekas [13] (see Lemma 1(a)), we immediately conclude a useful lemma.

Lemma 2.3. *Let F be a mapping on H. For any $x \in H$ and $\alpha > 0$, we have:*

$$\min\{1, \alpha\}\|e(x, 1)\| \leq \|e(x, \alpha)\| \leq \max\{1, \alpha\}\|e(x, 1)\|.$$

3. A self-adaptive projection algorithm

Based on Proposition 2.1, we construct a self-adaptive projection algorithms for the fixed point equations (2.1) and prove the weak convergence of the proposed algorithm.

Define the function $F : H_1 \times H_2 \to H_1$ by

$$F(x, y) = A^*(Ax - By),$$

and the function $G : H_1 \times H_2 \to H_2$ by

$$G(x, y) = B^*(By - Ax).$$

The self-adaptive projection algorithm is defined as follows:

Algorithm 3.1. *Given constants $\gamma > 0$, $\rho \in (0, 1)$, $\mu \in (0, 1)$. Let $x_0 \in H_1$ and $y_0 \in H_2$ be arbitrary. For $k = 0, 1, 2, \dots,$ compute*

$$\begin{cases} u_k = P_C(x_k - \tau_k F(x_k, y_k)), \\ v_k = P_Q(y_k - \tau_k G(x_k, y_k)), \end{cases}$$

where $\tau_k = \gamma\rho^{l_k}$ and l_k is the smallest nonnegative integer l such that

$$\|F(x_k, y_k) - F(u_k, v_k)\|^2 + \|G(x_k, y_k) - G(u_k, v_k)\|^2 \leq \mu^2 \frac{\|x_k - u_k\|^2 + \|y_k - v_k\|^2}{\tau_k^2}. \tag{3.1}$$

Set

$$\begin{cases} x_{k+1} = P_C(x_k - \tau_k F(u_k, v_k)), \\ y_{k+1} = P_Q(y_k - \tau_k G(u_k, v_k)). \end{cases} \tag{3.2}$$

Lemma 3.1. *The Armijo-like search rule (3.1) is well defined. Besides, $\underline{\tau} \leq \tau_k \leq \gamma$, where $\underline{\tau} = \min\left\{\gamma, \frac{\mu\rho}{\|A\|\sqrt{2(\|A\|^2+\|B\|^2)}}, \frac{\mu\rho}{\|B\|\sqrt{2(\|A\|^2+\|B\|^2)}}\right\}$.*

Proof. Obviously, $\tau_k \leq \gamma$. If $\tau_k = \gamma$, then this lemma is proved; otherwise, if $\tau_k < \gamma$, by the search rule (3.1), we know that τ_k/ρ must violate inequality (3.1), i.e.

$$\|F(x_k, y_k) - F(u_k, v_k)\|^2 + \|G(x_k, y_k) - G(u_k, v_k)\|^2 \geq \mu^2 \frac{\|x_k - u_k\|^2 + \|y_k - v_k\|^2}{\tau_k^2/\rho^2}.$$

On the other hand, we have

$$\begin{aligned} &\|F(x_k, y_k) - F(u_k, v_k)\|^2 + \|G(x_k, y_k) - G(u_k, v_k)\|^2 \\ &= \|A^*(Ax_k - By_k) - A^*(Au_k - Bv_k)\|^2 + \|B^*(By_k - Ax_k) - B^*(Bv_k - Au_k)\|^2 \\ &\leq (\|A\|^2 + \|B\|^2)(\|A\|\|x_k - u_k\| + \|B\|\|y_k - v_k\|)^2 \\ &\leq 2(\|A\|^2 + \|B\|^2)(\|A\|^2\|x_k - u_k\|^2 + \|B\|^2\|y_k - v_k\|^2) \\ &\leq 2(\|A\|^2 + \|B\|^2)\max\{\|A\|^2, \|B\|^2\}(\|x_k - u_k\|^2 + \|y_k - v_k\|^2). \end{aligned}$$

Consequently, we get

$$\tau_k \geq \frac{\mu\rho}{\sqrt{2(\|A\|^2 + \|B\|^2)}} \min\left\{\frac{1}{\|A\|}, \frac{1}{\|B\|}\right\},$$

which completes the proof. □

Theorem 3.1. *Let (x_k, y_k) be the sequence generated by the Algorithm 3.1. Then (x_k, y_k) converges weakly to a solution of the SEP (1.1).*

Proof. Let $(x^*, y^*) \in \Gamma$, i.e., $x^* \in C$, $y^* \in Q$, $Ax^* = By^*$. It is obvious that $F(x^*, y^*) = 0$ and $G(x^*, y^*) = 0$. Using the fact $Ax^* = By^*$, we have

$$\begin{aligned}
&\langle F(u_k, v_k), u_k - x^*\rangle + \langle G(u_k, v_k), v_k - y^*\rangle \\
&= \langle A^*(Au_k - Bv_k), u_k - x^*\rangle + \langle B^*(Bv_k - Au_k), v_k - y^*\rangle \\
&= \langle Au_k - Bv_k, Au_k - Ax^*\rangle + \langle Bv_k - Au_k, Bv_k - By^*\rangle \\
&= \|Au_k - Bv_k\|^2 \\
&\geq 0,
\end{aligned}$$

which implies

$$\langle F(u_k, v_k), x_k - x^*\rangle + \langle G(u_k, v_k), y_k - y^*\rangle \geq \langle F(u_k, v_k), x_k - u_k\rangle + \langle G(u_k, v_k), y_k - v_k\rangle. \tag{3.3}$$

Thus, we obtain

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &= \|P_C(x_k - \tau_k F(u_k, v_k)) - x^*\|^2 \\
&\leq \|x_k - \tau_k F(u_k, v_k) - x^*\|^2 - \|x_{k+1} - x_k + \tau_k F(u_k, v_k)\|^2 \\
&= \|x_k - x^*\|^2 - 2\tau_k\langle F(u_k, v_k), x_k - x^*\rangle \\
&\quad - \|x_{k+1} - x_k\|^2 - 2\tau_k\langle F(u_k, v_k), x_{k+1} - x_k\rangle,
\end{aligned} \tag{3.4}$$

where the first inequality follows from the property of projection mappings (Lemma 2.2 (ii)). Similarly, we have

$$\begin{aligned}
\|y_{k+1} - y^*\|^2 \leq\ &\|y_k - y^*\|^2 - 2\tau_k\langle G(u_k, v_k), y_k - y^*\rangle \\
&- \|y_{k+1} - y_k\|^2 - 2\tau_k\langle G(u_k, v_k), y_{k+1} - y_k\rangle.
\end{aligned} \tag{3.5}$$

Adding up (3.4) and (3.5) yields

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 \\
&\leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - 2\tau_k\langle F(u_k, v_k), x_k - x^*\rangle - 2\tau_k\langle G(u_k, v_k), y_k - y^*\rangle \\
&\quad - 2\tau_k\langle F(u_k, v_k), x_{k+1} - x_k\rangle - 2\tau_k\langle G(u_k, v_k), y_{k+1} - y_k\rangle \\
&\quad - \|x_{k+1} - x_k\|^2 - \|y_{k+1} - y_k\|^2 \\
&\leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - 2\tau_k\langle F(u_k, v_k), x_{k+1} - u_k\rangle - 2\tau_k\langle G(u_k, v_k), y_{k+1} - v_k\rangle \\
&\quad - \|x_{k+1} - u_k + u_k - x_k\|^2 - \|y_{k+1} - v_k + v_k - y_k\|^2 \\
&= \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - 2\tau_k\langle F(u_k, v_k), x_{k+1} - u_k\rangle - 2\tau_k\langle G(u_k, v_k), y_{k+1} - v_k\rangle \\
&\quad - \|x_{k+1} - u_k\|^2 - \|u_k - x_k\|^2 - \|y_{k+1} - v_k\|^2 - \|v_k - y_k\|^2 \\
&\quad - 2\langle x_{k+1} - u_k, u_k - x_k\rangle - 2\langle y_{k+1} - v_k, v_k - y_k\rangle \\
&= \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - \|x_{k+1} - u_k\|^2 - \|u_k - x_k\|^2 \\
&\quad - \|y_{k+1} - v_k\|^2 - \|v_k - y_k\|^2 \\
&\quad + 2\langle x_k - u_k - \tau_k F(u_k, v_k), x_{k+1} - u_k\rangle + 2\langle y_k - v_k - \tau_k G(u_k, v_k), y_{k+1} - v_k\rangle,
\end{aligned}$$

where the second inequality comes from (3.3).

Since $u_k = P_C(x_k - \tau_k F(x_k, y_k))$ and $x_{k+1} \in C$, we have by Lemma 2.1 that

$$\langle u_k - x_k + \tau_k F(x_k, y_k), x_{k+1} - u_k\rangle \geq 0.$$

Similarly, we get

$$\langle v_k - y_k + \tau_k G(x_k, y_k), y_{k+1} - v_k\rangle \geq 0.$$

It follows that

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 \\
&\leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - \|x_{k+1} - u_k\|^2 - \|u_k - x_k\|^2 - \|y_{k+1} - v_k\|^2 - \|v_k - y_k\|^2 \\
&\quad + 2\langle x_k - u_k - \tau_k F(u_k, v_k), x_{k+1} - u_k\rangle + 2\langle y_k - v_k - \tau_k G(u_k, v_k), y_{k+1} - v_k\rangle \\
&\quad + 2\langle u_k - x_k + \tau_k F(x_k, y_k), x_{k+1} - u_k\rangle + 2\langle v_k - y_k + \tau_k G(x_k, y_k), y_{k+1} - v_k\rangle \\
&= \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - \|x_{k+1} - u_k\|^2 - \|u_k - x_k\|^2 - \|y_{k+1} - v_k\|^2 - \|v_k - y_k\|^2 \\
&\quad + 2\tau_k\langle F(x_k, y_k) - F(u_k, v_k), x_{k+1} - u_k\rangle + 2\tau_k\langle G(x_k, y_k) - G(u_k, v_k), y_{k+1} - v_k\rangle \\
&\leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - \|x_{k+1} - u_k\|^2 - \|u_k - x_k\|^2 - \|y_{k+1} - v_k\|^2 - \|v_k - y_k\|^2 \\
&\quad + \tau_k^2\|F(x_k, y_k) - F(u_k, v_k)\|^2 + \|x_{k+1} - u_k\|^2 \\
&\quad + \tau_k^2\|G(x_k, y_k) - G(u_k, v_k)\|^2 + \|y_{k+1} - v_k\|^2 \\
&\leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - \|u_k - x_k\|^2 - \|v_k - y_k\|^2 + \mu^2(\|u_k - x_k\|^2 + \|v_k - y_k\|^2) \\
&= \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - (1 - \mu^2)(\|u_k - x_k\|^2 + \|v_k - y_k\|^2),
\end{aligned} \tag{3.6}$$

where the second inequality holds from $2\langle u, v\rangle \leq \|u\|^2 + \|v\|^2$, and the last inequality follows from the search rule (3.1).

Consequently, the sequence $\Gamma_k(x^*, y^*) := \|x_k - x^*\|^2 + \|y_k - y^*\|^2$ is decreasing and lower bounded by 0 for that $\mu \in (0, 1)$ and thus converges to some finite limit, say $l(x^*, y^*)$. Moreover, $(x_k), (y_k)$ are bounded. This implies

$$\lim_{k\to\infty} \|u_k - x_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|v_k - y_k\| = 0. \tag{3.7}$$

On the other hand,

$$\begin{aligned}
\|x_{k+1} - x_k\|^2 &\leq (\|x_{k+1} - u_k\| + \|u_k - x_k\|)^2 \\
&\leq 2\|x_{k+1} - u_k\|^2 + 2\|u_k - x_k\|^2 \\
&\leq 2\|x_k - \tau_k F(u_k, v_k) - x_k + \tau_k F(x_k, y_k)\|^2 + 2\|u_k - x_k\|^2 \\
&= 2\tau_k^2\|F(u_k, v_k) - F(x_k, y_k)\|^2 + 2\|u_k - x_k\|^2.
\end{aligned}$$

Similarly, we have

$$\|y_{k+1} - y_k\|^2 \leq 2\tau_k^2\|G(u_k, v_k) - G(x_k, y_k)\|^2 + 2\|v_k - y_k\|^2.$$

Adding up two inequalities gives

$$\begin{aligned}
\|x_{k+1} - x_k\|^2 + \|y_{k+1} - y_k\|^2 &\leq 2\tau_k^2(\|F(u_k, v_k) - F(x_k, y_k)\|^2 + \|G(u_k, v_k) - G(x_k, y_k)\|^2) \\
&\quad + 2(\|u_k - x_k\|^2 + \|v_k - y_k\|^2) \\
&\leq 2(\mu^2 + 1)(\|u_k - x_k\|^2 + \|v_k - y_k\|^2),
\end{aligned}$$

which results in

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|=0 \quad \text{and} \quad \lim_{k\to\infty}\|y_{k+1}-y_k\|=0. \tag{3.8}$$

Let $(\hat{x},\hat{y})\in\omega_w(x_k,y_k)$, then there exist two subsequences (x_{k_l}) and (y_{k_l}) of (x_k) and (y_k) which converge weakly to $\hat{x}$ and $\hat{y}$, respectively. We will show that $(\hat{x},\hat{y})$ is a solution of the SEP (1.1).

Let $f(x,y,\tau)=x-P_C(x-\tau F(x,y))$. Then $f(x_{k_l},y_{k_l},\tau_{k_l})=x_{k_l}-u_{k_l}$, and we get from lemmas 2.3, 3.1 and (3.7) that

$$\lim_{l\to\infty}\|f(x_{k_l},y_{k_l},1)\|\le\lim_{l\to\infty}\frac{\|x_{k_l}-u_{k_l}\|}{\min\{1,\tau_{k_l}\}}\le\lim_{l\to\infty}\frac{\|x_{k_l}-u_{k_l}\|}{\min\{1,\underline{\tau}\}}=0. \tag{3.9}$$

Let $g(x,y,\tau)=y-P_Q(y-\tau G(x,y))$. Similarly, we obtain

$$\lim_{l\to\infty}\|g(x_{k_l},y_{k_l},1)\|\le\lim_{l\to\infty}\frac{\|y_{k_l}-v_{k_l}\|}{\min\{1,\tau_{k_l}\}}\le\lim_{l\to\infty}\frac{\|y_{k_l}-v_{k_l}\|}{\min\{1,\underline{\tau}\}}=0. \tag{3.10}$$

On the other hand, given (x^*,y^*) as a solution point of the SEP (1.1), by using the property of the projection mappings (Lemma 2.1), and noting that $x^*\in C$, we have

$$\begin{aligned}0&\le\langle x_{k_l}-F(x_{k_l},y_{k_l})-P_C(x_{k_l}-F(x_{k_l},y_{k_l})),P_C(x_{k_l}-F(x_{k_l},y_{k_l}))-x^*\rangle\\&=\langle f(x_{k_l},y_{k_l},1)-F(x_{k_l},y_{k_l}),x_{k_l}-x^*-f(x_{k_l},y_{k_l},1)\rangle,\end{aligned} \tag{3.11}$$

which implies

$$\begin{aligned}\langle x_{k_l}-x^*,f(x_{k_l},y_{k_l},1)\rangle&\ge\|f(x_{k_l},y_{k_l},1)\|^2-\langle F(x_{k_l},y_{k_l}),f(x_{k_l},y_{k_l},1)\rangle\\&\quad+\langle F(x_{k_l},y_{k_l}),x_{k_l}-x^*\rangle\\&=\|f(x_{k_l},y_{k_l},1)\|^2-\langle F(x_{k_l},y_{k_l}),f(x_{k_l},y_{k_l},1)\rangle\\&\quad+\langle Ax_{k_l}-By_{k_l},Ax_{k_l}-Ax^*\rangle.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\langle y_{k_l}-y^*,g(x_{k_l},y_{k_l},1)\rangle&\ge\|g(x_{k_l},y_{k_l},1)\|^2-\langle G(x_{k_l},y_{k_l}),g(x_{k_l},y_{k_l},1)\rangle\\&\quad+\langle By_{k_l}-Ax_{k_l},By_{k_l}-By^*\rangle.\end{aligned}$$

Adding up two inequalities and using the fact $Ax^*=By^*$ yield

$$\begin{aligned}&\langle x_{k_l}-x^*,f(x_{k_l},y_{k_l},1)\rangle+\langle y_{k_l}-y^*,g(x_{k_l},y_{k_l},1)\rangle\\&\ge\|f(x_{k_l},y_{k_l},1)\|^2+\|g(x_{k_l},y_{k_l},1)\|^2\\&-\langle F(x_{k_l},y_{k_l}),f(x_{k_l},y_{k_l},1)\rangle-\langle G(x_{k_l},y_{k_l}),g(x_{k_l},y_{k_l},1)\rangle+\|Ax_{k_l}-By_{k_l}\|^2.\end{aligned} \tag{3.12}$$

Since

$$\|F(x_{k_l},y_{k_l})\|=\|F(x_{k_l},y_{k_l})-F(x^*,y^*)\|\le\|A\|(\|A\|\|x_{k_l}-x^*\|+\|B\|\|y_{k_l}-y^*\|),$$

and the subsequencea (x_{k_l}), (y_{k_l}) are bounded, we know that $(F(x_{k_l},y_{k_l}))$ is bounded. Similarly, $(G(x_{k_l},y_{k_l}))$ is bounded. Thereby, we get from (3.9), (3.10) and (3.12)

$$\lim_{l\to\infty}\|Ax_{k_l}-By_{k_l}\|=0, \tag{3.13}$$

The weak convergence of $(Ax_{k_l} - By_{k_l})$ to $A\hat{x} - B\hat{y}$ and lower semicontinuity of the squared norm imply

$$\|A\hat{x} - B\hat{y}\| \le \liminf_{l\to\infty} \|Ax_{k_l} - By_{k_l}\| = 0,$$

that is, $A\hat{x} = B\hat{y}$.

Combining (3.7) and (3.13) and using the fact that A, B are bounded operators, we get

$$\lim_{l\to\infty} \|Au_{k_l} - Bv_{k_l}\| = 0. \tag{3.14}$$

By noting that the two equalities in (3.2) can be rewritten as

$$\begin{cases} \dfrac{x_{k_l} - x_{k_{l+1}}}{\tau_{k_l}} - A^*(Au_{k_l} - Bv_{k_l}) \in N_C(x_{k_{l+1}}), \\ \dfrac{y_{k_l} - y_{k_{l+1}}}{\tau_{k_l}} - B^*(Bv_{k_l} - Au_{k_l}) \in N_Q(y_{k_{l+1}}), \end{cases}$$

that the graphs of the maximal monotone operators N_C, N_Q are weakly-strongly closed and by passing to the limit in the last inclusions, we obtain, from (3.8) and (3.14), that

$$\hat{x} \in C \quad \text{and} \quad \hat{y} \in Q.$$

Hence $(\hat{x}, \hat{y}) \in \Gamma$.

To show the uniqueness of the weak cluster points, we will use the same strick as in the celebrated Opial Lemma. Indeed, let $(\bar{x}, \bar{y})$ be other weak cluster point of (x_k, y_k). By passing to the limit in the relation

$$\Gamma_k(\hat{x}, \hat{y}) = \Gamma_k(\bar{x}, \bar{y}) + \|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2 + 2\langle x_k - \bar{x}, \bar{x} - \hat{x}\rangle + 2\langle y_k - \bar{y}, \bar{y} - \hat{y}\rangle,$$

we obtain

$$l(\hat{x}, \hat{y}) = l(\bar{x}, \bar{y}) + \|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2.$$

Reversing the role of $(\hat{x}, \hat{y})$ and $(\bar{x}, \bar{y})$, we also have

$$l(\bar{x}, \bar{y}) = l(\hat{x}, \hat{y}) + \|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2.$$

By adding the two last equalities, we obtain

$$\|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2 = 0.$$

Hence $(\hat{x}, \hat{y}) = (\bar{x}, \bar{y})$, this implies that the whole sequence (x_k, y_k) weakly converges to a solution of the SEP (1.1), which completes the proof. □

4. A relaxed self-adaptive projection algorithm

The computation of a projection onto a closed convex subset is generally difficult. To overcome this difficulty, Fukushima [12] suggested a way to calculate the projection onto a level set of a convex function by computing a sequence of projections onto half-spaces containing the original level set.

Define the closed convex sets C and Q as level sets:

$$C = \{x \in H_1 : c(x) \le 0\} \quad \text{and} \quad Q = \{y \in H_2 : q(y) \le 0\}, \tag{4.1}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex functions which are subdifferentiable on C and Q respectively and we assume that their subdifferentials are bounded on bounded sets.

Followed the ideas of Fukushima [12], Moudafi [16] introduced a relaxed alternating CQ algorithm for solving the SEP (1.1).

$$\begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_{Q_k}(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases} \tag{4.2}$$

where (C_k), (Q_k) are two sequences of closed convex sets defined by

$$C_k = \{x \in H_1 : c(x_k) + \langle \xi_k, x - x_k \rangle \leq 0\}, \tag{4.3}$$

where $\xi_k \in \partial c(x_k)$, and

$$Q_k = \{y \in H_2 : q(y_k) + \langle \eta_k, y - y_k \rangle \leq 0\}, \tag{4.4}$$

where $\eta_k \in \partial q(y_k)$. Note that the stepsize γ_k in (4.2) is chosen in $(0, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}))$.

It is easy to see that $C_k \supset C$ and $Q_k \supset Q$ for every $k \geq 0$. More importantly, since the projections onto half-spaces C_k and Q_k have closed forms, the relaxed alternating CQ-algorithm (4.2) is implementable. In what follows, we now introduce a relaxed self-adaptive projection algorithm for solving the SEP (1.1) where C and Q are given in (4.1). The stepsize is also chosen as in (3.1) which doesn't depend on the norms $\|A\|$ and $\|B\|$.

Algorithm 4.1. *Given constants $\gamma > 0$, $\rho \in (0,1)$, $\mu \in (0,1)$. Let $x_0 \in H_1$ and $y_0 \in H_2$ be arbitrary. For $k = 0, 1, 2, \dots,$ compute*

$$\begin{cases} u_k = P_{C_k}(x_k - \tau_k F(x_k, y_k)), \\ v_k = P_{Q_k}(y_k - \tau_k G(x_k, y_k)), \end{cases}$$

where $\tau_k = \gamma \rho^{l_k}$ and l_k is the smallest nonnegative integer l such that

$$\|F(x_k, y_k) - F(u_k, v_k)\|^2 + \|G(x_k, y_k) - G(u_k, v_k)\|^2 \leq \mu^2 \frac{\|x_k - u_k\|^2 + \|y_k - v_k\|^2}{\tau_k^2}. \tag{4.5}$$

Set

$$\begin{cases} x_{k+1} = P_{C_k}(x_k - \tau_k F(u_k, v_k)), \\ y_{k+1} = P_{Q_k}(y_k - \tau_k G(u_k, v_k)). \end{cases} \tag{4.6}$$

Following the proof of Lemma 3.1, it is easy to show that the Armijo-like search rule (4.5) is also well defined. Besides, $\underline{\tau} \leq \tau_k \leq \gamma$, where $\underline{\tau} = \min\{\gamma, \frac{\mu\rho}{\sqrt{2}(\|A\|^2 + \|B\|^2)}\}$.

Theorem 4.1. *Let (x_k, y_k) be the sequence generated by the Algorithm 4.1. Then (x_k, y_k) converges weakly to a solution of the SEP (1.1).*

Proof. Let $(x^*, y^*) \in \Gamma$, i.e., $x^* \in C$, $y^* \in Q$, $Ax^* = By^*$. Following the similar proof of Theorem 3.1, we obtain

$$\begin{aligned} \|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 \leq{}& \|x_k - x^*\|^2 + \|y_k - y^*\|^2 \\ & - (1 - \mu^2)(\|u_k - x_k\|^2 + \|v_k - y_k\|^2), \end{aligned} \tag{4.7}$$

Let $\Gamma_k(x^*, y^*) := \|x_k - x^*\|^2 + \|y_k - y^*\|^2$. Then the sequence $\Gamma_k(x^*, y^*)$ is decreasing and lower bounded by 0 for that $\mu \in (0, 1)$ and thus converges to some finite limit, say $l(x^*, y^*)$. Moreover, $(x_k), (y_k)$ are bounded. This implies

$$\lim_{k\to\infty} \|u_k - x_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|v_k - y_k\| = 0. \tag{4.8}$$

Next we show the sequence (x_k, y_k) generated by Algorithm 4.1 weakly converges to a solution of the SEP (1.1). Let $(\hat{x}, \hat{y}) \in \omega_w(x_k, y_k)$, then there exist two subsequences (x_{k_l}) and (y_{k_l}) of (x_k) and (y_k) which converge weakly to $\hat{x}$ and $\hat{y}$, respectively. Following the similar proof of (3.13), we obtain

$$\lim_{l\to\infty} \|Ax_{k_l} - By_{k_l}\| = 0. \tag{4.9}$$

The weak convergence of $(Ax_{k_l} - By_{k_l})$ to $A\hat{x} - B\hat{y}$ and the lower semicontinuity of the squared norm imply

$$\|A\hat{x} - B\hat{y}\| \le \liminf_{l\to\infty} \|Ax_{k_l} - By_{k_l}\| = 0,$$

that is, $A\hat{x} = B\hat{y}$.

Since $u_{k_l} \in C_{k_l}$, we have

$$c(x_{k_l}) + \langle \xi_k, u_{k_l} - x_{k_l} \rangle \le 0.$$

Thus

$$c(x_{k_l}) \le -\langle \xi_{k_l}, u_{k_l} - x_{k_l} \rangle \le \xi \|u_{k_l} - x_{k_l}\|,$$

where ξ satisfies $\|\xi_k\| \le \xi$ for all $k \in \mathbb{N}$. The lower semicontinuity of c and the first formula of (4.8) lead to

$$c(\hat{x}) \le \liminf_{l\to\infty} c(x_{k_l}) \le 0,$$

and therefore $\hat{x} \in C$.

Likewise, since $v_{k_l} \in Q_{k_l}$, we have

$$q(y_{k_l}) + \langle \eta_{k_l}, v_{k_l} - y_{k_l} \rangle \le 0.$$

Thus

$$q(y_{k_l}) \le -\langle \eta_{k_l}, v_{k_l} - y_{k_l} \rangle \le \eta \|v_{k_l} - y_{k_l}\|,$$

where η satisfies $\|\eta_k\| \le \eta$ for all $k \in \mathbb{N}$. Again, the lower semicontinuity of q and the second formula of (4.8) lead to

$$q(\hat{y}) \le \liminf_{l\to\infty} q(y_{k_l}) \le 0,$$

and therefore $\hat{y} \in Q$. Hence $(\hat{x}, \hat{y}) \in \Gamma$.

Following the same argument of Theorem 3.1, we can show the uniqueness of the weak cluster points and hence the whole sequence (x_k, y_k) weakly converges to a solution of the SEP (1.1), which completes the proof. □

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References

[1] A. Aleyner, S. Reich, *Block-iterative algorithms for solving convex feasibility problems in Hilbert and in Banach*, J. Math. Anal. Appl., **343**(2008), no. 1, 427-435.

[2] H. Attouch, A. Cabot, F. Frankel, J. Peypouquet, *Alternating proximal algorithms for constrained variational inequalities: Application to domain decomposition for PDE's*, Nonlinear Anal., **74**(2011), no. 18, 7455-7473.

[3] H. Attouch, J. Bolte, P. Redont, A. Soubeyran, *Alternating proximal algorithms for weakly coupled minimization problems. Applications to dynamical games and PDEs*, J. Convex Anal., **15**(2008), 485-506.

[4] H. Attouch, *Alternating minimization and projection algorithms. From convexity to nonconvexity*, Communication in Instituto Nazionale di Alta Matematica Citta Universitaria - Roma, Italy, June 8-12, 2009.

[5] H.H. Bauschke, J.M. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Rev., **38**(1996), 367-426.

[6] C. Byrne, *A Unified Treatment of Some Iterative Algorithms in Signal Processing and Image Reconstruction*, Marcel Dekker, New York, 1984.

[7] C. Byrne, A. Moudafi, *Extensions of the CQ algorithm for the split feasibility and split equality problems*, J. Nonlinear and Convex A, to appear.

[8] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol., **51**(2006), 2353-2365.

[9] Y. Censor, T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms, **8**(1994), 221-239.

[10] R. Chen, J. Li, Y. Ren, *Regularization method for the approximate split equality problem in infinite-dimensional Hilbert spaces*, Abstr. Appl. Anal., Volume 2013, Article ID 813635, 5 pages.

[11] Q.L. Dong, S. He, *Solving the split equality problem without prior knowledge of operator norms*, Optimization, 65(12) (2016), 2217-2226.

[12] M. Fukushima, *A relaxed projection method for variational inequalities*, Math. Program., **35**(1986), 58-70.

[13] E.M. Gafni, D.P. Bertsekas, *Two-metric projection methods for constrained optimization*, SIAM J. Control Optim., **22**(1984), 936-964.

[14] G. López, V. Martín-Márquez, F. Wang, H.K. Xu, *Solving the split feasibility problem without prior knowledge of matrix norms*, Inverse Probl., **27**(2012), 085004.

[15] A. Moudafi, *Alternating CQ-algorithm for convex feasibility and split fixed-point problems*, J. Nonlinear Convex Anal. 15(4) (2014), 809-818.

[16] A. Moudafi, *A relaxed alternating CQ-algorithm for convex feasibility problems*, Nonlinear Anal., **79**(2013), 117-121.

[17] B. Qu, N. Xiu, *A note on the CQ algorithm for the split feasibility problem*, Inverse Probl., **21**(2005), 1655-1665.

[18] Ph.L. Toint, *Global convergence of a class of trust region methods for nonconvex minimization in Hilbert space*, IMA J. Numer. Anal., **8**(1988), 231-252.

[19] Q. Yang, *The relaxed CQ algorithm for solving the split feasibility problem*, Inverse Probl., **20**(2004), 1261-1266.

[20] J. Zhao, J. Zhang Q. Yang, *A simple projection method for solving the multiple-sets split feasibility problem*, Inverse Probl. Sci. Eng., **21**(2013), no. 3, 537-546.

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