Fixed Point Theory, 18(2017), No. 1, 167-190
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

COMMON SOLUTIONS TO SOME SYSTEMS OF VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce an iterative scheme for approximating a common solution to a new system of unrelated split mixed vector variational inequality problems, multivalued variational inequality problems, and common fixed point problem for a family of nonexpansive mappings in a real Hilbert space. We prove a strong convergence theorem for the sequence generated by the proposed iterative scheme. The results presented in this paper generalize and unify previously known results in this area.

Key Words and Phrases: system of unrelated split mixed vector variational inequality problems, system of unrelated multivalued variational inequality problems, common fixed-point problem, iterative scheme.

2010 Mathematics Subject Classification: 49J30, 47H10, 47H17, 90C99.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$; let K_i and Q_i, where $i \in \{1, 2, 3, ..., N\}$ be nonempty, closed and convex subsets of H_1 and H_2, respectively, with $\cap_{i=1}^n K_i \neq \emptyset$, $\cap_{i=1}^n Q_i \neq \emptyset$. Let Y be a Hausdorff topological vector space, and for each i, let P_i be a pointed, proper, closed and convex cone in Y with $\text{int}P \neq \emptyset$, where $P = \cap_{i=1}^n P_i$. Let $L(H_1, Y)$ be the space of all continuous linear mappings from H_1 to Y.

The *classical scalar nonlinear variational inequality problem* (in short, VIP) is to find $x \in K_1$ such that

$$\langle A_1 x, y - x \rangle \geq 0, \ \forall y \in K_1, \tag{1.1}$$

where $A_1 : K_1 \to H_1$ is a nonlinear mapping.

Variational inequality theory introduced independently by Stampacchia [35] and Fichera [17] in the early sixties in potential theory and mechanics, respectively, constitutes a significant extension of variational principles. It was shown that the variational inequality theory provides a natural, descent, unified and efficient framework for the general treatment of a wide class of unrelated linear and nonlinear problems arising in elasticity, economics, transportation, optimization, control theory and engineering sciences [4, 5, 2, 11, 21, 28]. In the last decades, considerable interest was shown in developing various classes of variational inequality problems, both for its own sake and for its applications.

An important generalization of VIP (1.1) which represents the boundary value problem arising in the formulation of Signorini problem is the following:

find $x \in K_1$ such that

$$\langle A_1x, y - x\rangle + \phi_1(x, y) - \phi_1(x, x) \geq 0,\ \forall y \in K_1, \tag{1.2}$$

where $\phi_1 : H \times H \to \mathbb{R}$ is an appropriate nonlinear form. We call it a mixed variational inequality problem (in short, MVIP). This type of problems was studied in Duvaut and Lions [16] and Kikuchi and Oden [28]. For physical and mathematical formulation of the inequality (1.2), see for example Oden and Pires [32]. For related work, see also Baiocchi and Capelo [2] and Crank [11].

The vector version of MVIP(1.2) is called a mixed vector variational inequality problem (in short, MVVIP), which is to find $x \in K_1$ such that

$$\langle A_1x, y - x\rangle + \phi_1(x, y) - \phi_1(x, x) \in P_1,\ \forall y \in K_1, \tag{1.3}$$

where $A_1 : K_1 \to L(H_1, Y)$ and $\phi_1 : H_1 \times H_1 \to Y$ are nonlinear vector valued mappings.

Vector variational inequality theory, initiated by Giannessi [18], has emerged as a powerful tool for the study of a wide class of vector optimization problems and vector equilibrium problems. Further, vector variational inequality problem (in short, VVIP) provides a unified model of several problems, for example, vector optimization, vector complementarity problem, and vector saddle point problem, see [19, 20, 14].

Recently, Censor *et al.* [8] introduced the following split variational inequality problem (in short, SpVIP): Find $x \in K_1$ such that

$$\langle A_1x, z_1 - x\rangle \geq 0,\ \forall z_1 \in K_1, \tag{1.4}$$

and such that

$$y = Bx \in Q_1 \text{ solves } \langle A_2y, z_2 - y\rangle \geq 0,\ \forall z_2 \in Q_1, \tag{1.5}$$

where $A_1 : H_1 \to H_1$ and $A_2 : H_2 \to H_2$ are nonlinear mappings and $B : H_1 \to H_2$ is a bounded linear operator. They studied some iterative methods for SpVIP(1.4)-(1.5).

SpVIP(1.4)-(1.5) is an important generalization of VIP(1.1). It also includes as a special case, the split zero problem and split feasibility problem which has already been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning; see [7, 10]. For further related work, we refer the reader to Moudafi [30], Byrne *et al.* [6], Kazmi *et al.* [24, 25], Kazmi [26, 27].

In this paper, we introduce and study the following new system of unrelated split mixed vector variational inequality problems (in short, SSpMVVIP):

For each $i = 1, 2, 3, ..., N$, let $A_i : H_1 \to L(H_1, Y)$, $T_i : H_2 \to L(H_2, Y)$, $\phi_i : K_i \times K_i \to Y$, $\psi_i : Q_i \times Q_i \to Y$ be nonlinear mappings and $B : H_1 \to H_2$ be a bounded linear operator. The SSpMVVIP is to find $x^* \in \cap_{i=1}^N K_i$ such that

$$\langle A_i x^*, x_i - x^* \rangle + \phi_i(x_i, x^*) - \phi_i(x^*, x^*) \in P_i, \ \forall x_i \in K_i, \ 1 \le i \le N \tag{1.6}$$

and such that $y^* = Bx^* \in \cap_{i=1}^N Q_i$ solves

$$\langle T_i y^*, y_i - y^* \rangle + \psi_i(y_i, y^*) - \psi_i(y^*, y^*) \in P_i, \ \forall y_i \in Q_i, \ 1 \le i \le N. \tag{1.7}$$

When looked separately, (1.6) is the mixed vector variational inequality problem (in short, MVVIP(A_i, ϕ_i, K_i)), and we denote its solution set by Sol(MVVIP(A_i, ϕ_i, K_i)). The SSpMVVIP(1.6)-(1.7) constitutes a pair of mixed vector variational inequality problems that have to be solved so that the image $y^* = Bx^*$ of the solution x^* of MVVIP(A_i, ϕ_i, K_i) in H_1 under a given bounded linear operator B, is the solution of another MVVIP(T_i, ψ_i, Q_i)(1.7) in another space H_2. We denote the solution set of MVVIP(T_i, ψ_i, Q_i) by Sol(MVVIP(T_i, ψ_i, Q_i)).

When $i = 1$, SSpMVVIP(1.6)-(1.7) reduces to the split mixed vector variational inequality problem (SpMVVIP): Find $x^* \in K_1$ such that

$$\langle A_1 x^*, x_1 - x^* \rangle + \phi_1(x_1, x^*) - \phi_1(x^*, x^*) \in P_1, \ \forall x_1 \in K_1 \tag{1.8}$$

and such that $y^* = Bx^* \in Q_1$ solves

$$\langle T_1 y^*, y_1 - y^* \rangle + \psi_1(y_1, y^*) - \psi_1(y^*, y^*) \in P_1, \ \forall y_1 \in Q_1, \tag{1.9}$$

which appears to be new, and is a natural extension of MVIP(1.2) and MVVIP(1.3).

When $i = 1$ and $\phi_i, \psi_i = 0$, SSpMVVIP(1.6)-(1.7) reduces to the split vector variational inequality problem (SpVVIP): Find $x^* \in K_1$ such that

$$\langle A_1 x^*, x_1 - x^* \rangle \in P_1, \ \forall x_1 \in K_1 \tag{1.10}$$

and such that $y^* = Bx^* \in Q_1$ solves

$$\langle T_1 y^*, y_1 - y^* \rangle \in P_1, \ \forall y_1 \in Q_1, \tag{1.11}$$

which appears to be new, and is a vector version of SpVIP(1.4)-(1.5), and VVIP introduced by Giannessi [18].

For each $i = 1, 2, ..., N$, the solution set of SSpMVVIP(1.6)-(1.7) is denoted by $\Omega_i = \{p \in \text{Sol(MVVIP}(A_i, \phi_i, K_i)) : Bp \in \text{Sol(MVVIP}(T_i, \psi_i, Q_i))\}$. Thus the solution set of SSpMVVIP(1.6)-(1.7) is $\cap_{i=1}^N \Omega_i$.

Further, Censor *et al.* [9] considered and studied some iterative methods for the following system of unrelated multivalued variational inequality problem (in short, SMuVIP): For each $i = 1, 2, 3, ..., N$, let $G_i : H_1 \to 2^{H_1}$ be multivalued mappings, then find $x^* \in \cap_{i=1}^N K_i$ such that, for each $i = 1, 2, 3, ..., N$, there exists $w_i^* \in G_i(x^*)$ such that

$$\langle w_i^*, x_i - x^* \rangle \ge 0, \ \forall x_i \in K_i, \ i = 1, 2, 3, ..., N. \tag{1.12}$$

We denote by Sol(MuVIP(G_i, K_i)) the solution set of the multivalued variational inequality problem (in short, MuVIP(1.12)) corresponding to the mapping G_i and the set K_i. Then the set of solutions of SMuVIP (1.12) is given by $\cap_{i=1}^{N}$ Sol(MuVIP(G_i, K_i)). For related work, see Djafari-Rouhani *et al.* [15].

We also observe that if $A_i = 0, T_i = 0, \phi_i = 0, \psi_i = 0$ for all i then SSpMVVIP(1.6)-(1.7) reduces to the problem of finding a point $x^* \in \cap_{i=1}^{n} K_i$ such that $Bx^* \in \cap_{i=1}^{n} Q_i$ which is the well known split convex feasibilty problem (in short, SpCFP). If the sets K_i, Q_i are the fixed point sets of a family of operators $S_i : H_1 \to H_1$, $R_i : H_2 \to H_2$, respectively, then SpCFP is the split common fixed point problem (in short, SpCFPP). SpCFPP includes as a special case, the common fixed point problem (CFPP).

Recall that a mapping $S_i : K_i \to K_i$ is said to be nonexpansive if $\|S_i x - S_i y\| \leq \|x - y\|$, $\forall x, y \in K_i$. We denote the fixed point set of S_i by Fix(S_i) for each $i = 1, 2, ..., N$. We note that Fix(S_i) is closed and convex, possibly empty.

Motivated by the work of Nadezhkina and Takahashi [31], Censor *et al.* [9] and Djafari-Rouhani *et al.* [15], we introduce an iterative scheme for approximating a common solution to SSpMVVIP(1.6)-(1.7), SMuVIP(1.12) and CFPP for a finite family of nonexpansive mappings in a real Hilbert space. We establish a strong convergence theorem for the sequence generated by the proposed iterative scheme. The results presented in this paper extend and unify previously known results in this area.

2. Preliminaries

We recall some concepts and results needed in the sequel. The symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

In a real Hilbert space H_1, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H_1$ and $\lambda \in [0, 1]$.

It is also known that every Hilbert space satisfies the Opial's condition [33], i.e., for any sequence $\{x^n\}$ with $x^n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x^n - x\| < \liminf_{n\to\infty} \|x^n - y\| \tag{2.2}$$

holds for every $y \in H_1$ with $y \neq x$.

Furthermore, any Hilbert space H_1 has the Kadec-Klee property [22], i.e., if $\{x^n\}$ is a sequence in H such that $x^n \rightharpoonup x$ and $\|x^n\| \to \|x\|$ as $n \to \infty$, then $\|x^n - x\| \to 0$ as $n \to \infty$.

For every point $x \in H_1$, there exists a unique nearest point in K_1 denoted by $P_{K_1}x$ such that

$$\|x - P_{K_1}x\| \leq \|x - y\|, \ \forall y \in K_1. \tag{2.3}$$

The mapping P_{K_1} is called the metric projection of H_1 onto K_1. It is well known that P_{K_1} is nonexpansive and satisfies

$$\langle x-y, P_{K_1}x - P_{K_1}y\rangle \geq \|P_{K_1}x - P_{K_1}y\|^2,\ \forall x, y \in H_1. \tag{2.4}$$

Moreover, $P_{K_1}x$ is characterized by the fact that $P_{K_1}x \in K_1$ and

$$\langle x - P_{K_1}x, y - P_{K_1}x\rangle \leq 0,\ \forall y \in K_1. \tag{2.5}$$

This implies that:

$$\|x-y\|^2 \geq \|x - P_{K_1}x\|^2 + \|y - P_{K_1}x\|^2,\ \forall x \in H_1,\ \forall y \in K_1. \tag{2.6}$$

Further, it is well known that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leq \frac{1}{2}\|(T(x) - x) - (T(y) - y)\|^2 \tag{2.7}$$

and therefore, we get, for all $(x, y) \in H_1 \times \mathrm{Fix}(T)$,

$$\langle x - T(x), y - T(x)\rangle \leq \frac{1}{2}\|T(x) - x\|^2 \tag{2.8}$$

see, e.g. [[12], Theorem 3.1] and [[13], Theorem 2.1].

Definition 2.1. A mapping $A : H_1 \to H_1$ is said to be

(i) *monotone, if*

$$\langle Ax - Ay, x - y\rangle \geq 0,\ \forall x, y \in H_1;$$

(ii) *maximal monotone if it is monotone and its graph, denoted by graph(A), is not properly contained in the graph of any other monotone mapping.*

(iii) *firmly nonexpansive, if*

$$\langle Ax - Ay, x - y\rangle \geq \|Ax - Ay\|^2,\ \forall x, y \in H_1;$$

(iv) *β-Lipschitz continuous, if there exists a constant $\beta > 0$ such that*

$$\|Ax - Ay\| \leq \beta\|x - y\|,\ \forall x, y \in H_1.$$

Remark 2.1. The mapping $A : H_1 \to 2^{H_1}$ is maximal monotone if and only if A is monotone and we have:

$$\langle u - v, x - y\rangle \geq 0, \forall (y, v) \in \mathrm{graph}(A) \ \ \Rightarrow\ \ u \in Ax.$$

Definition 2.2. [36, 29] Let X and Y be two Hausdorff topological vector spaces and let D be a nonempty, convex subset of X and C be a pointed, proper, closed and convex cone of Y with $\mathrm{int}C \neq \emptyset$. Let 0 be the zero point of Y, $\mathbb{U}(0)$ be a base of neighborhoods of 0, $\mathbb{U}(x_0)$ be a base of neighborhoods of x_0 and $f : D \to Y$ be a mapping.

(i) *If, for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that*

$$f(x) \in f(x_0) + V + C,\ \forall x \in U \cap D,$$

then f is called upper C-continuous at x_0. If f is upper C-continuous for all $x \in D$, then f is called upper C-continuous on D;

(ii) *If, for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that*

$$f(x) \in f(x_0) + V - C, \ \forall x \in U \cap D,$$

then f is called lower C-continuous at x_0. If f is lower C-continuous for all $x \in D$, then f is called lower C-continuous on D;

(iii) *If, for any $x, y \in D$ and $t \in [0,1]$, the mapping f satisfies*

$$f(x) \in f(tx + (1-t)y) + C \text{ or } f(y) \in f(tx + (1-t)y) + C,$$

then f is called proper C-quasiconvex;

(iv) *If, for any $x, y \in D$ and $t \in [0,1]$, the mapping f satisfies*

$$tf(x) + (1-t)f(y) \in f(tx + (1-t)y) + C,$$

then f is called C-convex.

Lemma 2.1. [23] *Let X and Y be two real Hausdorff topological vector spaces, D be a nonempty, compact and convex subset of X, and C be a pointed, proper, closed and convex cone of Y with $\text{int}C \neq \emptyset$. Assume that $g : D \times D \to Y$ and $\Phi : D \to Y$ are two nonlinear mappings. Suppose that g and Φ satisfy*

(i) $g(x, x) \in C, \ \forall x \in D$;
(ii) Φ is *upper C-continuous* on D;
(iii) $g(., y)$ is *lower C-continuous*, $\forall y \in D$;
(iv) $g(x, .) + \Phi(.)$ is *proper C-quasiconvex*, $\forall x \in D$.

Then there exists a point $x \in D$ satisfying

$$G(x, y) \in C \setminus \{0\}, \ \forall y \in D,$$

where

$$G(x, y) = g(x, y) + \Phi(y) - \Phi(x), \ \forall x, y \in D.$$

Now, we recall two definitions; see Definitions 2.5 and 2.6 in Censor *et al.* [9]. Let K be a nonempty, closed and convex subset of H_1. Let $CB(K)$ denote the family of all nonempty, closed, convex and bounded subsets of K.

Definition 2.3. (Hausdorff Metric) Let $K_1, K_2 \in CB(K)$. The Hausdorff metric on $CB(K)$ is defined by

$$H(K_1, K_2) := max\{sup_{x \in K_2} d(x, K_1), sup_{y \in K_1} d(y, K_2)\}, \tag{2.9}$$

where the distance function is defined by $d(x, K) := inf\{||x - z||; z \in K\}$.

Definition 2.4. (Nonexpansive Mappings) Let $A : H_1 \to 2^{H_1}$ be a mapping such that $A(x) \in CB(H_1), \forall x \in H_1$. We say that

(i) *A is Lipschitz continuous with constant $L > 0$ if*

$$H(A(x), A(y)) \leq L||x - y||, \forall x, y \in H_1. \tag{2.10}$$

Therefore, given $x \in H_1$, $u \in A(x)$, and $y \in H_1$, there exists $v \in A(y)$ such that $||u - v|| \leq L||x - y||$.

(ii) *A is nonexpansive if it is Lipschitz continuous with $L = 1$.*

3. Mixed vector variational inequality problems

Let K be a nonempty, compact, and convex subset of a real Hilbert space H; let Y be a Hausdorff topological vector space, and let C be a proper, closed and convex cone of Y with $\mathrm{int}C \neq \emptyset$. Let $A : K \to L(H, Y)$ and $\psi : K \times K \to Y$ be two nonlinear mappings.

Consider the mixed vector variational inequality problem (MVVIP) of finding $x \in K$ such that

$$\langle Ax, y - x\rangle + \psi(y, x) - \psi(x, x) \in C, \ \forall y \in K. \tag{3.1}$$

Further, for any $z \in H$, we define a mapping $F_z : K \times K \to Y$ as follows:

$$F_z(x, y) = \langle Ax, y - x\rangle + \psi(y, x) - \psi(x, x) + \frac{e}{r}\langle y - x, x - z\rangle \tag{3.2}$$

where r is a positive real number and $e \in \mathrm{int}C$.

Then the auxiliary problem for MVVIP(3.1) is to find, for each $z \in H$, an $x \in K$ such that

$$F_z(x, y) \in C, \ \forall y \in K. \tag{3.3}$$

Assumption 3.1. Let F_z, A, and ψ satisfy the following conditions:

(1) *A is a weakly continuous and C-monotone mapping*, i.e.

$$\langle x - y, Ax - Ay\rangle \in C, \ \forall x, y \in K;$$

(2) *$F_z(., y)$ is lower C-continuous*, $\forall y \in K$ and $z \in H$;

(3) *$\psi(., .)$ is weakly continuous and $\psi(., y)$ is C-convex*, i.e.,

$$t\psi(x_1, y) + (1 - t)\psi(x_2, y) \in \psi(tx_1 + (1 - t)x_2, y) + C, \ \forall x_1, x_2 \in K, \ \forall t \in [0, 1];$$

(4) *$F_z(x, .)$ is proper C-quasiconvex, $\forall x \in K$ and $z \in H$*;

(5) *ψ is C-skew symmetric*, i.e.,

$$\psi(x, x) - \psi(x, y) - \psi(y, x) + \psi(y, y) \in C, \ \forall x, y \in K.$$

Remark 3.1. C-skew-symmetric bimappings are natural extensions of skew-symmetric bifunctions. The skew-symmetric bifunctions have the properties that can be considered analogous to the monotonicity of the gradient and the non-negativity of the second derivative for convex functions. For the properties and applications of the skew-symmetric bifunctions, we refer the reader to [1].

We now discuss the properties of the mapping $T_r^{(A,\psi)}$ defined below, which also show the existence and uniqueness of solutions to MVVIP(3.1).

Theorem 3.1. *Let H be a real Hilbert space; let K be a nonempty, compact, and convex subset of H; let Y be a Hausdorff topological vector space and let C be a proper, closed and convex cone of Y with* $\mathrm{int}C \neq \emptyset$. *Let $A : K \to L(H, Y)$ be a continuous and C-monotone mapping, where $L(H, Y)$ is the set of all bounded linear operators* *from* H *to* Y*; let* $\psi : K \times K \to Y$ *be a nonlinear mapping satisfying Assumption* 3.1*, along with* F_z *defined by* (3.2)*. Let* $T_r^{(A,\psi)} : H \to K$*, where* $r > 0$*, be defined as*

$$T_r^{(A,\psi)}(z) = \{x \in K : \langle Ax, y - x\rangle + \psi(y, x) - \psi(x, x) + \frac{e}{r}\langle y - x, x - z\rangle \in C,\ \forall y \in K\}.$$

Then the following hold:

(i) $T_r^{(A,\psi)}(z)$ *is nonempty for each* $z \in H$;
(ii) $T_r^{(A,\psi)}$ *is single-valued*;
(iii) $T_r^{(A,\psi)}$ *is firmly nonexpansive*;
(iv) Fix($T_r^{(A,\psi)}$)= Sol(MVVIP(3.1));
(v) Sol(MVVIP(3.1)) *is closed and convex.*

Proof. (i) For each $z \in H$, let $g(x, y) = F_z(x, y)$, where F_z is defined by (3.2), and let $\Phi(x) = 0$, for all $x \in K$. Then it is easy to see that $g(x, y)$ and $\Phi(x)$ satisfy all the conditions of Lemma 2.1. Hence there exists $x \in K$ such that

$$g(x, y) + \Phi(y) - \Phi(x) \in C,\ \forall y \in K,$$

and thus $T_r^{(A,\psi)}(z) \neq \emptyset$, for each $z \in H$.

(ii) Since, for each $z \in H$, $T_r^{(A,\psi)}(z) \neq \emptyset$, then if $x_1, x_2 \in T_r^{(A,\psi)}(z)$, we have:

$$\langle Ax_1, y - x_1\rangle + \psi(y, x_1) - \psi(x_1, x_1) + \frac{e}{r}\langle y - x_1, x_1 - z\rangle \in C,\ \forall y \in K \tag{3.4}$$

and

$$\langle Ax_2, y - x_2\rangle + \psi(y, x_2) - \psi(x_2, x_2) + \frac{e}{r}\langle y - x_2, x_2 - z\rangle \in C,\ \forall y \in K. \tag{3.5}$$

Letting $y = x_2$ in (3.4) and $y = x_1$ in (3.5), then adding, we get:

$$\langle Ax_1, x_2 - x_1\rangle + \langle Ax_2, x_1 - x_2\rangle + \psi(x_2, x_1) - \psi(x_1, x_1)$$
$$+\psi(x_1, x_2) - \psi(x_2, x_2) + \frac{e}{r}\langle x_2 - x_1, x_1 - x_2\rangle \in C$$
$$\langle Ax_1 - Ax_2, x_2 - x_1\rangle + \psi(x_2, x_1) - \psi(x_1, x_1)$$
$$+\psi(x_1, x_2) - \psi(x_2, x_2) + \frac{e}{r}\langle x_2 - x_1, x_1 - x_2\rangle \in C. \tag{3.6}$$

Since A is C-monotone, we have

$$\langle Ax_1 - Ax_2, x_2 - x_1\rangle \in -C. \tag{3.7}$$

Since ψ is C-skew symmetric

$$-\psi(x_1, x_1) + \psi(x_1, x_2) + \psi(x_2, x_1) - \psi(x_2, x_2) \in -C. \tag{3.8}$$

Using (3.7) and (3.8) in (3.6), we get:

$$\frac{e}{r}\langle x_2 - x_1, x_1 - x_2\rangle \in C.$$

Since $e \in \text{int}C$, we have $\frac{1}{r}\langle x_2 - x_1, x_1 - x_2\rangle \geq 0$ which implies that $x_1 = x_2$. Thus $T_r^{(A,\psi)}$ is single-valued.

(iii) For any $z_1, z_2 \in H$, let $x_1 = T_r^{(A,\psi)}(z_1)$ and $x_2 = T_r^{(A,\psi)}(z_2)$. Then

$$\langle Ax_1, y - x_1\rangle + \psi(y, x_1) - \psi(x_1, x_1) + \frac{e}{r}\langle y - x_1, x_1 - z_1\rangle \in C, \ \forall y \in K, \tag{3.9}$$

and

$$\langle Ax_2, y - x_2\rangle + \psi(y, x_2) - \psi(x_2, x_2) + \frac{e}{r}\langle y - x_2, x_2 - z_2\rangle \in C, \ \forall y \in K. \tag{3.10}$$

Letting $y = x_2$ in (3.9) and $y = x_1$ in (3.10), then adding, we get:

$$\begin{aligned}&\langle Ax_1 - Ax_2, x_2 - x_1\rangle + \psi(x_2, x_1) - \psi(x_1, x_1)\\ &+ \psi(x_1, x_2) - \psi(x_2, x_2) + \frac{e}{r}\langle x_2 - x_1, x_1 - x_2 - z_1 + z_2\rangle \in C.\end{aligned}$$

By using the C-monotonocity of A, C-skew symmetry of ψ, and the property of C, we have

$$\frac{e}{r}\langle x_2 - x_1, x_1 - x_2 - z_1 + z_2\rangle \in C.$$

Since $e \in \mathrm{int}C$, we have:

$$\begin{gathered}\frac{1}{r}\langle x_2 - x_1, x_1 - x_2 - z_1 + z_2\rangle \geq 0\\ \langle x_2 - x_1, x_1 - x_2\rangle + \langle x_2 - x_1, z_2 - z_1\rangle \geq 0\\ \langle x_2 - x_1, x_1 - x_2\rangle \geq -\langle x_2 - x_1, z_2 - z_1\rangle\\ \langle x_1 - x_2, x_1 - x_2\rangle \leq \langle x_1 - x_2, z_1 - z_2\rangle\end{gathered}$$

$$\langle T_r^{(A,\psi)}(z_1) - T_r^{(A,\psi)}(z_2), T_r^{(A,\psi)}(z_1) - T_r^{(A,\psi)}(z_2)\rangle \leq \langle T_r^{(A,\psi)}(z_1) - T_r^{(A,\psi)}(z_2), z_1 - z_2\rangle.$$

Thus $T_r^{(A,\psi)}$ is firmly nonexpansive.

(iv) Assume $x = T_r^{(A,\psi)}(x)$. Then

$$\langle Ax, y - x\rangle + \psi(y, x) - \psi(x, x) + \frac{e}{r}\langle y - x, x - x\rangle \in C, \ \forall y \in K,$$

and so

$$\langle Ax, y - x\rangle + \psi(y, x) - \psi(x, x) \in C, \ \forall y \in K.$$

Thus $x \in \mathrm{Sol(MVVIP(3.1))}$.

Conversely, let $x \in \mathrm{Sol(MVVIP(3.1))}$. Then

$$\langle Ax, y - x\rangle + \psi(y, x) - \psi(x, x) \in C, \ \forall y \in K,$$

and so

$$\langle Ax, y - x\rangle + \psi(y, x) - \psi(x, x) + \frac{e}{r}\langle y - x, x - x\rangle \in C, \ \forall y \in K.$$

Thus $x = T_r^{(A,\psi)}(x)$. Hence $\mathrm{Fix}(T_r^{(A,\psi)}) = \mathrm{Sol(MVVIP(3.1))}$.

(v) Since $T_r^{(A,\psi)}$ is firmly nonexpansive, $T_r^{(A,\psi)}$ is also nonexpansive, and hence $\mathrm{Sol(MVVIP(3.1))} = \mathrm{Fix}(T_r^{(A,\psi)})$ is closed and convex; see [22].

Next, we prove the following lemma which is used to prove our main result.

Lemma 3.1. *Let* A, ψ, and F_z *satisfy Assumption* 3.1 *and let* $T_r^{(A,\psi)}$ *be defined as in Theorem* 3.1. *Let* $z_1, z_2 \in H$ *and* $r_1, r_2 > 0$. *Then*:

$$\|T_{r_2}^{(A,\psi)}(z_2) - T_{r_1}^{(A,\psi)}(z_1)\| \leq \|z_2 - z_1\| + \frac{|r_2 - r_1|}{r_2}\|T_{r_2}^{(A,\psi)}(z_2) - z_2\|.$$

Proof. For any $z_1, z_2 \in H$ and $r_1, r_2 > 0$, let $x_1 = T_r^{(A,\psi)}(z_1)$ and $x_2 = T_r^{(A,\psi)}(z_2)$. Then we have:

$$\langle Ax_1, y - x_1\rangle + \psi(y, x_1) - \psi(x_1, x_1) + \frac{e}{r_1}\langle y - x_1, x_1 - z_1\rangle \in C, \ \forall y \in K, \tag{3.11}$$

and

$$\langle Ax_2, y - x_2\rangle + \psi(y, x_2) - \psi(x_2, x_2) + \frac{e}{r_2}\langle y - x_2, x_2 - z_2\rangle \in C, \ \forall y \in K. \tag{3.12}$$

Letting $y = x_2$ in (3.11) and $y = x_1$ in (3.12), then adding, we get:

$$\langle Ax_1 - Ax_2, x_2 - x_1\rangle + \psi(x_2, x_1) - \psi(x_1, x_1) + \psi(x_1, x_2) - \psi(x_2, x_2) + \frac{e}{r_1}\langle x_2 - x_1, x_1 - z_1\rangle$$
$$+\frac{e}{r_2}\langle x_1 - x_2, x_2 - z_2\rangle \in C$$

$$\langle Ax_1 - Ax_2, x_2 - x_1\rangle + \psi(x_2, x_1) - \psi(x_1, x_1) + \psi(x_1, x_2) - \psi(x_2, x_2) + e\langle x_2 - x_1, \frac{x_1 - z_1}{r_1}\rangle$$
$$+e\langle x_1 - x_2, \frac{x_2 - z_2}{r_2}\rangle \in C$$

$$e\langle x_2 - x_1, \frac{x_1 - z_1}{r_1} - \frac{x_2 - z_2}{r_2}\rangle \in P + \langle Ax_1 - Ax_2, x_1 - x_2\rangle$$
$$+\psi(x_1, x_1) - \psi(x_2, x_1) - \psi(x_1, x_2) + \psi(x_2, x_2).$$

Using the C-monotonocity of A and the C-skew symmetry of ψ, we have

$$e\langle x_2 - x_1, \frac{x_1 - z_1}{r_1} - \frac{x_2 - z_2}{r_2}\rangle \in C.$$

Since $e \in \text{int}C$, we have

$$\langle x_2 - x_1, x_1 - z_1 - \frac{r_1}{r_2}(x_2 - z_2)\rangle \geq 0,$$

which implies that

$$\langle x_2 - x_1, x_1 - x_2 + x_2 - z_1 - \frac{r_1}{r_2}(x_2 - z_2)\rangle \geq 0,$$

and so

$$\|x_2 - x_1\|^2 \leq \langle x_2 - x_1, x_2 - z_2 + z_2 - z_1 - \frac{r_1}{r_2}(x_2 - z_2)\rangle$$

$$\|x_2 - x_1\|^2 \leq \langle x_2 - x_1, z_2 - z_1 + (1 - \frac{r_1}{r_2})(x_2 - z_2)\rangle$$

$$\|x_2 - x_1\|^2 \leq \|x_2 - x_1\|[\|z_2 - z_1\| + \frac{|r_2 - r_1|}{r_2}\|x_2 - z_2\|].$$

This completes the proof.

4. Main Result

We establish a strong convergence theorem for approximating a common solution to SSpMVVIP(1.6)-(1.7), SMuVIP(1.12) and CFPP for a finite family of nonexpansive mappings.

Assume that $A_i : K_i \to L(H_1, Y)$, $\phi_i : K_i \times K_i \to Y$, and for any $r > 0$ and $z \in H_1$, $F_{i,z} : K_i \times K_i \to Y$ defined by

$$F_{i,z}(x_i, y_i) = \langle A_i x_i, y_i - x_i \rangle + \phi_i(y_i, x_i) - \phi_i(x_i, x_i) + \frac{e}{r}\langle y_i - x_i, x_i - z \rangle,$$

and $T_i : Q_i \to L(H_2, Y)$, $\psi_i : Q_i \times Q_i \to Y$, and for any $s > 0$ and $w \in H_2$, $G_{i,w} : Q_i \times Q_i \to Y$, defined by

$$G_{i,w}(u_i, v_i) = \langle T_i u_i, v_i - u_i \rangle + \psi_i(v_i, u_i) - \psi_i(u_i, u_i) + \frac{e}{s}\langle v_i - u_i, u_i - w \rangle,$$

satisfy Assumption 3.1.

For $r > 0$ and for all $z \in H_1$, define a mapping $T_r^{(A_i,\phi_i)} : H_1 \to K_i$ as follows:

$$T_r^{(A_i,\phi_i)}(z) = \{x_i \in K_i : F_{i,z}(x_i, y_i) \in P, \ \forall y_i \in K_i\}.$$

Then, it follows from Theorem 3.1 that $T_r^{(A_i,\phi_i)}(z) \neq \emptyset$ for each $z \in H_1$; $T_r^{(A_i,\phi_i)}$ is single-valued and firmly nonexpansive; Fix$(T_r^{(A_i,\phi_i)})$ =Sol(MVVIP(1.6)) which is closed and convex.

Further, for $s > 0$ and for all $w \in H_2$, define a mapping $T_s^{(T_i,\psi_i)} : H_2 \to Q_i$ as follows:

$$T_s^{(T_i,\psi_i)}(w) = \{u_i \in Q_i : G_{i,w}(u_i, v_i) \in P, \ \forall v_i \in Q_i\}.$$

Again, it follows from Theorem 3.1 that $T_s^{(T_i,\psi_i)}(z) \neq \emptyset$ for each $w \in H_2$; $T_s^{(T_i,\psi_i)}$ is single-valued and firmly nonexpansive; Fix$(T_s^{(T_i,\psi_i)})$ =Sol(MVVIP(4.1)) which is closed and convex, where Sol(MVVIP(4.1)) is the solution set of the following problem: Find $u_i \in Q_i$ such that

$$\langle T_i u_i, v_i - u_i \rangle + \psi_i(v_i, u_i) - \psi_i(u_i, u_i) \in P_i, \ \forall v_i \in Q_i. \tag{4.1}$$

We denote by I, the identity operator on H_1 as well as on H_2.

We easily observe that for each i, Ω_i is a closed and convex subset of H_1. Further, we note that for each $i = 1, 2, ...N$, Sol(MuVIP(G_i, K_i)) is a closed and convex subset of H_1 (see Lemma 2.4(ii) in [3]). Hence $(\cap_{i=1}^N \Omega_i) \cap (\cap_{i=1}^N(\text{Sol(MuVIP}(G_i, K_i))))$ is closed and convex. Assume that $(\cap_{i=1}^N \Omega_i) \cap (\cap_{i=1}^N(\text{Sol(MuVIP}(G_i, K_i)))) \neq \emptyset$.

Theorem 4.1. *Let H_1 and H_2 be two real Hilbert spaces; let $K_i \subseteq H_1$ and $Q_i \subseteq H_2$, where $i \in \{1, 2, 3, ..., N\}$ be nonempty, compact and convex subsets with $\cap_{i=1}^N K_i \neq \emptyset$, and $\cap_{i=1}^N Q_i \neq \emptyset$; let Y be a Hausdorff topological vector space, and for each i, let P_i be a proper, closed and convex cone of Y with* int$P \neq \emptyset$*, where $P = \cap_{i=1}^N P_i$. Let $B : H_1 \to H_2$ be a bounded linear operator; let $G_i : H_1 \to 2^{H_1}$ be a monotone and Lipschitz continuous mapping with constant β_i such that $G_i(x) \in CB(H_1), \forall x \in$*

H_1; let $A_i : K_i \to L(H_1, Y)$, $T_i : Q_i \to L(H_2, Y)$, $\phi_i : K_i \times K_i \to Y$, and $\psi_i : Q_i \times Q_i \to Y$ be nonlinear mappings satisfying the Assumption 3.1. *For each fixed i, let $S_i : K_i \to H_1$ be a nonexpansive mapping. Assume that $\Gamma := (\cap_{i=1}^N \Omega_i) \cap (\cap_{i=1}^N(\mathrm{Sol}(\mathrm{MuVIP}(G_i, K_i)))) \cap (\cap_{i=1}^N \mathrm{Fix}(S_i)) \neq \emptyset$.*

For a given $x^0 = x \in \cap_{i=1}^N K_i$, let the iterative sequence $\{x^n\}$ be generated by the following iterative scheme:

$$u_i^n = T_{r_i^n}^{(A_i,\phi_i)}(x^n + \delta B^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n) \tag{4.2}$$

$$y_i^n = P_{K_i}(u_i^n - \lambda_i^n w_i^n),\ w_i^n \in G_i(u_i^n) \tag{4.3}$$

find $v_i^n \in G_i(y_i^n)$ which satisfies Definition 2.4(i) *with w_i^n,*

$$z_i^n = \alpha_i^n x^n + (1 - \alpha_i^n) S_i P_{K_i}(u_i^n - \lambda_i^n v_i^n), \tag{4.4}$$

$$C_i^n = \{z \in H_1 : \|z_i^n - z\|^2 \leq \|x^n - z\|^2\}, \tag{4.5}$$

$$C^n = \cap_{i=1}^N C_i^n, \tag{4.6}$$

$$Q^n = \{z \in H_1 : \langle x^n - z, x - x^n\rangle \geq 0\}, \tag{4.7}$$

$$x^{n+1} = P_{C^n \cap Q^n} x, \tag{4.8}$$

for $n = 1, 2, ...,$ and for each $i = 1, 2, ..., N$. $\delta \in (0, 1/L)$, where L is the spectral radius of the operator B^*B and B^* is the adjoint of B. $\{\alpha_i^n\} \subseteq [0, c]$, for some $c \in [0, 1)$ and $\{r_i^n\} \subseteq [a, b]$ for some $a, b \in (0, \alpha^{-1})$, where $\alpha := \max_{1\leq i\leq N} \beta_i$ and $\{\lambda_i^n\} \subseteq [a, b]$. Then, the sequence $\{x^n\}$ converges strongly to $z = P_\Gamma x$.

Proof. We divide the proof into four claims.

Claim 4.1. The projection $P_\Gamma(x)$ and $\{x^n\}$ are well defined. Further, the sequences $\{x^n\}$, $\{u_i^n\}$, $\{t_i^n\}$ and $\{z_i^n\}$ are bounded, where $\{t_i^n\} := P_{K_i}(u_i^n - \lambda_i^n v_i^n)$.

Proof. Evidently, $P_\Gamma(x)$ is well defined, since Γ is a nonempty, closed and convex subset of H_1. Now we show that $\{x^n\}$ is well defined. Indeed, let $\hat{x} \in \Gamma$; then $\hat{x} \in \Omega_i$, and hence $\hat{x} = T_{r_i^n}^{(A_i,\phi_i)}\hat{x}$ and $B\hat{x} = T_{r_i^n}^{(T_i,\psi_i)}(B\hat{x})$. We estimate

$$\begin{aligned} \|u_i^n - \hat{x}\|^2 &= \|T_{r_i^n}^{(A_i,\phi_i)}(x^n + \delta B^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n) - \hat{x}\|^2 \\ &= \|T_{r_i^n}^{(A_i,\phi_i)}(x^n + \delta B^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n) - T_{r_i^n}^{(A_i,\phi_i)}\hat{x}\|^2 \\ &\leq \|x^n + \delta B^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n - \hat{x}\|^2 \\ &= \|x^n - \hat{x}\|^2 + \delta^2\|B^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\|^2 \\ &\quad +2\delta\langle x^n - \hat{x}, B^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\rangle. \end{aligned} \tag{4.9}$$

Thus, we have

$$\begin{aligned} \|u_i^n - \hat{x}\|^2 \leq\ & \|x^n - \hat{x}\|^2 + \delta^2\langle (T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n, BB^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\rangle \\ & +2\delta\langle x^n - \hat{x}, B^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\rangle. \end{aligned} \tag{4.10}$$

Now, we have

$$\begin{aligned}\delta^2\langle (T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n \quad , \quad & BB^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\rangle \\ \leq \quad & L\delta^2\langle (T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n, (T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\rangle \\ = \quad & L\delta^2\|(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\|^2.\end{aligned} \tag{4.11}$$

Denoting $\Lambda = 2\delta\langle x^n - \hat{x}, B^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\rangle$ and using (2.8), we have

$$\begin{aligned}\Lambda &= 2\delta\langle B(x^n - \hat{x}), (T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\rangle \\ &= 2\delta\langle B(x^n - \hat{x}) + (T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n - (T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n, (T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\rangle \\ &= 2\delta\{\langle T_{r_i^n}^{(T_i,\psi_i)}Bx^n - B\hat{x}, (T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\rangle - \|(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\|^2\} \\ &\leq 2\delta\{\tfrac{1}{2}\|(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\|^2 - \|(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\|^2\} \\ &= -\delta\|(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\|^2.\end{aligned} \tag{4.12}$$

Using (4.11) and (4.12) in (4.10), we have

$$\|u_i^n - \hat{x}\|^2 \leq \|x^n - \hat{x}\|^2 + \delta(L\delta - 1)\|(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n\|^2. \tag{4.13}$$

Since $\delta \in (0, 1/L)$, we obtain

$$\|u_i^n - \hat{x}\|^2 \leq \|x^n - \hat{x}\|^2. \tag{4.14}$$

Again, since $\hat{x} \in \Gamma$, then there is $w_i^* \in G_i(\hat{x})$ such that w_i^* satisfies (1.12). Setting $t_i^n = P_{K_i}(u_i^n - \lambda_i^n v_i^n)$ and applying (2.6) with $u_i^n - \lambda_i^n v_i^n$ and $\hat{x}$, we get

$$\begin{aligned}\|t_i^n - \hat{x}\|^2 &= \|P_{K_i}(u_i^n - \lambda_i^n v_i^n) - \hat{x}\|^2 \\ &\leq \|u_i^n - \lambda_i^n v_i^n - \hat{x}\|^2 - \|u_i^n - \lambda_i^n v_i^n - t_i^n\|^2 \\ &= \|u_i^n - \hat{x}\|^2 - \|u_i^n - t_i^n\|^2 + 2\lambda_i^n\langle v_i^n, \hat{x} - t_i^n\rangle \\ &= \|u_i^n - \hat{x}\|^2 - \|u_i^n - t_i^n\|^2 + 2\lambda_i^n[\langle v_i^n - w_i^*, \hat{x} - y_i^n\rangle \\ &\quad + \langle w_i^*, \hat{x} - y_i^n\rangle + \langle v_i^n, y_i^n - t_i^n\rangle] \\ &\leq \|u_i^n - \hat{x}\|^2 - \|u_i^n - t_i^n\|^2 + 2\lambda_i^n\langle v_i^n, y_i^n - t_i^n\rangle \\ &= \|u_i^n - \hat{x}\|^2 - \|u_i^n - y_i^n\|^2 - 2\langle u_i^n - y_i^n, y_i^n - t_i^n\rangle \\ &\quad - \|y_i^n - t_i^n\|^2 + 2\lambda_i^n\langle v_i^n, y_i^n - t_i^n\rangle \\ &= \|u_i^n - \hat{x}\|^2 - \|u_i^n - y_i^n\|^2 - \|y_i^n - t_i^n\|^2 \\ &\quad + 2\langle u_i^n - \lambda_i^n v_i^n - y_i^n, t_i^n - y_i^n\rangle.\end{aligned} \tag{4.15}$$

By (2.5), we have

$$\begin{aligned}\langle u_i^n - \lambda_i^n v_i^n - y_i^n, t_i^n - y_i^n\rangle &= \langle u_i^n - \lambda_i^n w_i^n - y_i^n, t_i^n - y_i^n\rangle + \lambda_i^n\langle w_i^n - v_i^n, t_i^n - y_i^n\rangle \\ &\leq \lambda_i^n\langle w_i^n - v_i^n, t_i^n - y_i^n\rangle.\end{aligned}$$

By the Cauchy-Schwartz inequality, it follows that

$$\langle u_i^n - \lambda_i^n v_i^n - y_i^n, t_i^n - y_i^n\rangle \leq \lambda_i^n\|w_i^n - v_i^n\|\|t_i^n - y_i^n\|.$$

Each mapping G_i, $i = 1, 2, 3, ..., N$, is Lipschitz continuous with constant β_i. Therefore G_i is obviously Lipschitz continuous with constant α. Using this fact in the above inequality, we have

$$\langle u_i^n - \lambda_i^n v_i^n - y_i^n, t_i^n - y_i^n\rangle \leq \lambda_i^n\alpha\|u_i^n - y_i^n\|\|t_i^n - y_i^n\|.$$

Using it in (4.15), we have

$$\|t_i^n - \hat{x}\|^2 \leq \|u_i^n - \hat{x}\|^2 - \|u_i^n - y_i^n\|^2 - \|y_i^n - t_i^n\|^2 + 2\lambda_i^n \alpha \|u_i^n - y_i^n\| \|t_i^n - y_i^n\|. \quad (4.16)$$

Since

$$\begin{aligned} 0 &\leq (\lambda_i^n \alpha \|u_i^n - y_i^n\| - \|t_i^n - y_i^n\|)^2 \\ &= (\lambda_i^n \alpha)^2 \|u_i^n - y_i^n\|^2 - 2\lambda_i^n \alpha \|u_i^n - y_i^n\| \|t_i^n - y_i^n\| + \|t_i^n - y_i^n\|^2, \end{aligned}$$

we obtain that

$$2\lambda_i^n \alpha \|u_i^n - y_i^n\| \|t_i^n - y_i^n\| \leq (\lambda_i^n \alpha)^2 \|u_i^n - y_i^n\|^2 + \|t_i^n - y_i^n\|^2.$$

Thus from (4.16), we have

$$\begin{aligned} \|t_i^n - \hat{x}\|^2 &\leq \|u_i^n - \hat{x}\|^2 - \|u_i^n - y_i^n\|^2 - \|y_i^n - t_i^n\|^2 \\ &\quad + (\lambda_i^n \alpha)^2 \|u_i^n - y_i^n\|^2 + \|t_i^n - y_i^n\|^2 \\ &\leq \|u_i^n - \hat{x}\|^2 - \|u_i^n - y_i^n\|^2 + (\lambda_i^n \alpha)^2 \|u_i^n - y_i^n\|^2 \\ &= \|u_i^n - \hat{x}\|^2 - (1 - (\lambda_i^n \alpha)^2) \|u_i^n - y_i^n\|^2 \\ &\leq \|u_i^n - \hat{x}\|^2. \end{aligned} \quad (4.17)$$

Using (4.14), we also have

$$\|t_i^n - \hat{x}\|^2 \leq \|x^n - \hat{x}\|^2 \quad (4.18)$$

Since $\hat{x} \in \Gamma$, then for each fixed i, $\hat{x} = S_i \hat{x}$. Next, using (2.1) and (4.18), we get the following estimates:

$$\begin{aligned} \|z_i^n - \hat{x}\|^2 &= \|\alpha_i^n x^n + (1 - \alpha_i^n) S_i t_i^n - \hat{x}\|^2 \\ &= \|\alpha_i^n (x^n - \hat{x}) + (1 - \alpha_i^n)(S_i t_i^n - \hat{x})\|^2 \\ &= \alpha_i^n \|x^n - \hat{x}\|^2 + (1 - \alpha_i^n) \|S_i t_i^n - \hat{x}\|^2 - \alpha_i^n (1 - \alpha_i^n) \|S_i t_i^n - \hat{x}\|^2 \\ &\leq \alpha_i^n \|x^n - \hat{x}\|^2 + (1 - \alpha_i^n) \|S_i t_i^n - \hat{x}\|^2 \\ &\leq \alpha_i^n \|x^n - \hat{x}\|^2 + (1 - \alpha_i^n) \|t_i^n - \hat{x}\|^2 \\ &\leq \alpha_i^n \|x^n - \hat{x}\|^2 + (1 - \alpha_i^n) \|x^n - \hat{x}\|^2 \\ &= \|x^n - \hat{x}\|^2. \end{aligned} \quad (4.19)$$

Therefore $\hat{x} \in C_i^n$ for each fixed i, and hence $\hat{x} \in C^n$. Thus $\Gamma \subseteq C^n$, for every $n = 0, 1, 2, ...$ Further, since $\Gamma \subseteq C^0$ and $\Gamma \subseteq Q^0 = H_1$, it follows that $\Gamma \subseteq C^0 \cap Q^0$, and hence $C^0 \cap Q^0$ ia a nonempty, closed, and convex set. Thus $x^1 = P_{C^0 \cap Q^0} x$ is well defined. Now, suppose that $\Gamma \subseteq C^{n-1} \cap Q^{n-1}$ for some $n > 1$. Let $\hat{x} \in \Gamma$; it follows from (2.5) that

$$\langle x - x^n, x^n - \hat{x} \rangle = \langle x - P_{C^{n-1} \cap Q^{n-1}} x, P_{C^{n-1} \cap Q^{n-1}} x - \hat{x} \geq 0,$$

we conclude that $\hat{x} \in Q^n$. Therefore $\Gamma \subseteq C^n \cap Q^n$ for every $n = 0, 1, 2, ..$ and hence $x^{n+1} = P_{C^n \cap Q^n} x$ is well defined for every $n = 0, 1, 2, ..$. Thus the sequence $\{x^n\}$ is well defined.

Next we show that the sequences $\{x^n\}$, $\{u_i^n\}$, $\{y_i^n\}$ and $\{z_i^n\}$ are bounded. Indeed, let $d = P_\Gamma x$. From $x^{n+1} = P_{C^n \cap Q^n} x$ and $d \in \Gamma \subseteq C^n \cap Q^n$, we have

$$\|x^{n+1} - x\| \leq \|d - x\|, \quad (4.20)$$

for every $n = 0, 1, 2, ...$ Therefore $\{x^n\}$ is bounded.

Further, it follows from (4.14), (4.18), and (4.19) that the sequences $\{u_i^n\}$, $\{t_i^n\}$, and $\{z_i^n\}$ are bounded for each $i = 1, 2, ..., N$.

Claim 4.2. We have

$$\lim_{n\to\infty} \|z_i^n - x^n\| = \lim_{n\to\infty} \|u_i^n - y_i^n\| = \lim_{n\to\infty} \|(T_{r_i^n}^{T_i,\psi_i} - I)Bx^n\| = 0,$$

$$\lim_{n\to\infty} \|t_i^n - y_i^n\| = \lim_{n\to\infty} \|S_i t_i^n - t_i^n\| = \lim_{n\to\infty} \|y_i^n - x^n\| = 0,$$

$\forall i = 1, 2, ..., N$, where $t_i^n = P_{K_i}(u_i^n - \lambda_i^n v_i^n)$.

Proof. From (4.7) and (4.8), we have that $x^{n+1} \in C^n \cap Q^n$ and $x^n = P_{Q^n} x$. Therefore

$$\|x^n - x\| \leq \|x^{n+1} - x\|, \text{ for every } n = 0, 1, 2, ... \tag{4.21}$$

It follows from (4.20), and (4.21) that the sequence $\{\|x^n - x\|\}$ is nondecreasing and bounded, hence convergent. Therefore $\lim_{n\to\infty} \|x^n - x\|$ exists.

Since $x^n = P_{Q^n} x$ and $x^{n+1} \in Q^n$, using (2.4), we have

$$\|x^{n+1} - x^n\|^2 \leq \|x^{n+1} - x\|^2 - \|x^n - x\|^2, \text{ for every } n = 0, 1, 2, ...$$

This implies that

$$\lim_{n\to\infty} \|x^{n+1} - x^n\| = 0. \tag{4.22}$$

Since for every $i = 1, 2, ..., N$, $x^{n+1} \in C_i^n$, it follows from (4.5) that

$$\begin{aligned} \|z_i^n - x^n\|^2 &\leq 2\langle z_i^n - x^n, x^{n+1} - x^n\rangle \\ &\leq 2\|z_i^n - x^n\|\|x^{n+1} - x^n\|. \end{aligned}$$

Therefore

$$\|z_i^n - x^n\|^2 \leq 2\|x^{n+1} - x^n\|$$

and hence using (4.22), we have

$$\lim_{n\to\infty} \|z_i^n - x^n\| = 0, \text{ for every } i = 1, 2, ..., N. \tag{4.23}$$

It follows from (4.17) and (4.19) that

$$\begin{aligned} (1 - (\lambda_i^n \alpha)^2)\|u_i^n - y_i^n\|^2 &\leq \|x^n - \hat{x}\|^2 - \|t_i^n - \hat{x}\|^2 \\ &\leq \|x^n - \hat{x}\|^2 + \tfrac{\alpha_i^n}{1-\alpha_i^n}\|x^n - \hat{x}\|^2 - \tfrac{1}{1-\alpha_i^n}\|z_i^n - \hat{x}\|^2 \\ &= \tfrac{1}{1-\alpha_i^n}(\|x^n - \hat{x}\|^2 - \|z_i^n - \hat{x}\|^2) \end{aligned}$$

Now,

$$\begin{aligned} \|u_i^n - y_i^n\|^2 &\leq [(1 - \alpha_i^n)(1 - (\lambda_i^n \alpha)^2)]^{-1}(\|x^n - \hat{x}\|^2 - \|z_i^n - \hat{x}\|^2) \\ &= [(1 - \alpha_i^n)(1 - (\lambda_i^n \alpha)^2)]^{-1}(\|x^n - \hat{x}\| - \|z_i^n - \hat{x}\|) \\ &\quad (\|x^n - \hat{x}\| + \|z_i^n - \hat{x}\|) \\ &\leq [(1 - \alpha_i^n)(1 - (\lambda_i^n \alpha)^2)]^{-1}\|x^n - z_i^n\|(\|x^n - \hat{x}\| + \|z_i^n - \hat{x}\|). \end{aligned}$$

Since $\{x^n\}$ and $\{z_i^n\}$, $i = 1, 2, ..., N$, all are bounded and $\lim_{n\to\infty} \|z_i^n - x^n\| = 0$, we have

$$\lim_{n\to\infty} \|u_i^n - y_i^n\| = 0, \text{ for every } i = 1, 2, ..., N. \tag{4.24}$$

By the same process as in (4.16), we have

$$\begin{aligned}\|t_i^n-\hat{x}\|^2 &\leq \|u_i^n-\hat{x}\|^2-\|u_i^n-y_i^n\|^2-\|y_i^n-t_i^n\|^2\\ &\quad +2\lambda_i^n\alpha\|u_i^n-y_i^n\|\|t_i^n-y_i^n\|\\ &\leq \|u_i^n-\hat{x}\|^2-\|u_i^n-y_i^n\|^2-\|y_i^n-t_i^n\|^2\\ &\quad +\|u_i^n-y_i^n\|^2+(\lambda_i^n\alpha)^2\|t_i^n-y_i^n\|^2\\ &\leq \|u_i^n-\hat{x}\|^2-(1-(\lambda_i^n\alpha)^2)\|t_i^n-y_i^n\|^2\\ &\leq \|x^n-\hat{x}\|^2-(1-(\lambda_i^n\alpha)^2)\|t_i^n-y_i^n\|^2.\end{aligned} \tag{4.25}$$

Now, from (4.19) and (4.24), we get

$$\begin{aligned}\|z_i^n-\hat{x}\|^2 &\leq \alpha_i^n\|x^n-\hat{x}\|^2+(1-\alpha_i^n)\|t_i^n-\hat{x}\|^2\\ &\leq \alpha_i^n\|x^n-\hat{x}\|^2+(1-\alpha_i^n)[\|x^n-\hat{x}\|^2-(1-(\lambda_i^n\alpha)^2)\|t_i^n-y_i^n\|^2]\\ &\leq \|x^n-\hat{x}\|^2-(1-\alpha_i^n)(1-(\lambda_i^n\alpha)^2)\|t_i^n-y_i^n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|t_i^n-y_i^n\|^2 &\leq [(1-\alpha_i^n)(1-(\lambda_i^n\alpha)^2)]^{-1}(\|x^n-\hat{x}\|^2-\|z_i^n-\hat{x}\|^2)\\ &= [(1-\alpha_i^n)(1-(\lambda_i^n\alpha)^2)]^{-1}(\|x^n-\hat{x}\|-\|z_i^n-\hat{x}\|)\\ &\quad (\|x^n-\hat{x}\|+\|z_i^n-\hat{x}\|)\\ &\leq [(1-\alpha_i^n)(1-(\lambda_i^n\alpha)^2)]^{-1}\|x^n-z_i^n\|(\|x^n-\hat{x}\|+\|z_i^n-\hat{x}\|).\end{aligned} \tag{4.26}$$

Again, since the sequences $\{x^n\}$ and $\{z_i^n\}$, $i=1,2,...,N$, are bounded and $\lim_{n\to\infty}\|z_i^n-x^n\|=0$, it follows from (4.26) that

$$\lim_{n\to\infty}\|t_i^n-y_i^n\|=0,\ \text{for every } i=1,2,...,N. \tag{4.27}$$

Further, it follows from (4.24), (4.27), and the triangle inequality that

$$\lim_{n\to\infty}\|u_i^n-t_i^n\|=0,\ \text{for every } i=1,2,...,N. \tag{4.28}$$

Next, we show that

$$\lim_{n\to\infty}\|S_it_i^n-t_i^n\|=0,\ \text{for every } i=1,2,...,N.$$

From (4.12), (4.13) and (4.17), we have

$$\begin{aligned}\|z_i^n-\hat{x}\|^2 &= \|\alpha_i^nx^n+(1-\alpha_i^n)S_it_i^n-\hat{x}\|^2\\ &= \|\alpha_i^n(x^n-\hat{x})+(1-\alpha_i^n)(S_it_i^n-\hat{x})\|^2\\ &\leq [\alpha_i^n\|x^n-\hat{x}\|+(1-\alpha_i^n)\|S_it_i^n-\hat{x}\|]^2\\ &\leq [\alpha_i^n\|x^n-\hat{x}\|+(1-\alpha_i^n)\|t_i^n-\hat{x}\|]^2\\ &\leq [\alpha_i^n\|x^n-\hat{x}\|+(1-\alpha_i^n)\|u_i^n-\hat{x}\|]^2\\ &= (\alpha_i^n)^2\|x^n-\hat{x}\|^2+(1-\alpha_i^n)^2\|u_i^n-\hat{x}\|^2\\ &\quad +2\alpha_i^n(1-\alpha_i^n)\|x^n-\hat{x}\|\|u_i^n-\hat{x}\|\\ &\leq (\alpha_i^n)^2\|x^n-\hat{x}\|^2+(1-\alpha_i^n)^2[\|x^n-\hat{x}\|^2\\ &\quad +\delta(L\delta-1)\|(T_{r_i^n}^{T_i,\psi_i}-I)Bx^n\|^2]\\ &\quad +2\alpha_i^n(1-\alpha_i^n)\|x^n-\hat{x}\|\|u_i^n-\hat{x}\|\\ &= (\alpha_i^n)^2\|x^n-\hat{x}\|^2+(1-\alpha_i^n)^2\|x^n-\hat{x}\|^2\\ &\quad +(1-\alpha_i^n)^2\delta(L\delta-1)\|(T_{r_i^n}^{T_i,\psi_i}-I)Bx^n\|^2\\ &\quad +2\alpha_i^n(1-\alpha_i^n)\|x^n-\hat{x}\|\|u_i^n-\hat{x}\|.\end{aligned}$$

Now

$$\begin{aligned}(1-\alpha_i^n)^2\delta(1-L\delta)\|(T_{r_i^n}^{T_i,\psi_i} &- I)Bx^n\|^2\\ &\leq ((\alpha_i^n)^2+(1-\alpha_i^n)^2)\|x^n-\hat{x}\|^2-\|z_i^n-\hat{x}\|^2\\ &\quad +2\alpha_i^n(1-\alpha_i^n)\|x^n-\hat{x}\|\|u_i^n-\hat{x}\|\\ &\leq ((\alpha_i^n)^2+(1-\alpha_i^n)^2)\|x^n-\hat{x}\|^2-\|z_i^n-\hat{x}\|^2\\ &\quad +2\alpha_i^n(1-\alpha_i^n)\|x^n-\hat{x}\|^2\\ &= \|x^n-\hat{x}\|^2-\|z_i^n-\hat{x}\|^2\\ &= (\|x^n-\hat{x}\|-\|z_i^n-\hat{x}\|)\\ &\quad (\|x^n-\hat{x}\|+\|z_i^n-\hat{x}\|)\\ &\leq \|x^n-z_i^n\|(\|x^n-\hat{x}\|+\|z_i^n-\hat{x}\|).\end{aligned}$$

Since $\delta(1-L\delta)>0$ and $\|x^n-z_i^n\|\to 0$ as $n\to\infty$, therefore we have

$$\lim_{n\to\infty}\|(T_{r_i^n}^{T_i,\psi_i}-I)Bx^n\|=0,\ \text{for every}\ i=1,2,...,N. \tag{4.29}$$

Now, we estimate:

$$\begin{aligned}&\|u_i^n-\hat{x}\|^2=\|T_{r_i^n}^{(A_i,\phi_i)}(x^n+\delta B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n-\hat{x}\|^2\\ &=\|T_{r_i^n}^{(A_i,\phi_i)}(x^n+\delta B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n-T_{r_i^n}^{(A_i,\phi_i)}\hat{x}\|^2\\ &\leq\langle u_i^n-\hat{x},x^n+\delta B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n-\hat{x}\rangle\\ &=\tfrac{1}{2}\{\|u_i^n-\hat{x}\|^2+\|x^n-\hat{x}\|^2+2\delta\langle x^n-\hat{x},B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\rangle\\ &+\delta^2\|B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|^2-\|u_i^n-x^n-\delta B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|^2\}\\ &=\tfrac{1}{2}\{\|u_i^n-\hat{x}\|^2+\|x^n-\hat{x}\|^2+2\delta\langle x^n-\hat{x},B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\rangle\\ &+\delta^2\|B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|^2-[\|u_i^n-x^n\|^2+\delta^2\|B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|^2\\ &-2\delta\langle u_i^n-x^n,B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\rangle]\}\\ &=\tfrac{1}{2}\{\|u_i^n-\hat{x}\|^2+\|x^n-\hat{x}\|^2-\|u_i^n-x^n\|^2+2\delta\langle u_i^n-\hat{x},B^*(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\rangle\}.\end{aligned}$$

Hence by using Cauchy-Schwartz inequality, we obtain

$$\|u_i^n-\hat{x}\|^2\leq\|x^n-\hat{x}\|^2-\|u_i^n-x^n\|^2+2\delta\|B(u_i^n-\hat{x})\|\|(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|. \tag{4.30}$$

Further, we estimate:

$$\begin{aligned}\|z_i^n-\hat{x}\|^2 &= \|\alpha_i^nx^n+(1-\alpha_i^n)S_it_i^n-\hat{x}\|^2\\ &= \|\alpha_i^n(x^n-\hat{x})+(1-\alpha_i^n)(S_it_i^n-\hat{x})\|^2\\ &\leq [\alpha_i^n\|x^n-\hat{x}\|+(1-\alpha_i^n)\|S_it_i^n-\hat{x}\|]^2\\ &\leq [\alpha_i^n\|x^n-\hat{x}\|+(1-\alpha_i^n)\|t_i^n-\hat{x}\|]^2\\ &= (\alpha_i^n)^2\|x^n-\hat{x}\|^2+(1-\alpha_i^n)^2\|t_i^n-\hat{x}\|^2\\ &\quad +2\alpha_i^n(1-\alpha_i^n)\|x^n-\hat{x}\|\|t_i^n-\hat{x}\|\\ &\leq (\alpha_i^n)^2\|x^n-\hat{x}\|^2+(1-\alpha_i^n)^2[\|u_i^n-\hat{x}\|^2\\ &\quad -(1-(\lambda_i^n\alpha)^2)\|u_i^n-y_i^n\|^2]\\ &\quad +2\alpha_i^n(1-\alpha_i^n)\|x^n-\hat{x}\|\|t_i^n-\hat{x}\|\ \text{using}(4.17)\\ &\leq (\alpha_i^n)^2\|x^n-\hat{x}\|^2+(1-\alpha_i^n)^2[\|x^n-\hat{x}\|^2-\|u_i^n-x^n\|^2\\ &\quad +2\delta\|B(u_i^n-\hat{x})\|\|(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|]\\ &\quad -(1-\alpha_i^n)^2(1-(\lambda_i^n\alpha)^2)\|u_i^n-y_i^n\|^2\\ &\quad +2\alpha_i^n(1-\alpha_i^n)\|x^n-\hat{x}\|\|t_i^n-\hat{x}\|\ \text{using}(4.30).\end{aligned}$$

Using (4.25), this implies that:

$$\begin{aligned}(1-\alpha_i^n)^2\|u_i^n-x^n\|^2 &\leq ((\alpha_i^n)^2+(1-\alpha_i^n)^2)\|x^n-\hat{x}\|^2-\|z_i^n-\hat{x}\|^2\\ &\quad +2\delta(1-\alpha_i^n)^2\|B(u_i^n-\hat{x})\|\|(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|\\ &\quad -(1-\alpha_i^n)^2(1-(\lambda_i^n\alpha)^2)\|u_i^n-y_i^n\|^2\\ &\quad +2\alpha_i^n(1-\alpha_i^n)\|x^n-\hat{x}\|\|t_i^n-\hat{x}\|\end{aligned}$$

$$\begin{aligned}&\leq ((\alpha_i^n)^2+(1-\alpha_i^n)^2+2\alpha_i^n(1-\alpha_i^n))\|x^n-\hat{x}\|^2-\|z_i^n-\hat{x}\|^2\\ &\quad +2\delta(1-\alpha_i^n)^2\|B(u_i^n-\hat{x})\|\|(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|\\ &\quad -(1-\alpha_i^n)^2(1-(\lambda_i^n\alpha)^2)\|u_i^n-y_i^n\|^2\\ &= \|x^n-\hat{x}\|^2-\|z_i^n-\hat{x}\|^2+2\delta(1-\alpha_i^n)^2\|B(u_i^n-\hat{x})\|\|(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|\\ &\quad -(1-\alpha_i^n)^2(1-(\lambda_i^n\alpha)^2)\|u_i^n-y_i^n\|^2\\ &\leq (\|x^n-\hat{x}\|+\|z_i^n-\hat{x}\|)(\|x^n-\hat{x}\|-\|z_i^n-\hat{x}\|)\\ &\quad +2\delta(1-\alpha_i^n)^2\|B(u_i^n-\hat{x})\|\|(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|\\ &\quad -(1-\alpha_i^n)^2(1-(\lambda_i^n\alpha)^2)\|u_i^n-y_i^n\|^2\\ &\leq \|x^n-z_i^n\|(\|x^n-\hat{x}\|+\|z_i^n-\hat{x}\|)\\ &\quad +2\delta(1-\alpha_i^n)^2\|B(u_i^n-\hat{x})\|\|(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|\\ &\quad -(1-\alpha_i^n)^2(1-(\lambda_i^n\alpha)^2)\|u_i^n-y_i^n\|^2.\end{aligned}$$

Since $\|x^n-z_i^n\|\to 0$, $\|u_i^n-y_i^n\|\to 0$, and $\|(T_{r_i^n}^{(T_i,\psi_i)}-I)Bx^n\|\to 0$ as $n\to\infty$, therefore, we have

$$\lim_{n\to\infty}\|u_i^n-x^n\|=0. \tag{4.31}$$

From (4.28) and (4.31), we have

$$\|t_i^n-x^n\| \leq \|t_i^n-u_i^n\|+\|u_i^n-x^n\|$$

Hence:

$$\lim_{n\to\infty}\|t_i^n-x^n\|=0. \tag{4.32}$$

Now,

$$\begin{aligned}z_i^n-x^n &= \alpha_i^n x^n+(1-\alpha_i^n)S_it_i^n-x^n\\ &= (1-\alpha_i^n)(S_it_i^n-x^n),\end{aligned}$$

we have

$$\begin{aligned}(1-c)\|S_it_i^n-x^n\| &\leq (1-\alpha_i^n)\|S_it_i^n-x^n\|\\ &= \|z_i^n-x^n\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|z_i^n-x^n\|=0$, therefore we have

$$\lim_{n\to\infty}\|S_it_i^n-x^n\|=0. \tag{4.33}$$

From (4.32) and (4.33), we have

$$\|S_it_i^n-t_i^n\| \leq \|S_it_i^n-x^n\|+\|t_i^n-x^n\|,$$

hence:

$$\lim_{n\to\infty}\|S_it_i^n-t_i^n\|=0. \tag{4.34}$$

Also from(4.27) and (4.32), we have

$$\lim_{n\to\infty}\|y_i^n-x^n\|=0. \tag{4.35}$$

Claim 4.3. The weak limit of every weakly convergent subsequence of the sequences $\{x^n\}$, $\{u_i^n\}$, $\{y_i^n\}$ and $\{z_i^n\}$ belongs to Γ.

Proof. Since $\{x^n\}$ is bounded, it has a weakly convergent subsequence $\{x^{n_k}\}$, say $x^{n_k} \rightharpoonup \hat{w}$. Then it follows from (4.31) that $u_i^{n_k} \rightharpoonup \hat{w}$ and $t_i^{n_k} \rightharpoonup \hat{w}$, for each $i = 1, 2, ..., N$.

Since K_i is closed and convex, $u_i^{n_k} \in K_i$ and $u_i^{n_k} \rightharpoonup \hat{w}$ for each $i = 1, 2, ..., N$, it follows that $\hat{w} \in K_i$. Hence $\hat{w} \in \cap_{i=1}^N K_i$.

First, let's show that $\hat{w} \in \cap_{i=1}^N \mathrm{Fix}(S_i)$. Assume by contradiction that $\hat{w} \notin \cap_{i=1}^N \mathrm{Fix}(S_i)$ for some $i = 1, 2, ..., N$. Since $S_i\hat{w} \neq \hat{w}$, then by Opial's condition (2.2) and (4.34), we have

$$\begin{aligned}
\liminf_{k\to\infty} \|t_i^{n_k} - \hat{w}\| &< \liminf_{k\to\infty} \|t_i^{n_k} - S_i\hat{w}\| \\
&\leq \liminf_{k\to\infty} \{\|t_i^{n_k} - S_i t_i^{n_k}\| + \|S_i t_i^{n_k} - S_i\hat{w}\|\} \\
&= \liminf_{k\to\infty} \|t_i^{n_k} - \hat{w}\|,
\end{aligned}$$

which is a contradiction. Thus $\hat{w} \in \cap_{i=1}^N \mathrm{Fix}(S_i)$.

Now we show that $\hat{w} \in \Omega_i$ for each $i = 1, 2, ..., N$. Since

$$u_i^n = T_{r_i^n}^{(A_i,\phi_i)}(x^n + \delta B^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n),$$

by Theorem 3.1, we have

$$\begin{aligned}
&\langle y - u_i^n, A_i u_i^n\rangle + \phi_i(y, u_i^n) - \phi_i(u_i^n, u_i^n) \\
&+\frac{e}{r_i^n}\langle y - u_i^n, u_i^n - (x^n + \delta B^*(T_{r_i^n}^{(T_i,\psi_i)} - I)Bx^n)\rangle \in P_i, \ \forall y \in K_i
\end{aligned} \tag{4.36}$$

and hence

$$\begin{aligned}
&\langle y - u_i^{n_k}, A_i u_i^{n_k}\rangle + \phi_i(y, u_i^{n_k}) - \phi_i(u_i^{n_k}, u_i^{n_k}) \\
&+\frac{e}{r_i^{n_k}}\langle y - u_i^{n_k}, u_i^{n_k} - (x^{n_k} + \delta B^*(T_{r_i^{n_k}}^{(T_i,\psi_i)} - I)Bx^{n_k})\rangle \in P_i, \ \forall y \in K_i.
\end{aligned} \tag{4.37}$$

Set $v_{i,t} = tv_i + (1-t)\hat{w}$ for all $t \in [0,1]$ and $v_i \in K_i, \ i = 1, 2, ..., N$. Then we have $v_{i,t} \in K_i, \ i = 1, 2, ..., N$. From (4.37), it follows that

$$\begin{aligned}
&\langle v_{i,t} - u_i^{n_k}, A_i u_i^{n_k}\rangle + \phi_i(v_{i,t}, u_i^{n_k}) - \phi_i(u_i^{n_k}, u_i^{n_k}) \\
&+\frac{e}{r_i^{n_k}}\langle v_{i,t} - u_i^{n_k}, u_i^{n_k} - (x^{n_k} + \delta B^*(T_{r_i^{n_k}}^{(T_i,\psi_i)} - I)Bx^{n_k})\rangle \in P_i \\
&\langle v_{i,t} - u_i^{n_k}, A_i u_i^{n_k} + A_i v_{i,t} - A_i v_{i,t}\rangle + \phi_i(v_{i,t}, u_i^{n_k}) - \phi_i(u_i^{n_k}, u_i^{n_k}) \\
&+\frac{e}{r_i^{n_k}}\langle v_{i,t} - u_i^{n_k}, u_i^{n_k} - (x^{n_k} + \delta B^*(T_{r_i^{n_k}}^{(T_i,\psi_i)} - I)Bx^{n_k})\rangle \in P_i \\
&\langle v_{i,t} - u_i^{n_k}, A_i v_{i,t}\rangle \in \langle v_{i,t} - u_i^{n_k}, A_i v_{i,t} - A_i u_i^{n_k}\rangle - \phi_i(v_{i,t}, u_i^{n_k}) + \phi_i(u_i^{n_k}, u_i^{n_k}) \\
&-\frac{e}{r_i^{n_k}}\langle v_{i,t} - u_i^{n_k}, u_i^{n_k} - (x^{n_k} + \delta B^*(T_{r_i^{n_k}}^{(T_i,\psi_i)} - I)Bx^{n_k})\rangle + P_i \\
&\langle v_{i,t} - u_i^{n_k}, A_i v_{i,t}\rangle \in \langle v_{i,t} - u_i^{n_k}, A_i v_{i,t} - A_i u_i^{n_k}\rangle - \phi_i(v_{i,t}, u_i^{n_k}) + \phi_i(u_i^{n_k}, u_i^{n_k})
\end{aligned}$$

$$-e\langle v_{i,t}-u_i^{n_k}, \frac{u_i^{n_k}-x^{n_k}}{r_i^{n_k}}\rangle + e\langle v_{i,t}-u_i^{n_k}, \frac{\delta B^*(T_{r_i^{n_k}}^{(T_i,\psi_i)}-I)Bx^{n_k}}{r_i^{n_k}}\rangle + P_i.$$

From (4.29) and (4.31), we obtain that $\frac{u_i^{n_k}-x^{n_k}}{r_i^{n_k}} \to 0$ and $\frac{\delta B^*(T_{r_i^{n_k}}^{(T_i,\psi_i)}-I)Bx^{n_k}}{r_i^{n_k}} \to 0$ as $k\to\infty$. Since A_i is monotone, we also have that $\langle v_{i,t}-u_i^{n_k}, A_iv_{i,t}-A_iu_i^{n_k}\rangle \in P_i$. Thus, it follows that

$$\langle v_{i,t}-\hat{w}, A_iv_{i,t}\rangle \in \phi_i(\hat{w},\hat{w})-\phi_i(v_{i,t},\hat{w})+P_i$$
$$\langle tv_i+(1-t)\hat{w}-\hat{w}, A_iv_{i,t}\rangle \in \phi_i(\hat{w},\hat{w})-\phi_i(tv_i+(1-t)\hat{w},\hat{w})+P_i$$
$$t\langle v_i-\hat{w}, A_iv_{i,t}\rangle \in \phi_i(\hat{w},\hat{w})-t\phi_i(v_i,\hat{w})-(1-t)\phi_i(\hat{w},\hat{w})+P_i,$$
$$(\because \phi_i \text{ is convex in first argument})$$
$$t\langle v_i-\hat{w}, A_iv_{i,t}\rangle \in t\phi_i(\hat{w},\hat{w})-t\phi_i(v_i,\hat{w})+P_i$$
$$\langle v_i-\hat{w}, A_iv_{i,t}\rangle \in \phi_i(\hat{w},\hat{w})-\phi_i(v_i,\hat{w})+P_i$$

Letting $t\to 0$ and using the weak continuity of A_i in the above, we get:

$$\langle v_i-\hat{w}, A_i\hat{w}\rangle \in \phi_i(\hat{w},\hat{w})-\phi_i(v_i,\hat{w})+P_i,\ \forall v_i\in K_i$$
$$\langle v_i-\hat{w}, A_i\hat{w}\rangle + \phi_i(v_i,\hat{w})-\phi_i(\hat{w},\hat{w}) \in P_i,\ \forall v_i\in K_i.$$

This implies that $\hat{w}\in \mathrm{Sol}(\mathrm{MVVIP}(A_i,K_i))$, for each $i=1,2,...,N$.

Next, we show that $B\hat{w}\in \mathrm{Sol}(\mathrm{MVVIP}(T_i,Q_i))$. Since B is a bounded linear operator, we have $Bx^{n_k}\rightharpoonup B\hat{w}$.

Now setting

$$v^{n_k}=Bx^{n_k}-T_{r_i^{n_k}}^{(T_i,\psi_i)}Bx^{n_k} \tag{4.38}$$

and using (4.29) in (4.38), we get

$$\lim_{n\to\infty} v^{n_k}=0 \text{ and } Bx^{n_k}-v^{n_k}=T_{r_i^{n_k}}^{(T_i,\psi_i)}Bx^{n_k}.$$

Thus from Theorem 3.1, we have

$$\langle z-(Bx^{n_k}-v^{n_k}), T_i(Bx^{n_k}-v^{n_k})\rangle + \psi_i(z,Bx^{n_k}-v^{n_k})-\psi_i(Bx^{n_k}-v^{n_k},Bx^{n_k}-v^{n_k})$$
$$+\frac{e}{r_i^{n_k}}\langle z-(Bx^{n_k}-v^{n_k}), (Bx^{n_k}-v^{n_k})-Bx^{n_k}\rangle \in P_i,\ \forall z\in Q_i.$$

Letting $k\to\infty$, we get:

$$\langle z-B\hat{w}, T_i(B\hat{w})\rangle + \psi_i(z,B\hat{w})-\psi_i(B\hat{w},B\hat{w}) \in P_i,\ \forall z\in Q_i,$$

which means that $B\hat{w}\in \mathrm{Sol}(\mathrm{MVVIP}(T_i,Q_i))$, and hence $\hat{w}\in\Omega_i$ for each $i=1,2,...,N$.

It has been proved in [9] that $\hat{w}\in\cap_{i=1}^N \mathrm{Sol}(\mathrm{MuVIP}(G_i,K_i))$. Thus $\hat{w}\in\Gamma$.

Claim 4.4. The sequences $\{x^n\}$, $\{u_i^n\}$, $\{y_i^n\}$ and $\{z_i^n\}$ converge strongly to $P_\Gamma(x)$.

Proof. Since $x^{n+1}=P_{C^n\cap Q^n}x$, we have for any $s\in C^n\cap Q^n$

$$\|x^{n+1}-x\|\le\|s-x\|. \tag{4.39}$$

Since $\Gamma \subseteq C^n \cap Q^n$, and $P_\Gamma(x)$ and the sequence $\{x^n\}$ are well-defined, for $s = P_\Gamma(x)$ we get:

$$\|x^n - x\| \leq \|P_\Gamma(x) - x\| \tag{4.40}$$

and furthermore,

$$\lim_{n\to\infty} \|x^n - x\| \leq \|P_\Gamma(x) - x\|. \tag{4.41}$$

Now, since we already proved that the weak limit $\hat{w}$ of every subsequence $\{x^{n_k}\}_{k\in N}$ of $\{x^n\}_{n\in N}$ belongs to Γ, it follows from (4.41) that:

$$\begin{aligned} \|\hat{w} - x\| &\leq \liminf_{k\to\infty} \|x^{n_k} - x\| \\ &= \lim_{n\to\infty} \|x^n - x\| \\ &\leq \|P_\Gamma(x) - x\|. \end{aligned}$$

Since $\hat{w} \in \Gamma$, it follows that $\hat{w} = P_\Gamma(x)$.

Since every weak cluster point of the sequence $\{x^n\}_{n\in N}$ is equal to $\hat{w}$, it follows that

$$w - \lim_{n\to\infty} x^n = \hat{w} = P_\Gamma(x).$$

Finally

$$\|\hat{w} - x\| \leq \liminf_{k\to\infty} \|x^{n_k} - x\| = \lim_{n\to\infty} \|x^n - x\| = \|\hat{w} - x\|.$$

Since $w - \lim_{n\to\infty} (x^n - x) = \hat{w} - x$ and $\lim_{n\to\infty} \|x^n - x\| = \|\hat{w} - x\|$, it follows from the Kadec-Klee property that $\lim_{n\to\infty} \|x^n - \hat{w}\| = 0$.

This completes the proof of Theorem 4.1.

The following corollary is due to Censor *et al.* [9].

Corollary 4.1. *Let H_1 be a real Hilbert space; let $K_i \subseteq H_1$, where $i \in \{1, 2, 3, ..., N\}$, be a nonempty, closed and convex subset with $\cap_{i=1}^n K_i \neq \emptyset$; let $G_i : H_1 \to 2^{H_1}$ be a monotone and Lipschitz continuous mapping with constant β_i such that $G_i(x) \in CB(H_1, \forall x \in H_1$. Assume that $\Gamma := \cap_{i=1}^N(\mathrm{Sol}(\mathrm{MuVIP}(G_i, K_i))) \neq \emptyset$. For a given $x^0 = x \in \cap_{i=1}^N K_i$, let the iterative sequence $\{x^n\}$ be generated by the following iterative scheme*:

$$y_i^n = P_{K_i}(x^n - \lambda_i^n w_i^n),\ w_i^n \in G_i(x^n)$$

find $v_i^n \in G_i(y_i^n)$ which satisfies Definition 2.4(i) *with w_i^n,*

$$\begin{aligned} z_i^n &= P_{K_i}(x^n - \lambda_i^n v_i^n), \\ C_i^n &= \{z \in H_1 : \|z_i^n - z\|^2 \leq \|x^n - z\|^2\}, \\ C^n &= \cap_{i=1}^N C_i^n, \\ Q^n &= \{z \in H_1 : \langle x^n - z, x - x^n\rangle \geq 0\}, \\ x^{n+1} &= P_{C^n \cap Q^n} x, \end{aligned}$$

for $n = 1, 2, ...,$ and for each $i = 1, 2, ..., N$, and $\{\lambda_i^n\} \subseteq [a, b]$ for some $a, b \in (0, \alpha^{-1})$, where $\alpha := \max_{1\leq i\leq N} \beta_i$. Then, the sequence $\{x^n\}$ converges strongly to $z = P_\Gamma x$.

Proof. The proof follows by taking $H_2 = H_1$, $Y = R$, $P_i = (0, +\infty)$, $A_i = 0$, $T_i = 0$, $\phi_i = 0$, $\psi_i = 0$, $S_i = I$, and $\alpha_i^n = 0$ in Theorem 4.1.

The following corollary is due to Nadezhkina and Takahashi [31].

Corollary 4.2. *Let H_1 be a real Hilbert space; let $K_1 \subseteq H_1$ be a nonempty, closed and convex subset; let $G_1 : K_1 \to K_1$ be a monotone and Lipschitz continuous mapping with constant α. Let $S_1 : K_1 \to K_1$ be a nonexpansive mapping. Assume that $\Gamma := \mathrm{Sol}(\mathrm{VIP}(1.1)) \cap \mathrm{Fix}(S_1)) \neq \emptyset$. For a given $x^0 = x \in K_1$, let the iterative sequence $\{x^n\}$ be generated by the following iterative scheme*:

$$\begin{aligned} y_1^n &= P_{K_1}(x^n - \lambda_1^n G_1(x^n)), \\ z_1^n &= \alpha_1^n x^n + (1-\alpha_1^n) S_1 P_{K_1}(x^n - \lambda_1^n G_1(y_1^n)), \\ C^n &= \{z \in H_1 : \|z_1^n - z\|^2 \leq \|x^n - z\|^2\}, \\ Q^n &= \{z \in H_1 : \langle x^n - z, x - x^n \rangle \geq 0\}, \\ x^{n+1} &= P_{C^n \cap Q^n} x, \end{aligned}$$

for $n = 1, 2, ...,$ and $\{\alpha_1^n\} \subseteq [0, c]$, for some $c \in [0, 1)$ and $\{\lambda_1^n\} \subseteq [a, b]$ for some $a, b \in (0, \alpha^{-1})$. Then, the sequence $\{x^n\}$ converges strongly to $z = P_\Gamma x$.

Proof. The proof follows by taking $i = 1$, $H_2 = H_1$, $Y = R$, $P_i = (0, +\infty)$, $A_i = 0$, $T_i = 0$, $\phi_i = 0$, $\psi_i = 0$ and G_1 as a single-valued mapping in Theorem 4.1.

Remark 4.1. Theorem 3.1 and Lemma 3.1 can be proved in the framework of convex Banach space, while the proof of Theorem 4.1 needs further research effort because many of the properties specific to Hilbert spaces were used in the paper, in particular specific to the metric projection mapping.

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Received: April 4, 2014; Accepted: February 19, 2015.