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N-ORDER UNIFORMLY NONCREASY BANACH LATTICES AND THE SUZUKI NONEXPANSIVE-TYPE MAPPINGS

ANNA BETIUK-PILARSKA

Institute of Mathematics, Maria Curie-Skłodowska University
20-031 Lublin, Poland
E-mail: abetiuk@hektor.umcs.lublin.pl

Abstract. We show that if K is a nonempty weakly compact convex subset of weakly orthogonal N-order uniformly noncreasy Banach lattice and $T : K \to K$ satisfies condition (C) or is continuous and satisfies condition (C_λ) for some $\lambda \in (0, 1)$, then T has a fixed point. This generalizes a result from [2].

Key Words and Phrases: Nonexpansive mapping, Fixed point, Weakly orthogonal lattice, Mapping satisfying condition (C_λ), N-order uniformly noncreasy Banach lattice.
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1. Introduction

Let X be a Banach space. By B_X we denote the closed unit ball of X. For notation and terminology concerning Banach lattices we refer the reader to [11]. Let us recall that if an inequality involves lattice and algebraic operations, then it is enough to check its validity for real numbers to be sure that it holds for vectors in arbitrary Banach lattice (see [11] p.1). In the next lemma we collect some lattice inequalities which will be used in the sequel.

Lemma 1.1. *Let X be a Banach lattice. Then*

(i) *for every $x, y \in X$*
$$|x| - |x| \wedge |y| \le |x - y|,$$
(ii) *for every $z, x_1, \ldots, x_N \in X$ and $N \ge 2$, $N \in \mathbb{N}$*
$$|z| \le \bigwedge_{\substack{i,j=1,\ldots,N \\ i \neq j}} (|z - x_i| \vee |z - x_j|) + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} |x_i| \wedge |x_j| \qquad \textit{(see [2])}.$$

Given a Banach lattice X and $\varepsilon \in [0, 1]$, we put
$$\delta_{m,X}(\varepsilon) = \inf\{1 - \|x - y\| : 0 \le y \le x,\ \|x\| \le 1,\ \|y\| \ge \varepsilon\}.$$
We say that X is uniformly monotone if $\delta_{m,X}(\varepsilon) > 0$ for all $\varepsilon \in (0, 1]$. The coefficient
$$\varepsilon_{0,m}(X) = \sup\{\varepsilon \in [0, 1) : \delta_{m,X}(\varepsilon) = 0\}$$
is called the characteristic of monotonicity of the lattice X.

In [10], the modulus of order smoothness of a Banach lattice X was introduced as follows $\rho_{m,X}(t) = \sup\{\|x \vee ty\| - 1 : x, y \in B_X,\ x, y \geq 0\}$, where $t \in [0,1]$.

A Banach lattice X is called order uniformly smooth if $\lim_{t\to 0} \frac{\rho_{m,X}(t)}{t} = 0$. The coefficient

$$\rho_{m,X}(1) + 1 = \sup\{\|x \vee y\| : x, y \in B_X,\ x, y \geq 0\}$$

is called the Riesz angle of X and denoted by $\alpha(X)$ (see [4]).

Kurc [10] proved the duality formulae

$$\rho_{m,X^*}(t) = \sup_{0\leq\varepsilon\leq 1} (\varepsilon t - \delta_{m,X}(\varepsilon)) \text{ and } \delta_{m,X}(\varepsilon) = \sup_{0\leq t\leq 1} (\varepsilon t - \rho_{m,X^*}(t)),$$

for all $\varepsilon, t \in [0,1]$. They show that a Banach lattice X is uniformly monotone (resp. order uniformly smooth) if and only if the dual lattice X^* is order uniformly smooth (resp. uniformly monotone). Using the above formulae it is also easy to see that $\rho_{m,X^*}(1) < 1$ if and only if $\varepsilon_{0,m}(X) < 1$.

In [1] it was proved that if $\rho_{m,X}(1) < 1$ and $\varepsilon_{0,m}(X) < 1$, then X is superreflexive. In particular, if a Banach lattice X is order uniformly smooth and uniformly monotone, then X is superreflexive.

In [2] a generalization of Riesz angle was introduced in the following way.

Definition 1.2. Let X be a Banach lattice and $N \in \mathbb{N}$, $N \geq 2$. The N-dimensional Riesz angle of X is defined as

$$\alpha_N(X) = \sup\left\{ \left\| \bigwedge_{\substack{i,j=1,\ldots,N\\ i\neq j}} (x_i \vee x_j) \right\| :\ x_1, \ldots, x_N \in B_X,\ x_1, \ldots, x_N \geq 0 \right\}.$$

Of course $\alpha_N(X) \geq 1$. Moreover, $\alpha_2(X) = \alpha(X)$ and $\alpha_N(X) \leq \alpha_{N-1}(X)$ for every $N \geq 3$. Hence $\alpha_N(X) \leq \alpha(X)$. This estimate is not sharp. It was shown in [2] that, for every Banach lattice X and every natural $N \geq 2$, we have

$$\alpha_N(X) \leq \frac{N}{N-1}.$$

The following class of Banach lattices was introduced in [2].

Definition 1.3. Let $r \in (0,1]$. A Banach lattice X is said to be r-N-order uniformly noncreasy (r-N-OUNC) if for all $u_1, \ldots, u_N \in \frac{N-1}{N} B_X$ such that $\|u_i - u_j\| \leq 1$ we have either

$$\left\| \bigwedge_{\substack{i,j=1,\ldots,N\\ i\neq j}} (|u_i| \vee |u_j|) \right\| \leq r$$

or there exist $i \neq j$ such that for every $y \in X$ the conditions $|y| \leq |u_i - u_j|$, $\|y\| \geq r$ imply $\||u_i - u_j| - |y|\| \leq r$. A Banach lattice X is N-order uniformly noncreasy (N-OUNC) if it is r-N-OUNC for some $r \in (0,1)$.

For $N = 2$ this definition coincides with the definition of an order uniformly noncreasy Banach lattice given in [1]. Of course, each Banach lattice X is r-N-OUNC with $r = \frac{N-1}{N}\alpha_N(X)$. Therefore, if $\alpha_N(X) < \frac{N}{N-1}$, then X is N-OUNC. It is also easy to see that X is r-N-OUNC where $r = \max\{\varepsilon, 1 - \delta_{m,X}(\varepsilon)\}$ for any $\varepsilon \in (0,1)$. It follows that if $\varepsilon_{0,m}(X) < 1$, then X is N-OUNC. The class of all N-OUNC Banach lattices contains therefore all Banach lattices X with $\alpha_N(X) < \frac{N}{N-1}$ and all uniformly monotone lattices. In [1] and [2] we can find examples showing that this class is essentially bigger, i.e. there exist Banach lattices which are N-OUNC, but they are not uniformly monotone and $\alpha_N(X) = \frac{N-1}{N}$. Moreover it is shown that for different values of N the classes of N-OUNC Banach lattices are essentially different.

Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for $x, y \in K$. We say that a Banach space X has the weak fixed point property if every nonexpansive mapping defined on a nonempty weakly compact convex subset of X has a fixed point. There is a large literature concerning fixed point theory of nonexpansive mappings and their generalizations (see [13] and references therein). Recently, Suzuki [16] defined a class of generalized nonexpansive mappings as follows.

Definition 1.4. A mapping $T : K \to K$ is said to satisfy condition (C) if for all $x, y \in K$,

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \le \|x - y\|.$$

Subsequently, the above definition has been extended in [14].

Definition 1.5. Let $\lambda \in (0,1)$. A mapping $T : K \to K$ is said to satisfy condition (C_λ) if for all $x, y \in K$,

$$\lambda\|x - Tx\| \le \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \le \|x - y\|.$$

We say that X has the weak fixed point property for continuous mappings satisfying condition (C_λ) if every such mapping defined on a nonempty weakly compact convex subset of X has a fixed point.

It is not difficult to see that if $\lambda_1 < \lambda_2$, then condition (C_{λ_1}) implies condition (C_{λ_2}). Several examples of mappings satisfying condition (C_λ) are given in [14, 16]. Moreover, if K is convex and $T : K \to K$ satisfies condition (C_λ) for some $\lambda \in (0,1)$, then for every $\gamma \in [\lambda, 1)$ the mapping $T_\gamma : K \to K$ defined by $T_\gamma x = \gamma Tx + (1-\gamma)x$ satisfies condition $(C_{\frac{\lambda}{\gamma}})$.

A sequence (x_n) is an approximate fixed point sequence for T (in short afps) if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

2. Basic lemmas

Recall that a mapping $T : M \to M$ acting on a metric space (M,d) is said to be asymptotically regular if

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0, \text{ for all } x \in M.$$

Lemma 2.1. [14, Theorem 4] *Let K be a bounded convex subset of a Banach space X. Assume that $T : K \to K$ satisfies condition (C_λ) for some $\lambda \in (0,1)$. For $\gamma \in [\lambda, 1)$ define a sequence (x_n) in K by taking $x_1 \in K$ and $x_{n+1} = \gamma Tx_n + (1-\gamma)x_n$ for $n \ge 1$.*

Then (x_n) is an approximate fixed point sequence for T, that is T_γ is asymptotically regular.

In [3] the following theorem was proven which is the uniform version of the above theorem.

Theorem 2.2. *Let K be a bounded convex subset of a Banach space X. Fix $\lambda \in (0,1), \gamma \in [\lambda, 1)$ and let $\mathcal{F}$ denote the collection of all mappings which satisfy condition (C_λ). Let $T_\gamma = (1-\gamma)I + \gamma T$ for $T \in \mathcal{F}$. Then for every $\varepsilon > 0$, there exists a positive integer n_0 such that $\left\|T_\gamma^{n+1}x - T_\gamma^n x\right\| < \varepsilon$ for every $n \geq n_0, x \in K$ and $T \in \mathcal{F}$.*

Let D be a nonempty weakly compact convex subset of a Banach space X and $T : D \to D$. It follows from the Kuratowski-Zorn lemma that there exists a minimal (in the sense of inclusion) convex and weakly compact set $K \subset D$ which is invariant under T. The lemma below is a counterpart of the Goebel-Karlovitz lemma (see [8, 9]). It was proved by Dhompongsa and Kaewcharoen [6, Theorem 4.14] in the case of mappings which satisfy condition (C), and from Butsan, Dhompongsa and Takahashi result in[5, Lemma 3.2] and Lloréns Fuster and Moreno Gálvez result in [12, Th. 4.7] we have the same conclusion in the case of continuous mappings satisfying condition (C_λ) for some $\lambda \in (0,1)$. Denote by

$$r(K,(x_n)) = \inf\{\limsup_{n\to\infty} \|x_n - x\| : x \in K\}$$

the asymptotic radius of a sequence (x_n) relative to K.

Lemma 2.3. *Let K be a nonempty convex weakly compact subset of a Banach space X which is minimal invariant under $T : K \to K$. If T is continuous and satisfies condition (C_λ) for some $\lambda \in (0,1)$, then there exists an approximate fixed point sequence (x_n) for T such that*

$$\lim_{n\to\infty} \|x_n - x\| = \inf\{r(K,(y_n)) : (y_n) \text{ is an afps in } K\}$$

for every $x \in K$. In the case $\lambda = \frac{1}{2}$ continuity assumption can be dropped.

Now let $(x_n^1), \dots, (x_n^N)$ be sequences in K. Put

$$v_n = \frac{1}{N}x_n^1 + \ldots + \frac{1}{N}x_n^N.$$

The following technical lemma deals with the behavior of sequences $(T_\gamma^k v_n)_{n\in\mathbb{N}}$, $k = 1, 2, \dots, N$.

Lemma 2.4. *Let K be a convex subset of Banach lattice X and let $T : K \to K$ satisfy condition (C_λ) for some $\lambda \in (0,1)$. Fix $\gamma \in (\lambda, 1)$, integers $M > 1$, $N \geq 3$ and $\varepsilon > 0$ such that $(M+3)\varepsilon < \frac{1}{N}$. Suppose that $(x_n^1), \dots, (x_n^N)$ are sequences in K such that*

$$\lim_{n\to\infty} \|x_n^i\| = 1, \quad \lim_{n\to\infty} \|x_n^i - x_n^j\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \||x_n^i| \wedge |x_n^j|\| = 0$$

for every $i, j = 1, \dots, N, i \neq j$ and the following conditions are satisfied for every $n \in \mathbb{N}$

(i) $\min_{i,j=1,\dots,N, i\neq j} \{\|x_n^i\|, \|x_n^i - x_n^j\|\} > 1 - \varepsilon$

(ii) $\|Tx_n^i - x_n^i\| < \varepsilon$, $i = 1, \ldots, N$.

Let $v_n = \frac{1}{N}x_n^1 + \ldots + \frac{1}{N}x_n^N$. *Then there exists* $n_0 \in \mathbb{N}$ *such that for every natural* $n \geq n_0$, $k = 1, \ldots, M$ *and* $i = 1, \ldots, N$

$$\frac{1}{N} - (k+2)\varepsilon \leq \|T_\gamma^k v_n - x_n^i\| \leq \frac{N-1}{N} + (k+1)\varepsilon,$$

where T_γ^k *is the* k*th iterate of* T_γ.

Proof. First, note that if N is even then

$$\begin{aligned}\lim_{n\to\infty} \|v_n\| \leq& \frac{1}{N} \lim_{n\to\infty} \left(\||x_n^1| + |x_n^2|\| + \||x_n^3| + |x_n^4|\| + \ldots + \||x_n^{N-1}| + |x_n^N|\|\right)\\ =& \frac{1}{N} \lim_{n\to\infty} \left(\|x_n^1 - x_n^2\| + \|x_n^3 - x_n^4\| + \ldots + \|x_n^{N-1} - x_n^N\|\right)\\ =& \frac{1}{N} \cdot \frac{N}{2} = \frac{1}{2} \leq \frac{N-1}{N}.\end{aligned}$$

It follows from the fact that if $\lim_{n\to\infty} \||x_n^i| \wedge |x_n^j|\| = 0$, then $\lim_{n\to\infty} \||x_n^i| + |x_n^j|\| = \lim_{n\to\infty} \|x_n^i - x_n^j\|$. For odd N

$$\begin{aligned}\lim_{n\to\infty} \|v_n\| \leq& \frac{1}{N} \lim_{n\to\infty} \left(\||x_n^1| + |x_n^2|\| + \ldots + \||x_n^{N-2}| + |x_n^{N-1}|\| + \|x_n^N\|\right)\\ =& \frac{1}{N} \lim_{n\to\infty} \left(\|x_n^1 - x_n^2\| + \ldots + \|x_n^{N-2} - x_n^{N-1}\| + \|x_n^N\|\right)\\ =& \frac{1}{N} \cdot \frac{N+1}{2} \leq \frac{N-1}{N}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\lim_{n\to\infty} \|v_n - x_n^i\| = \lim_{n\to\infty} \left\|\frac{1}{N}x_n^1 + \ldots + \frac{1}{N}x_n^N - x_n^i\right\|\\ &\leq \frac{1}{N} \lim_{n\to\infty} (\|x_n^1 - x_n^i\| + \ldots + \|x_n^{i-1} - x_n^i\| + \|x_n^{i+1} - x_n^i\| + \ldots + \|x_n^N - x_n^i\|)\\ &\leq \frac{N-1}{N}.\end{aligned}$$

There exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$\|v_n\| \leq \frac{N-1}{N} + \varepsilon \quad \text{and} \quad \|v_n - x_n^i\| \leq \frac{N-1}{N} + \varepsilon,\ i = 1, \ldots, N.$$

Fix $n \geq n_0$. We have

$$\|v_n - x_n^i\| \geq \|x_n^i\| - \|v_n\| \geq 1 - \varepsilon - \left(\frac{N-1}{N} + \varepsilon\right) = \frac{1}{N} - 2\varepsilon.$$

Thus

$$\frac{1}{N} - 2\varepsilon \leq \|v_n - x_n^i\| \leq \frac{N-1}{N} + \varepsilon.$$

Since for every $i = 1, \ldots, N$

$$\lambda\|Tx_n^i - x_n^i\| \leq \|Tx_n^i - x_n^i\| < \varepsilon < \frac{1}{N} - 2\varepsilon \leq \|v_n - x_n^i\|$$

it follows from condition (C_λ) that

$$\|Tx_n^i - Tv_n\| \le \|x_n^i - v_n\|$$

so

$$\|T_\gamma x_n^i - T_\gamma v_n\| \le \gamma\|Tx_n^i - Tv_n\| + (1-\gamma)\|x_n^i - v_n\| \le \|x_n^i - v_n\|.$$

Now we proceed by induction on k. For $k = 1$, notice that

$$\begin{aligned}\|T_\gamma v_n - x_n^i\| \le& \|T_\gamma v_n - T_\gamma x_n^i\| + \|T_\gamma x_n^i - x_n^i\| \\ \le& \|v_n - x_n^i\| + \varepsilon \le \frac{N-1}{N} + 2\varepsilon.\end{aligned}$$

To prove the lower estimate note that for $i \neq j$

$$\begin{aligned}\|T_\gamma v_n - x_n^i\| \ge& \|x_n^i - x_n^j\| - \|T_\gamma v_n - x_n^j\| \\ \ge& 1 - \varepsilon - \left(\frac{N-1}{N} + 2\varepsilon\right) = \frac{1}{N} - 3\varepsilon.\end{aligned}$$

Now we suppose that the lemma is true for a fixed $k < M$. Then by induction assumption

$$\frac{\lambda}{\gamma}\|T_\gamma x_n^i - x_n^i\| \le \|T_\gamma x_n^i - x_n^i\| < \varepsilon < \frac{1}{N} - (k+2)\varepsilon \le \|x_n^i - T_\gamma^k v_n\|$$

and hence

$$\|T_\gamma^{k+1} v_n - T_\gamma x_n^i\| \le \|T_\gamma^k v_n - x_n^i\|.$$

We thus get

$$\begin{aligned}\|T_\gamma^{k+1} v_n - x_n^i\| \le& \|T_\gamma^{k+1} v_n - T_\gamma x_n^i\| + \|T_\gamma x_n^i - x_n^i\| \\ \le& \|T_\gamma^k v_n - x_n^i\| + \varepsilon \le \frac{N-1}{N} + (k+2)\varepsilon.\end{aligned}$$

Now we prove the lower estimate. For $i \neq j$

$$\begin{aligned}\|T_\gamma^{k+1} v_n - x_n^i\| \ge& \|x_n^i - x_n^j\| - \|T_\gamma^{k+1} v_n - x_n^j\| \\ \ge& 1 - \varepsilon - \left(\frac{N-1}{N} + (k+2)\varepsilon\right) = \frac{1}{N} - (k+3)\varepsilon.\end{aligned}$$

□

In the sequel we will need the following lemma.

Lemma 2.5. *Let K be a convex weakly compact subset of a Banach lattice X. Suppose that a mapping $T : K \to K$ satisfies condition (C_λ) for some $\lambda \in (0,1)$, $N \ge 3$ is a natural number and $(x_n^1), \dots, (x_n^N)$ are weakly null approximate fixed point sequences for T in K such that for every $i = 1, \dots, N$ and $x \in K$*

$$r = \lim_{n\to\infty} \|x_n^i - x\| = \inf\{r(K,(y_n)) :\ (y_n) \text{ is an afps in } K\} \tag{2.1}$$

and for every $i, j = 1, \dots, N, i \neq j$

$$\lim_{n\to\infty} \||x_n^i| \wedge |x_n^j|\| = 0.$$

Then for every $\varepsilon > 0$ there exist subsequences of $(x_n^1), \dots, (x_n^N)$, denoted again $(x_n^1), \dots, (x_n^N)$, and a sequence (z_n) in K such that

(i) $\|z_n\| > r(1-\varepsilon/2)$
(ii) $\|z_n - x_n^i\| \le r\frac{N-1}{N} + \varepsilon,\ i = 1,\dots,N$
for every $n \in \mathbb{N}$.

Proof. Let us first notice that if $S : \frac{1}{a}K \to \frac{1}{a}K$ is defined by $Sy = \frac{1}{a}T(ay)$, then

$$\|Sy - y\| = \frac{1}{a}\|T(ay) - ay\|$$

and S satisfies condition (C_λ). It follows that if the sequences $(x_n^1),\dots,(x_n^N)$ satisfy the assumptions of Lemma 2.5, then the sequences $(\frac{x_n^1}{r}),\dots,(\frac{x_n^N}{r})$ satisfy these assumptions with S and $a = 1$, i.e., $(\frac{x_n^1}{r}),\dots,(\frac{x_n^N}{r})$ are weakly null afps for $S : \frac{1}{r}K \to \frac{1}{r}K$ and

$$1 = \lim_{n\to\infty} \|\frac{x_n^i}{r} - y\| = \inf\{r(\frac{1}{r}K,(z_n)) :\ (z_n) \text{ is an afps for } S \text{ in } K\}$$

for every $y \in \frac{1}{r}K$. Therefore it suffices to prove the lemma for $r = 1$.

We claim that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in K$ and $\|Tx - x\| < \delta(\varepsilon)$, then $\|x\| > 1 - \varepsilon/2$. Indeed, otherwise, arguing as in [7], there exists $\varepsilon_0 > 0$ such that we can find $w_n \in K$ with $\|Tw_n - w_n\| < \frac{1}{n}$ and $\|w_n\| \le 1 - \varepsilon_0/2$ for every $n \in \mathbb{N}$. Then the sequence (w_n) is an approximate fixed point sequence in K, but $\limsup_{n\to\infty} \|w_n\| \le 1 - \varepsilon_0/2$, which contradicts our assumption that $\limsup_{n\to\infty} \|w_n\| \ge 1$.

Fix $\varepsilon > 0$ and $\gamma \in (\lambda, 1)$. From Theorem 2.2, there exists $M > 1$ such that

$$\|T_\gamma^{M+1}x - T_\gamma^M x\| < \gamma\delta(\varepsilon) \tag{2.2}$$

for every $x \in K$. Choose $\eta > 0$ such that $(M+3)\eta < \frac{1}{N}$ and $(M+1)\eta < \varepsilon$. Put $v_n = \frac{1}{N}x_n^1 + \ldots + \frac{1}{N}x_n^N$ and consider sequences $(T_\gamma^k v_n)_{n\in\mathbb{N}}$ for $k = 1,\dots,M$. Applying (2.1) (with $r = 1$) and passing to subsequences, we can assume that the assumptions of Lemma 2.4 are satisfied, i.e., for every $i = 1,\dots,N$ $\lim_{n\to\infty}\|x_n^i\| = 1$ and for $i,j = 1,\dots,N, i \ne j$

$$\lim_{n\to\infty} \|x_n^i - x_n^j\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \||x_n^i| \wedge |x_n^j|\| = 0$$

and for every $n \in \mathbb{N}$

(i) $\min\limits_{i,j=1,\dots,N, i\ne j} \{\|x_n^i\|, \|x_n^i - x_n^j\|\} > 1 - \eta,$
(ii) $\|Tx_n^i - x_n^i\| < \eta,\ i = 1,\dots,N.$

Denote $z_n = T_\gamma^M v_n$. It follows from Lemma 2.4 that there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ and $i = 1,\dots,N$ we have

$$\|z_n - x_n^i\| = \|T_\gamma^M v_n - x_n^i\| \le \frac{N-1}{N} + (M+1)\eta \le \frac{N-1}{N} + \varepsilon. \tag{2.3}$$

Furthermore, by (2.2)

$$\|Tz_n - z_n\| = \frac{1}{\gamma}\|T_\gamma^{M+1}v_n - T_\gamma^M v_n\| < \delta(\varepsilon)$$

and consequently,

$$\|z_n\| > 1 - \varepsilon/2. \tag{2.4}$$

Passing again to subsequences we can assume that (2.3) and (2.4) hold for every $n \in \mathbb{N}$. □

3. Fixed point theorems

Definition 3.1. A Banach lattice X is said to be weakly orthogonal if

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \| |x_n| \wedge |x_m| \| = 0$$

whenever (x_n) is a sequence in X which converges weakly to 0.

This definition was given by Borwein and Sims in [4], but the reader should be aware that also a different property is called weak orthogonality in the literature (see [15]). It is easy to show that c_0, c, l_p $(1 \le p < \infty)$ are weakly orthogonal while l_∞ and $L_p([0,1])$ do not have this property.

The Banach-Mazur distance of two isomorphic Banach spaces X and Y is defined by the formula

$$d(X,Y) = \inf \|S\| \|S^{-1}\|$$

where the infimum is taken over all linear isomorphisms S of X onto Y. Borwein and Sims in [4] proved that a Banach space X has the weak fixed point property for nonexpansive mappings if there exists a weakly orthogonal Banach lattice Y such that

$$d(X,Y)\alpha(Y) < 2.$$

This result was generalized in [1]. It was shown that a Banach space X has the weak fixed point property if there exists a weakly orthogonal 2-r-OUNC Banach lattice Y such that $d(X,Y)r < 1$. S. Dhompongsa and A. Kaewcharoen in [6] proved the same result for continuous mappings satisfying condition (C). Their result was generalized for the class of N-OUNC Banach lattices in [2]. Now we are ready to prove our main theorem which is a generalization of Theorem 3.9 from [2].

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a Banach lattice. Assume that there exist a norm $\|\cdot\|_1$ on X and a constant $d > 0$ such that*

$$\|x\|_1 \le \|x\| \le d\|x\|_1$$

for every $x \in X$ and $Y = (X, \|\cdot\|_1)$ is weakly orthogonal $r - N - OUNC$ Banach lattice satisfying $dr < 1$. Then X has the weak fixed point property for continuous mappings satisfying condition (C_λ) for some $\lambda \in (0,1)$. In the case $\lambda = \frac{1}{2}$ continuity assumption can be dropped.

Proof. Assume that the conclusion of the theorem is false. Then there exist a nonempty weakly compact convex subset K of X and a mapping $T : K \to K$ satisfying condition (C) or continuous mapping satisfying (C_λ) for some $\lambda \in (0,1)$ which has no fixed point. We can assume that K is minimal T-invariant. By Lemma 2.3 there exists an approximate fixed point sequence (x_n) for T in K such that $r = \lim_{n\to\infty} \|x_n - x\| = \inf\{r(K,(y_n)) : \ (y_n) \ \text{ is an afps in } \ K\}$ for every $x \in K$. There is no loss of generality in assuming that $r = 1$ and (x_n) converges weakly to 0. In particular $0 \in K$.

Let $Y = (X, \|\cdot\|_1)$ be a weakly orthogonal $r - N - OUNC$ Banach lattice such that $\|x\|_1 \le \|x\| \le d\|x\|_1$ for every $x \in X$ and let $dr < 1$. Choose $\varepsilon > 0$ such that $dr < \frac{1-\varepsilon}{1+\varepsilon}$. Similarly as in [2] we find subsequences $(x_n^1), \dots, (x_n^N)$ of (x_n) satisfying

$$\lim_{n\to\infty} \|x_n^i - x_n^j\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \||x_n^i| \wedge |x_n^j|\|_1 = 0$$

for $i, j = 1, \dots, N, i \ne j$. From Lemma 2.5 there exist subsequences of $(x_n^1), \dots, (x_n^N)$, denoted again $(x_n^1), \dots, (x_n^N)$, and a sequence (z_n) such that for $i = 1, \dots, N$ and every $n \in \mathbb{N}$

$$\|z_n - x_n^i\| \le \frac{N-1}{N}(1+\varepsilon) \text{ and } \|z_n\| > 1 - \varepsilon/2.$$

There exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ and $i, j = 1, \dots, N, i \ne j$

$$\|x_n^i\| > 1 - \varepsilon/2 \quad \text{and} \quad \||x_n^i| \wedge |x_n^j|\|_1 < \frac{\varepsilon}{2N(N-1)d} < \frac{\varepsilon}{2d}.$$

Fix $n \ge n_0$ and put $u_n^i = (z_n - x_n^i)/(1+\varepsilon)$ and $y_n^{ij} = (|x_n^i| - |x_n^i| \wedge |x_n^j|)/(1+\varepsilon)$ for $i, j = 1, \dots, N, i < j$. Then for all $i, j = 1, \dots, N, i < j$ we have

$$\|u_n^i\|_1 = \|z_n - x_n^i\|_1/(1+\varepsilon) \le \|z_n - x_n^i\|/(1+\varepsilon) \le \frac{N-1}{N}$$

and

$$\begin{aligned}\|y_n^{ij}\|_1 &= \||x_n^i| - |x_n^i| \wedge |x_n^j|\|_1/(1+\varepsilon) \ge (\|x_n^i\|_1 - \||x_n^i| \wedge |x_n^j|\|_1)/(1+\varepsilon) \\ &> \left(\frac{1}{d}\|x_n^i\| - \frac{\varepsilon}{2d}\right)/(1+\varepsilon) > (1-\varepsilon)/(d(1+\varepsilon)) > r.\end{aligned}$$

By Lemma 1.1 $|y_n^{ij}| \le |u_n^i - u_n^j|$ for $i, j = 1, \dots, N, i < j$. Moreover,

$$\begin{aligned}\||u_n^i - u_n^j| - |y_n^{ij}|\|_1 =& \||x_n^i - x_n^j| - |x_n^i| + |x_n^i| \wedge |x_n^j|\|_1/(1+\varepsilon) \\ \ge& \|||x_n^i| - |x_n^j|| - |x_n^i| + |x_n^i| \wedge |x_n^j|\|_1/(1+\varepsilon) \\ =& \||x_n^j| - |x_n^i| \wedge |x_n^j|\|_1/(1+\varepsilon) \\ \ge& (\|x_n^j\|_1 - \||x_n^i| \wedge |x_n^j|\|_1)/(1+\varepsilon) \\ \ge& (\frac{1}{d}\|x_n^i\| - \frac{\varepsilon}{2d})/(1+\varepsilon) \\ \ge& (1-\varepsilon)/(d(1+\varepsilon)) > r\end{aligned}$$

and

$$\begin{aligned}&(1-\varepsilon/2)/(1+\varepsilon) < \|z_n\|/(1+\varepsilon) \le d\|z_n\|_1/(1+\varepsilon) \\ &\le d\left(\left\|\bigwedge_{\substack{i,j=1,\dots \\ i\ne j}} (|z_n - x_n^i| \vee |z_n - x_n^j|)\right\|_1 + \sum_{\substack{i,j=1 \\ i \ne j}}^{N} \||x_n^i| \wedge |x_n^j|\|_1\right)/(1+\varepsilon) \\ &\le d\left\|\bigwedge_{\substack{i,j=1,\dots,N \\ i\ne j}} (|u_n^i| \vee |u_n^j|)\right\|_1 + \varepsilon/(2(1+\varepsilon)).\end{aligned}$$

Thus

$$\left\| \bigwedge_{\substack{i,j=1,\ldots,N \\ i\neq j}} (|u_n^i| \vee |u_n^j|) \right\|_1 > \frac{1}{d} \cdot \frac{1-\varepsilon}{1+\varepsilon} > r.$$

Hence $Y = (X, \|\cdot\|_1)$ is not $r-N-OUNC$. □

Corollary 3.3. *Every weakly orthogonal N-OUNC Banach lattice has the weak fixed point property for continuous mappings satisfying condition (C_λ) for some $\lambda \in (0,1)$. In the case $\lambda = \frac{1}{2}$ continuity assumption can be dropped.*

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