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FIXED POINT THEOREMS IN CONES UNDER LOCAL CONDITIONS

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Abstract. We provide in this paper new fixed theorems for operators leaving invariant a cone in a Banach space. The main conditions in these new results are imposed on the behavior of the operator at 0 and ∞.

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1. Introduction

The problem of seeking positive solutions for boundary value problems (bvps for short) associated with differential equations having positive nonlinearities, is usually converted to that of finding solutions in the cone of nonnegative functions C of some functional space X to the abstract Hammerstein equation,

$$u = LFu \tag{1.1}$$

where $L \in L(X)$ is compact and positive $(L(C) \subset C)$ and $F : C \to C$ is continuous and bounded (maps bounded sets into bounded sets). Note that the mapping $T = LF$ leave invariant the cone C.

This approach has motivated many works, where existence results for fixed point for operators leaving invariant a cone have been proved. Krasnosel'skii's theorems of compression and expansion of a cone in a Banach space (see Theorems 4.12 and 4.14 in [11] and Theorems 2.3.3 and 2.3.4 in [10]), are the most famous and the most used in the literature.

Krasnosel'skii has provided in [11] many others interesting fixed point theorems. Among these results, Theorems 4.10, 4.11 and 4.16 have attracted the attention of Amann in [1] where he generalized these results for strict set-contraction leaving invariant a cone in a Banach space. Roughly speaking, these theorems and their generalization, state that if such an operator is approximatively linear at 0 and ∞ such that the spectral radius of the linear approximations are oppositely located with respect to 1, then it has a fixed point.

In this paper, we will prove new fixed point theorems for operators leaving invariant a cone in a Banach space, and as in the above Krasnosel'kii's theorems, the main assumptions are on the behavior of the operator at 0 and ∞. More precisely, we will assume that our operator has an approximative linear minorant at 0 and an approximative linear majorant at ∞ or conversely; existence of the fixed point is obtained under additional conditions: it required that, the approximative linear minorant has the strongly index-jump property and the positive spectrums of the approximative linear majorant and minorant are oppositely located with respect to 1. The concepts of index-jump and the strongly index-jump will be introduced in Section 2, where we prove that a linear positive compact operator has the index-jump property (IJP for short) if and only if it has at least one positive eigenvalue, and we present some classes of operators having the strongly index-jump property (SIJP for short).

The most interesting property of the SIJP consists in the fact that it is conserved by limits of nondecreasing sequences of operators having the SIJP (see the proof of Theorem 3.2). In order to indicate the interst of this property, let us return to bvps. In the case where the nonlinearity has a singular weight, the operator L in formulation (1.1) will contain this singular weight. Technically, one can see that such an operator is a limit of a nondecreasing sequence of operators (L_n) having the SIJP. This what makes interesting the above property.

The spirit of hypotheses in this work meet that in many results in the literature. Theorem 7.B in [15] state that if a positive mapping T has a linear minorant having a eigensubsolution, then T has eigensolutions. Webb in [13] has obtained fixed calculations for a positive mapping A under the condition that A has a specific linear majorant or minorant (see Theorems 4.4, 4.5 and 4.7 in [13]); he has also provided nonexistence results under similar conditions (see Theorem 4.9 in [13]). Main ideas of this work are inspired from the works in [2], [7] and [8].

The paper is organized as follows. Section 2 is devoted for the needed background. In Section 3, we present the main results and their needed preliminaries. In the last section, we present some applications of our main results to a class of fourth-order bvps.

2. Background

Let X be a real Banach space and denote by $L_c(X)$, the set of all linear compact self-mapping of X. A nonempty closed convex subset C of X is called a cone if $C \cap (-C) = \{0_X\}$ and $(tC) \subset C$ for all $t \geq 0$. It is well known that a cone C induces a partial order in the Banach space X. We write for all $x, y \in X,\ x \preceq y$ if $y - x \in C,\ x \prec y$ if $y - x \in C$ and $y \neq x$ and $x \npreceq y$ if $y - x \notin C$. Notations $\succeq,\ \succ$, and $\nsucceq$ denote respectively the inverse situations.

A cone C of X is said to be: solid if $int(C) \neq \emptyset$; normal if there exists a positive constant N such that for all $u, v \in C,\ u \leq v$ implies $\|u\| \leq N\|v\|$ and total if $\overline{C - C} = X$.

Definition 2.1. Let C be a cone of X and $L \in L_c(X)$. L is said to be positive if $L(C) \subset C$ and strongly positive if C is solid and $L(C \setminus \{0\}) \subset int\,(C)$.

Definition 2.2. Let C be a cone of X and let $L \in L_c(X)$ be positive. L is said to be lower bounded on C if $c = \inf\{\|Lu\| : u \in \partial\,(C \cap B\,(0,1))\} > 0$. In this case we have $\|Lu\| \geq c\,\|u\|$ for all $u \in C$.

Definition 2.3. Let C be a cone of X and $L \in L_c(X)$ be positive. A real number λ is said to be a positive eigenvalue of L if $\lambda > 0$ and there exists $\phi \succ 0_X$ such that $L\phi = \lambda\phi$.

The two following theorems are known as Krein-Rutman theorems. They present situations where the spectral radius $r(L) = \lim \|L^n\|^{1/n}$ of a positive linear compact operator L, is a positive eigenvalue of L.

Theorem 2.4. [9], [15] Assume that the cone C is total and let $L \in L_C(X)$ be positive with $r(L) > 0$. Then $r(L)$ is a positive eigenvalue of L.

Theorem 2.5. [9], [15] Let $L \in L_c(X)$ be strongly positive. Then $r(L)$ is the unique positive eigenvalue of L.

Proposition 2.6. [9], [15] Let $L \in L_c(X)$ be strongly positive and consider the nonhomogenuous equation

$$u - \gamma Lu = v \tag{2.1}$$

with $v \succ 0_X$. Then Equation (2.1) has a unique positive solution if $\gamma < 1/r(L)$ and no positive solution if $\gamma > 1/r(L)$.

Definition 2.7. Let C be a cone in X and $T_1, T_2 : C \to C$ be continuous mapping. We write $T_1 \preceq T_2$ if $T_1x \preceq T_2x$ for all $x \in C$.

Definition 2.8. Let $\chi : X \times X \to \mathbb{R}$ be a bilinear form and C a cone of X. χ is said to be:

a) continuous if there exists a positive constant c_1 such that $|\chi(u,v)| \leq c_1 \|u\| \|v\|$ for $u, v \in X$

b) positive if $\chi(u,v) > 0$ for all $u, v \in C \setminus \{0_X\}$,

c) increasing if for all $u_1, u_2, v_1, v_2 \in C$, $(u_1 \preceq u_2, v_1 \preceq v_2)$ implies $\chi(u_1, v_1) \leq \chi(u_2, v_2)$.

d) coercive on the cone C if there exists a positive constant c_2 such that $\chi(u,v) \geq c_2 \|u\| \|v\|$ for all $u, v \in C$.

We will use extensively in this work the fixed point index theory. For sake of completeness, let us recall some lemmas providing fixed point index computations. Let C be a cone in X. Let for $R > 0$, $C_R = C \cap B(0,R)$ where $B(0,R)$ is the open ball of radius R centered at 0 and ∂C_R and consider a compact mapping $f : \overline{C_R} \to C$ with $fx \neq x$ for all $x \in \partial C_R$.

Lemma 2.9. If $fx \neq \lambda x$ for all $x \in \partial C_R$ and $\lambda \geq 1$ then $i(f, C_R, C) = 1$.

Lemma 2.10. If $fx \neq \lambda x$ for all $x \in \partial C_R$ and $\lambda \in (0,1]$ and $\inf\{\|fx\| : x \in \partial C_R\} > 0$, then $i(f, C_R, C) = 0$.

Lemma 2.11. If $fx \not\succeq x$ for all $x \in \partial C_R$ then $i(f, C_R, C) = 1$.

Lemma 2.12. If $fx \not\preceq x$ for all $x \in \partial C_R$ then $i(f, C_R, C) = 0$.

For more details and proofs we refer the reader to [10].

We end this section by the following important lemma, which roughly speaking state that the positive spectrum of a linear compact operator is nonempty whenever this operator is a limit of a sequence of linear compact operators with nonempty positive spectrum.

Lemma 2.13. Let for all integer n, L_n be a positive operator in $L_c(X)$ having a positive eigenvalue λ_n. If $L_n \to L$ in operator norm and (λ_n) converges to some real number $\lambda > 0$ then λ is a positive eigenvalue of L.

Proof. Let ϕ_n be the eigenvector associated with λ_n such that $\|\phi_n\| = 1$ and set $\psi_n = L\phi_n$. Since L is compact and the sequence (ϕ_n) is bounded, we have up to a subsequence $\psi_n \to \psi \in K$. Thus, we obtain the following estimates,

$$\begin{aligned}\|\lambda_n\phi_n - \psi\| &= \|L_n\phi_n - \psi\| \\ &\leq \|L_n\phi_n - L\phi_n\| + \|L\phi_n - \psi\| \\ &\leq \|L_n - L\| + \|\psi_n - \psi\|\end{aligned}$$

leading to

$$\lim \lambda_n\phi_n = \psi \text{ and } \|\psi\| = \lim \|\lambda_n\phi_n\| = \lim \lambda_n = \lambda > 0.$$

Also, we have

$$\begin{gathered}\|L_n\phi_n - (L\psi/\lambda)\| = \|(L_n(\lambda_n\phi_n)/\lambda_n) - (L\psi/\lambda)\| \\ \leq \|(1/\lambda_n) L_n(\lambda_n\phi_n) - (1/\lambda) L_n(\lambda_n\phi_n)\| + \|(1/\lambda) L_n(\lambda_n\phi_n) - (1/\lambda) L(\lambda_n\phi_n)\| \\ + (1/\lambda)\|L(\lambda_n\phi_n) - L\psi\| \\ \leq |(1/\lambda_n) - (1/\lambda)| \lambda_n \|L_n\| + (\lambda_n/\lambda)\|L_n - L\| + \|L/\lambda\| \|\lambda_n\phi_n - \psi\|\end{gathered}$$

leading to

$$\lim L_n\phi_n = L\psi/\lambda.$$

Thus, letting $n \to \infty$ in equation $L_n\phi_n = \lambda_n\phi_n$ we obtain $L\psi = \lambda\psi$ that is λ is a positive eigenvalue of L. This ends the proof

3. Main results

3.1. **Preliminaries.** In all this section, we let E be a real Banach space, K, P be two cones in E with $P \subset K$. Also we set

$$L_c^P(E) = \{L \in L_c(E) : L(K) \subset P\},$$

$$\Sigma^+ = \{\chi : E \times E \to \mathbb{R} : \chi \text{ is bilinear, positive and increasing}\},$$

$$\Sigma_{co}^+ = \{\chi : E \times E \to \mathbb{R} : \chi \text{ is bilinear, positive, increasing continuous and coercive}\},$$

$$\Pi_{lb}^P = \left\{\begin{array}{c} L \in L_c^P(E) : L \text{ is lower bounded on } P \text{ and there exists} \\ \chi_L \in \Sigma^+ \text{ such that } \chi_L(Lu, v) = \chi_L(u, Lv) \text{ for all } u, v \in P \end{array}\right\},$$

$$\Pi_{co}^P = \left\{\begin{array}{c} L \in L_c^P(E) : \Lambda_{L,K} \neq \emptyset \text{ and there exists } \chi_L \in \Sigma_{co}^+ \\ \text{such that } \chi_L(Lu, v) = \chi_L(u, Lv) \text{ for all } u, v \in P. \end{array}\right\}$$

and

$$\Pi^P = \left\{\begin{array}{c} L \in L_c^P(E) : \Lambda_{L,K} \neq \emptyset \text{ and there exists } \chi_L \in \Sigma^+ \\ \text{such that } \chi_L(Lu, v) = \chi_L(u, Lv) \text{ for all } u, v \in P. \end{array}\right\},$$

where for all $L \in L_c^P(E)$, $\Lambda_{L,K}$ denotes the set of all positive eigenvalues of L and we set so,

$$\lambda_{L,K}^- = \inf \Lambda_{L,K} \text{ and } \lambda_{L,K}^+ = \begin{cases} \sup \Lambda_{L,K} \text{ if } \Lambda_{L,K} \neq \emptyset \\ 0 \text{ if } \Lambda_{L,K} \neq \emptyset. \end{cases}$$

It is natural to ask why we have considered the two cones K, P in the definition of positive operator? The answer is that the cone K is a natural cone which is related to the space X and the cone P is related to the operator L and it represents in some manner, the regularity of L. So, natural conditions, as the normality or totality, are required on the cone K and regularity conditions on L, as lower boundness and coercivity, are required on the cone P.

Also, we have from the permanence property of the fixed point index that for all compact mapping, $f : \overline{K_R} \to P$ with $f(x) \neq x$ for all $x \in \partial K_R$, $i(f, K_R, K) = i(f, P_R, P)$.

Also, for $L \in L_c^P(E)$, we define the subsets

$$\Theta_P^+(L) = \{\theta \geq 0 : \text{ there exists } u \in P \setminus \{0_E\} \text{ such that } Lu \preceq \theta u \},$$

$$\Theta_P^-(L) = \{\theta \geq 0 : \text{ there exists } u \in P \setminus \{0_E\} \text{ such that } Lu \succeq \theta u\}$$

and note that

- $0 \in \Theta_P^-(L)$ and if $\theta \in \Theta_P^-(L)$ then $[0, \theta] \subset \Theta_P^-(L)$,
- If $\theta \in \Theta_P^+(L)$ then $[\theta, +\infty[\subset \Theta_P^+(L)$.

When these two quantities exists, we set

$$\theta_{L,P}^+ = \inf \Theta_P^+(L) \text{ and } \theta_{L,P}^- = \sup \Theta_P^-(L).$$

Lemma 3.1. Assume that $0 < \theta_{L,P}^-, \theta_{L,P}^+ < \infty$ then for all $R > 0$ we have

$$i(\gamma L, K_R, K) = \begin{cases} 1, & if\ \gamma\theta_{L,P}^- < 1, \\ 0, & if\ \gamma\theta_{L,P}^+ > 1. \end{cases}$$

Proof. Let $\gamma > 0$ be such that $\gamma\theta_{L,P}^- < 1$. Suppose that for some $u \in \partial P_R$, $\gamma L u \succeq u$ then we have $Lu \succeq u/\gamma$ and $1/\gamma \in \Theta_P^-(L)$, leading to the contradiction $\gamma\theta_{L,P}^- \geq 1$. So, we have proved that $\gamma L u \not\succeq u$ for all $u \in \partial P_R$ and Lemma 2.11 leads to, $i(\gamma L, K_R, K) = i(\gamma L, P_R, P) = 1$.

The case $\gamma\theta_{L,P}^+ > 1$ is checked similarly by means of Lemma 2.12

Lemma 3.2. For all $L \in L_c^P(E)$ we have $\theta_{L,P}^+ \leq \theta_{L,P}^-$.

Proof. Indeed, if $\theta_{L,P}^+ > \theta_{L,P}^-$ we have from Lemma 3.1, for $\gamma \in \left(1/\theta_{L,P}^+, 1/\theta_{L,P}^-\right)$, the contradiction

$$i(\gamma L, K_R, K) = \begin{cases} 1, & \text{since } \gamma\theta_{L,P}^- < 1, \\ 0, & \text{since } \gamma\theta_{L,P}^+ > 1. \end{cases}$$

Remark 3.3. Clearly, we have for all $L \in L_c^P(E)$, $\Lambda_{L,K} \subset \left[\theta_{L,P}^+, \theta_{L,P}^-\right]$.

Lemma 3.4. [7] For all $L \in L_c^P(E)$, the set $\Theta_P^+(L)$ is not empty.

Lemma 3.5. [14, Theorem 2.7] For all operator $L \in L_c^P(E)$ the set $\Theta_P^-(L)$ is bounded from above by $r(L)$

The following proposition showing the importance of the constant $\theta_{L,P}^+$ is easy to prove.

Proposition 3.6. Let $L \in L_c^P(E)$ with $\theta_{L,P}^+ > 0$ and consider for $y \in P \setminus \{0_E\}$ the equation

$$\lambda u - Lu = y. \tag{3.1}$$

Then Equation (3.1) has no solution in $P \setminus \{0_E\}$ for all $\lambda \in \left(0, \theta_{L,P}^+\right)$.

The condition for nonexistence of positive solutions to Equation (3.1) in Proposition 3.6 is more natural to that given in Theorem 2.16 in [11].

Remark 3.7. Let $L \in L_c^P(E)$ and set

$$\theta_{L,K}^+ = \inf\{\theta \geq 0 \text{ there exists } u \succ 0_E \text{ such that } Lu \preceq \theta u\}$$

and

$$\theta_{L,K}^- = \inf\{\theta \geq 0 \text{ there exists } u \succ 0_E \text{ such that } Lu \succeq \theta u\}.$$

Note that if $\theta_{L,K}^+ > 0$ then for all $y \succ 0_E$, Equation (3.1) has no positive solution.
Note also that if $N(L) = \{0_E\}$ then

$$\theta_{L,P}^+ = \theta_{L,K}^+ \text{ and } \theta_{L,P}^- = \theta_{L,K}^-.$$

3.2. **The index jump property.** Let $L \in L_c^P(E)$ and $\gamma \in (0, +\infty) \setminus \Lambda_{L,K}$. The integer $i(\gamma L, K_R, K)$ is defined for all $R > 0$ and the excision property of the fixed point index, make it independant of R. This justifies the following definition.

Definition 3.8. An operator $L \in L_c^P(E)$ is said to have the IJP if there exists $\mu_L > 0$ such that for all $R > 0$ and all $\gamma \in (0, +\infty) \setminus \Lambda_{L,K}$, we have

$$i(\gamma L, K_R, K) = \begin{cases} 1, & if\ \gamma\mu_L < 1, \\ 0, & if\ \gamma\mu_L > 1. \end{cases}$$

Clearly the real number μ_L in Definition 3.8 is unique. Now, let us answer to the question: which are operators in $L_c^P(E)$ having the IJP?

Theorem 3.9. Let $L \in L_c^P(E)$. Then L has the IJP if and only if $\Lambda_{L,K} \neq \emptyset$. Moreover, we have that $\mu_L = \lambda_{L,K}^+$.

Proof. Let $L \in L_c^P(E)$ having the IJP at μ_L and by the contrary suppose that μ_L is not an eigenvalue. Then $i(L/\mu_L, K_R, K)$ is defined and from the continuity property of the fixed point index, yields the contradiction

$$0 = \lim_{\gamma \overset{>}{\to} 1/\mu_L} i(\gamma L, K_R, K) = i(L/\mu_L, K_R, K) = \lim_{\gamma \overset{<}{\to} 1/\mu_L} i(\gamma L, K_R, K) = 1$$

Thus, we have proved that $\Lambda_{L,K} \neq \emptyset$.

Now, we need to prove that if μ_0 is a positive eigenvalue of L, Then $i(\gamma L, K_R, K) = 0$ for all $\gamma \in (1/\mu_0, +\infty) \setminus \Lambda_{L,K}$ and $R > 0$. To this aim, let $e > 0_E$ be the eigenvector associated with the eigenvalue μ_0. We claim that for all $\lambda \in (0, \mu_0) \setminus \sigma(L)$ and all $t > 0$ equation

$$\lambda u - Lu = te \tag{3.2}$$

admits no positive solution. Indeed, from the Riesz-Schauder theory, there is two subspaces $N(\mu_0)$ and $R(\mu_0)$ such that $\dim(N(\mu_0)) < \infty$, $R(\mu_0)$ is closed, $E =$

$N(\mu_0) \oplus R(\mu_0)$, $L(N(\mu_0)) \subset N(\mu_0)$, $L(R(\mu_0)) \subset R(\mu_0)$ and μ_0 is the unique eigenvalue of L_{μ_0}, the restriction of L to $N(\mu_0)$. Moreover, if P, Q are respectively the projections of E on $N(\mu_0)$ and $R(\mu_0)$, we have that $PL = LP$ and $QL = LQ$.

Thus, Equation (3.2) is equivalent to the system

$$\begin{cases} \lambda v - Lv = te \\ \lambda w - Lw = 0. \end{cases} \tag{3.3}$$

where $v = Pu$ and $w = Qu$. Since $\lambda \notin \sigma(L)$, the second equation in System (3.3) has $w = Qu = 0$ as a unique solution.

For the first equation in System (3.3), there exists a basis $B = \{e_i\}_{i=1}^{i=n}$ where $n = \dim(N(\mu_0))$ and $e_1 = e$ in which the matrix M_{μ_0} of L_{μ_0} has the Jordan form

$$\begin{pmatrix} \mu_0 & m_{1,2} & 0 & \cdot & \cdot & 0 \\ 0 & \mu_0 & m_{2,3} & 0 & & \cdot \\ \cdot & 0 & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & & 0 \\ \cdot & & & \cdot & \cdot & m_{n-1,n} \\ 0 & 0 & \cdot & \cdot & 0 & \mu_0 \end{pmatrix}$$

where for $i = 1, \cdots, n-1$, $m_{i,i+1} = 1$ or 0.

Therefore, if X and b are respectively the coordinate matrices of $v = P(u)$ and te in the basis B, then, the first equation in System (3.3) take the matricial form

$$(\lambda I - M_{\mu_0}) X = b$$

having the unique solution

$$X = \begin{pmatrix} t/(\lambda - \mu_0) \\ 0 \\ \cdot \\ 0 \end{pmatrix}$$

and so, $u = \frac{t}{\lambda - \mu_0} e \notin K$ is the unique solution of Equation (3.2). The claim is proved.

Let $\gamma \in (1/\mu_0, +\infty)$ with $1/\gamma \notin \Lambda_{L,K}$ and let us compute $i(\gamma L, K_R, K)$. We distinguish two cases:

-) $1/\gamma \in (0, \mu_0) \setminus \sigma(L)$, in this case if (T_n) is a sequence of positive operators such that $T_n(u) = \gamma L(u) + t_n e$ where (t_n) is a sequence of positive real numbers with $\lim t_n = 0$, then we have since the equation

$$u - \gamma L u = t_n e$$

has no solution in $\overline{K_R}$,

$$i(\gamma L, K_R, K) = \lim i(T_n, K_R, K) = 0.$$

=) $1/\gamma \in (\sigma(L) \setminus \Lambda_{L,K}) \cap (0, \mu_0)$, then there is a sequence (γ_n) such that $1/\gamma_n \in (0, \mu_0) \setminus \sigma(L)$ and $\lim \gamma_n = \gamma$; thus, we have

$$i(\gamma L, K_R, K) = \lim i(\gamma_n L, K_R, K) = 0.$$

Reciprocally, suppose that $\Lambda_{L,K} \neq \emptyset$ and let $\gamma > 0$. We have from the above that $i(\gamma L, K_R, K) = 0$ if $1/\gamma \in \left(0, \lambda_{L,K}^+\right) \setminus \Lambda_{L,K}$, so, let us discuss the case $1/\gamma \in$

$\left(\lambda_{L,K}^{+},+\infty\right)$. Assume that for some $\lambda \geq 1$ and $u \in \partial K_R$, $\gamma Lu = \lambda u$. Then λ/γ is a positive eigenvalue of L and we have the contradiction

$$1/\gamma \leq \lambda/\gamma \leq \lambda_{L,K}^{+} < 1/\gamma.$$

Therefore, Lemma 2.9 leads to $i(\gamma L, K_R, K) = 1$. Thus, we have proved that L has the IJP at $\lambda_{L,K}^{+}$ and by uniqueness, we have $\lambda_{L,K}^{+} = \mu_L$, ending the proof

Remark 3.10. Let $L \in L_c^p(E)$ and assume that the cone K is total and $r(L) > 0$. We have from Lemma 1 in [12] that L has the IJP at $r(L)$. Clearly, Theorem 3.9 generalize this lemma to the case where the cone K is not total.

Remark 3.11. Let $L \in L_c^p(E)$, the Schauder index has the jump property (see Corollary 14.6 in [15]) and the jump happens at any eigenvalue of L having an odd algebraic multiplicity. This means that the Schauder index can jump ifintely many times. However, for the fixed point index, the jump can happens at most one time and this happens only at the largest positive eigenvalue of L.

Here below, we will see how the IJP acts and give us the following Krasnosel'skii's minorant principal proved in [15] (see Proposition 7.25).

Corollary 3.12. Assume that $L \in L_c^P(E)$. Then $\Lambda_{L,K} \neq \emptyset$ if and only if $\theta_{L,K}^{-} > 0$ (i.e. there exists $\theta > 0$ and $u \succ 0_E$ such that $Lu \succeq \theta u$).

Proof. Let $\theta_0 > 0$ and $e \succ 0_E$ be such that $Le \succeq \theta_0 e$ and consider the cone

$$K^0 = \{u \in K : Lu \succeq \theta_0 u\}.$$

Since $K^0 \neq \{0_E\}$ and $L(K_0) \subset K_0$, the constants θ_{L,K^0}^{+}, θ_{L,K^0}^{-} are well defined and one can check easily that

$$0 < \theta_0 \leq \theta_{L,K^0}^{+} \leq \theta_{L,K^0}^{-} \leq r(L).$$

Thus, we understand from Lemma 3.1 that L has the IJP on the cone K^0, then we have from Theorem 3.9 that $\sigma_{K^0}(L) \neq \emptyset$. Ending the proof

Now, we need to examinate if the IJP is conserved by limits in the space $L_c^P(E)$.

Proposition 3.13. Let $(L_n) \subset L_c^P(E)$ be such that for all integer n, L_n has the IJP at μ_n and assume that $L_n \to L$ in operator norm. Then either
i) $\lim \mu_n = 0$ or
ii) L has the IJP at some $\mu > 0$.

Proof. First, since $\lim \|L_n\| = \|L\|$, there exists $c > 0$ such that

$$0 < \mu_n \leq \|L_n\| \leq \|L\| + c.$$

Clearly if $\lim \mu_n \neq 0$, the real number $\mu = \limsup \mu_n$ is positive. Assume that is the case and let (μ_{n_k}) be a subsequence of (μ_n) converging to μ. We have from Lemma 2.13 that μ is a positive eigenvalue of L. So, let us compute $i(\gamma L, P_R, P)$ for any $R > 0$ and $\gamma \in (0,+\infty) \setminus \sigma_K(L)$. If $\gamma \in (0, 1/\mu) \setminus \sigma_K(L)$, then there exists $k_0 \in \mathbb{N}$ such that $\gamma < 1/\mu_{n_k}$ for all $k \geq k_0$ and in this case, $i(\gamma L_{n_k}, P_R, P) = 1$ for all $k \geq k_0$ and we have

$$i(\gamma L, P_R, P) = \lim i(\gamma L_{n_k}, P_R, P) = 1.$$

If $\gamma \in (1/\mu, +\infty) \setminus \sigma_K(L)$ then there exists $k_1 \in \mathbb{N}$ such that $\gamma > 1/\mu_{n_k}$ for all $k \geq k_0$ and in this case $i(\gamma L_{n_k}, P_R, P) = 0$ for all $k \geq k_0$ and we have

$$i(\gamma L, P_R, P) = \lim i(\gamma L_{n_k}, P_R, P) = 0.$$

So, L has the IJP at its largest positive eigenvalue μ and this ends the proof

Definition 3.14. An operator $L \in L_c^P(E)$ is said to have the SIJP if $\theta_{L,P}^+ > 0$. In the particular case where $\theta_{L,P}^+ = \theta_{L,P}^+ = \mu > 0$ we say that L has the SIJP at μ.

Remark 3.15. Clearly, If $L \in L_c^P(E)$ has the SIJP, then L has the IJP.

In the following, we present classes of operators in $L_c^P(E)$ having the SIJP and consequently the IJP.

Proposition 3.16. Let $L \in L_c^P(E)$ be strongly positive. Then L has the SIJP at $r(L)$.

Proof. First, we have from Theorem 2.5, Remark 3.3 and Lemma 3.5 that

$$0 \leq \theta_{L,P}^+ \leq r(L) \leq \theta_{L,P}^- \leq r(L)$$

that is $0 < \theta_{L,P}^- = r(L)$.

Now, assume that $\theta_{L,P}^+ < r(L)$ and let $\theta_0 \in \left(\theta_{L,P}^+, r(L)\right)$ and $u_0 \in P \setminus \{0_E\}$ be such that $L(u_0) \leq \theta_0 u_0$. In fact, we have that $L(u_0) < \theta_0 u_0$ indeed, if $L(u_0) = \theta_0 u_0$, then uniqueness in Theorem 2.5 leads to the contradiction $r(L) = \theta_0 < r(L)$. Thus, one has that the equation

$$\lambda u - Lu = y$$

has a positive solution for $\lambda = \theta_0 < r(L)$ and $y = \theta_0 u_0 - Lu_0$, contradicting Proposition 2.6. This completes the proof

Proposition 3.17. Let $L \in L_c^P(E)$ and assume that L is lower bounded on the cone P. Then L has the SIJP.

Proof. Because of Lemmas 3.1 and 3.1, we have to show that $\theta_{L,P}^+ > 0$. Set $c_{L,P} = \inf\{\|Lu\| : u \in \partial P_1\} > 0$ and suppose that there exists sequences (θ_n) and $(u_n) \subset P \setminus \{0_E\}$ with $\lim \theta_n = 0$ and $\|u_n\| = 1$ such that

$$Lu_n \preceq \theta_n u_n. \tag{3.4}$$

Since $\|\theta_n u_n\| = \theta_n$ we have that $\lim \theta_n u_n = 0_E$. Consequently up to a subsequence $\lim Lu_n = 0_E$. So the contradiction

$$0 < c_{L,P} \leq \lim \|Lu_n\| = 0.$$

This shows that $\theta_{L,P}^+ > 0$, ending the proof

Proposition 3.18. Let $L \in L_c^P(E)$ and assume that there exists $L_1 \in L_c^P(E)$ having the SIJP such that $L_1 \preceq L$. Then L has the SIJP and we have $0 < \theta_{L_1,P}^+ \leq \theta_{L,P}^+$ and $\theta_{L_1,P}^- \leq \theta_{L,P}^-$

Proof. Indeed, we have

$$\Theta_P^+(L) \subset \Theta_P^+(L_1) \text{ and } \Theta_P^-(L_1) \subset \Theta_P^-(L).$$

Theorem 3.19. Assume that the operator $L \in \Pi^P$. Then L admits a unique positive eigenvalue at which it has the SIJP.

Proof. Let λ be a positive eigenvalue of L associated with an eingenvector ϕ. If $\theta > 0$ and $u \in P \setminus \{0_E\}$ are such that $Lu \preceq \theta u$, we have then

$$\lambda \chi_L (\phi, u) = \chi_L (L\phi, u) = \chi_L (\phi, Lu) \leq \theta \chi_L (\phi, u)$$

leading to $\lambda \leq \theta$.

Since θ is arbitrary, we have $\lambda \leq \theta^+_{L,P}$.

Similarly, if $\sigma > 0$ and $v \in P \setminus \{0_E\}$ are such that $Lv \succeq \sigma v$, we have then

$$\lambda \chi_L (\phi, u) = \chi_L (L\phi, u) = \chi_L (\phi, Lu) \geq \sigma \chi_L (\phi, u)$$

leading to $\sigma < \lambda$.

Since σ is arbitrary, we have $\lambda \geq \theta^-_{L,P}$.

At the end, taking in acount Remark 3.3, we obtain from the above

$$0 < \lambda \leq \theta^+_{L,P} \leq \lambda \leq \theta^-_{L,P} \leq \lambda$$

that is $\lambda = \theta^+_{L,P} = \theta^-_{L,P} > 0$ and L has the SIJP at its unique positive eigenvalue λ. This ends the proof

Remark 3.20. Clearly if the cone K in Theorem 3.2 is total we have from Theorem 2 that L has the SIJP at $r(L)$.

Corollary 3.21. Assume that $L \in \Pi^P_{lb}$. Then L admits a unique positive eigenvalue at which it has the SIJP.

Proof. We have from Proposition 3.17 that L has the SIJP and $\Lambda_{L,K} \neq \emptyset$. This shows that $\Pi^P_{lb} \subset \Pi^P$ and Theorem 3.19 implies that L admits a unique positive eigenvalue at which it has the SIJP.

Remark 3.22. In fact we have that

$$\Pi^P_{co} \subset \Pi^P_{lb} \subset \Pi^P.$$

Indeed if $L \in \Pi^P_{co}$ then L admits a unique positive eigenvalue λ_L associated with an eigenvalue $\phi \in P \setminus \{0_E\}$ with $\|\phi\| = 1$. Thus, if χ_L is the bilinear form making of L an operator in Π^P_{co}, we have then for any $u \in P$

$$c^{co}_L \lambda_L \|u\| \leq \lambda_L \chi_L (\phi, u) = \chi_L (L\phi, u) = \chi_L (\phi, Lu) \leq c^c_L \|Lu\|$$

leading to

$$\|Lu\| \geq \frac{c^{co}_L \lambda_L}{c^c_L} \|u\|$$

where c^{co}_L and c^c_L are respectively constants of coercivity and continuity of χ_L.

In what remains, we let $\Gamma (E)$ be the class of operators $L \in L^P_c (E)$ such that there exists a sequence of cones (P^n) and an increasing sequence of operators (L_n), such that for all $n \in \mathbb{N}$, $L_n (K) \subset P^n \subset P$, L_n has the SIJP at λ_n and $L_n \to L$ in operator norm.

Clearly, all the above classes of positive operators considered in Propositions 3.16, 3.17, Theorem 3.19, and Corollary 3.21 are contained in $\Gamma(E)$. So, let us prove that operators in $\Gamma(E)$ have also the SIJP.

Theorem 3.23. Assume that $L \in \Gamma(E)$. Then L has the SIJP and $\theta^{+}_{L,P}$ is the unique positive eigenvalue of L (at which it has the IJP). Moreover if the cone K is total then L has the SIJP at $r(L)$.

Proof. Let $(P^n), (L_n)$ and (λ_n) be the sequences making of L an operator in the class $\Gamma(E)$ and let ϕ_n be the normalized eigenvector associated with λ_n.

First, we have that $\Theta^{-}_{P^n}(L_n) = \Theta^{-}_{P}(L_n)$. Indeed; it is obvious that $\Theta^{-}_{P^n}(L_n) \subset \Theta^{-}_{P}(L_n)$ and if $\theta > 0,\ u \in P \setminus \{0_E\}$ are such that $L_n u \succeq \theta u$ then $L_n(u) \in P^n \setminus \{0_E\}$ and $L_n(L_n u) \succeq \theta L_n u$. This shows that $\theta \in \Theta^{-}_{P^n}(L)$ and $\Theta^{-}_{P}(L_n) \subset \Theta^{-}_{P^n}(L_n)$.

Since L_n has the SIJP at λ_n, we have $\lambda_n = \theta^{-}_{L_n,P^n} = \theta^{-}_{L_n,P}$, then from Proposition 3.6 , (λ_n) is a nondecreasing bounded sequence ($\lambda_n \leq \|L\| + C$ for some $C > 0$). Set $\lambda_L = \lim \lambda_n$. We have from Proposition 3.13 that λ_L is the largest positive eigenvalue of L. Also, we have from Proposition 3.6 that

$$\theta^{+}_{L_n,P} \leq \theta^{+}_{L_n,P^n} = \lambda_n \leq \theta^{+}_{L,P}$$

in which letting $n \to \infty$ we get since λ_L is an eigenvalue of L,

$$\theta^{+}_{L,P} \leq \lambda_L = \lim \lambda_n = \lim \theta^{+}_{L_n,P^n} \leq \theta^{+}_{L,P}$$

that is $\lambda_L = \theta^{+}_{L,P}$.

We conclude from all the above that

$$0 < \theta^{+}_{L,P} = \lambda_L \leq \theta^{-}_{L,P} \leq r(L)$$

that is L has the SIJP and $\theta^{+}_{L,P} = \lambda_L$ is the unique positive eigenvalue of L.

Moreover, if the cone K is total then we have from Theorem 2.4 that $r(L)$ is a positive eigenvalue of L and so,

$$0 < \theta^{+}_{L,P} = \lambda_L = \theta^{-}_{L,P} = r(L).$$

This ends the proof

3.3. **Fixed point theorems.** We need first to introduce the following class of operators. Set

$$SIJP(E) = \left\{ L \in L^{P}_{c}(E) : L \text{ has the SIJP} \right\}.$$

Theorem 3.24. Let $T : K \to K$ be a completely continuous mapping and assume that the cone K is normal and there exists three operators $L_1, L_2 \in L^{P}_{c}(E)$, three functions $F_1, F_2, F_3 : K \to K$ and $\gamma > 0$ such that $L_2 \in SIJP(E)$, $\theta^{-}_{L_1,P} < 1 < \theta^{+}_{L_2,P}$ and for all $u \in K$

$$\begin{aligned} Tu &\preceq L_1 u + F_1 u, \\ L_2 u - F_2 u \preceq Tu &\preceq \gamma L_2 u + F_3 u. \end{aligned} \tag{3.5}$$

If either

$$F_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_i u = \circ(\|u\|) \text{ as } u \to \infty,\ i = 2, 3 \tag{3.6}$$

or

$$F_1 u = \circ (\|u\|) \text{ as } u \to \infty \text{ and } F_i u = \circ (\|u\|) \text{ as } u \to 0, \; i = 2, 3, \tag{3.7}$$

then T has a positive fixed point.

Proof. We present the proof in the case where (3.6) holds, the other case is checked similarly. We have to prove existence of $0 < r < R$ such that

$$i(T, P_r, P) = 1 \text{ and } i(T, P_R, P) = 0.$$

In such a situation, additivity and solution properties of the fixed point index imply that

$$i(T, P_R \setminus \overline{P_r}, P) = i(T, P_R, P) - i(T, P_r, P) = -1$$

and T has a positive fixed point u with $r < \|u\| < R$.

Now, consider the function $H_1 : [0, 1] \times K \to K$ defined by $H_1(t, u) = (1 - t)Tu + tL_2 u$ and let us prove existence of $R > 0$ large enough, such that for all $t \in [0, 1]$ equation $H_1(t, u) = u$ has no solution in ∂P_R. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0, 1]$ and $u_n \in \partial P_n$ such that

$$u_n = (1 - t_n)Tu_n + t_n L_2 u_n.$$

Note that $v_n = u_n / \|u_n\| \in \partial P_1$ and satisfies

$$v_n = (1 - t_n) (Tu_n / \|u_n\|) + t_n L_2 v_n.$$

Thus, the inequalities

$$L_2 v_n - (F_2 u_n / \|u_n\|) \preceq (Tu_n / \|u_n\|) \preceq \gamma L_2 v_n + (F_3 u_n / \|u_n\|)$$

combined with the normality of the cone K and the fact that $F_i(u_n) = \circ (\|u_n\|)$ as $n \to \infty$ for $i = 2, 3$, implies that the sequence $(Tu_n / \|u_n\|)$ is bounded. This and because of the compactness of L_2, there exists a subsequence denoted also (v_n) such that $\lim L_2 v_n = v \succ 0_E$ and $v \succeq L_2 v$. Therefore, we have $1 \geq \theta^+_{L_2,P}$, contradicting the hypothesis $\theta^+_{L_2,P} > 1$.

For such a radius $R > 0$, homotopy property of the fixed point index leads to

$$i(T, P_R, P) = i(H_1(0, \cdot), P_R, P) = i(H_1(1, \cdot), P_R, P) = i(L_2, P_R, P) = 0.$$

In similar way, consider the function $H_2 : [0, 1] \times K \to K$ defined by $H_2(t, u) = (1 - t)Tu + tL_1 u$ and let us prove existence of $r > 0$ small enough, such that for all $t \in [0, 1]$ equation $H_2(t, u) = u$ has no solution in ∂P_r. By the contrary suppose that for all integer $n \geq 1$ there exist $t_n \in [0, 1]$ and $u_n \in \partial P_{1/n}$ such that

$$u_n = (1 - t_n)Tu_n + t_n L_1 u_n.$$

Note that $v_n = u_n / \|u_n\| \in \partial P_1$ and satisfies

$$v_n = (1 - t_n) (Tu_n / \|u_n\|) + t_n L_1 v_n.$$

Thus, the inequality

$$(Tu_n / \|u_n\|) \preceq L_1 (v_n) + (F_1 u_n / \|u_n\|)$$

combined with the normality of the cone K and the fact that $F_1(u_n) = \circ (\|u_n\|)$ as $n \to \infty$ implies that the sequence $(F_1 u_n / \|u_n\|)$ is bounded. This and because of the compactness of L_1, there exists a subsequence denoted also (v_n) which converges to $v \in \partial P_1$ satisfying $v \preceq L_1 v$. Therefore, we have$1 \leq \theta^-_{L_1,P}$ contradicting $\theta^-_{L_1,P} < 1$.

For such a radius $r > 0$, homotopy property of the fixed point index leads to

$$i(T, P_r, P) = i(H_2(0, \cdot), P_r, P) = i(H_2(1, \cdot), P_r, P) = i(L, P_r, P) = 1.$$

This completes the proof

Note that in Theorem 3.24, inequalities in (3.5) means that the mapping T is asymptoticaly controled by two operators in $L^P_c(E)$. In the following two theorems, we will omitsuch a control but we will require that the mapping T has asymptoticaly a majorant and a minorant in special classes of operators in $SIJP(E)$.

Theorem 3.25. Let $T : K \to K$ be a completely continuous mapping and assume that there exists two operators $L_1, L_2 \in L^P_c(E)$ and two functions $F_1, F_2 : K \to K$ such that L_1 is lower bounded on P, $\theta^-_{L_2,P} < 1 < \theta^+_{L_1,P}$ and for all $u \in K$

$$L_1 u - F_1 u \preceq Tu \preceq L_2 u + F_2 u.$$

If either

$$F_1 u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to 0 \tag{3.8}$$

or

$$F_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to \infty, \tag{3.9}$$

then T has a positive fixed point.

Proof. We present the proof in the case where (3.8) holds, the other case is checked similarly. As in proof of Theorem 3.24 , we have to prove existence of $0 < r < R$ such that $i(T, P_r, P) = 1$ and $i(T, P_R, P) = 0$.

Consider the function $H_3 : [0, 1] \times K \to K$ defined by $H_3(t, u) = (1 - t)Tu + tL_1 u$ and let us prove existence of $R > 0$ large enough, such that for all $t \in [0, 1]$, equation $H_3(t, u) = u$ has no solution in ∂P_R. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0, 1]$ and $u_n \in \partial P_n$ such that

$$u_n = (1 - t_n)Tu_n + t_n L_1 u_n.$$

Note that $v_n = u_n / \|u_n\| \in \partial P_1$ satisfies

$$v_n = (1 - t_n)(Tu_n / \|u_n\|) + t_n L_1 v_n$$

then

$$L_1 v_n = (1 - t_n)L_1 (Tu_n / \|u_n\|) + t_n L_1 (L_1 v_n). \tag{3.10}$$

Because of the lower boundeness of L_1 we have

$$\|L_1 v_n\| \geq c_{L_1,P} > 0$$

where $c_{L_1,P} = \inf \{\|L_1 u\|, u \in \partial P_1\}$. We distinguish two cases.

Either (t_n) admits a subsequence denoted also (t_n) such that $t_n \to 1$. In this case letting $n \to \infty$ in (3.10), we get from the compacteness of L_1 and the boundeness of $(L_1 v_n)$ that $v = \lim L_1 v_n$ satisfies

$$v = L_1 v \text{ and } \|v\| = \lim \|L_1 v_n\| \geq c_{L_1,P} > 0.$$

This leads to the contradiction

$$1 < \theta^+_{L_1,K} \leq \lambda^-_{L_1,K} \leq 1 \leq \lambda^+_{L_1,K}.$$

Or there exists $\epsilon \in (0,1)$ such that $t_n < 1-\epsilon$ for all $n \in \mathbb{N}$. In this case we have from (3.10)

$$\|Tu_n/\|u_n\|\| \leq (1-t_n)^{-1}(1+t_n\|L_1\|) \leq \epsilon^{-1}(1+\|L_1\|)$$

and the sequence $(Tu_n/\|u_n\|)$ is bounded. As above, $v = \lim L_1 v_n$ satisfies

$$v \succeq L_1 v \text{ and } \|v\| = \lim \|L_1 v_n\| \geq c_{L_1,P} > 0$$

leading to $\theta^+_{L_1,P} \leq 1$ which contradicts the hypothesis $1 < \theta^+_{L_1,P}$.

Thus, there exists $R > 0$ large such that $H_5(t,u) \neq u$ for all $t \in [0,1]$ and $u \in \partial P_R$ and for such a radius $R > 0$, homotopy property of the fixed point index implies that

$$i(T, P_R, P) = i(H_3(0,\cdot), P_R, P) = i(H_3(1,\cdot), P_R, P) = i(L_1, P_R, P) = 0.$$

Arguing as in proof of Theorem 3.24, we prove existence of $r > 0$ small enough, such that $i(T, P_r, P) = 1$ and this completes the proof

Theorem 3.26. Let $T : K \to K$ be a completely continuous mapping and assume that there exists two operators $L_1, L_2 \in L^P_c(E)$ and two functions $F_1, F_2 : K \to K$ such that $L_1 \in \Pi^P_{co}$, $\lambda^+_{L_2,K} < 1 < \lambda_{L_1}$ and for all $u \in K$

$$L_1 u - F_1 u \preceq Tu \preceq L_2 u + F_2 u$$

where λ_{L_1} is the unique positive eigenvalue of L_1.

If either

$$F_1 u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to 0 \tag{3.11}$$

or

$$F_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to \infty, \tag{3.12}$$

then T has a positive fixed point.

Proof. We present the proof in the case where (3.11) holds, the other case is checked similarly. As in proof of Theorem 3.24 , we have to prove existence of $0 < r < R$ such that $i(T, P_r, P) = 1$ and $i(T, P_R, P) = 0$.

Consider the function $H_4 : [0,1] \times K \to K$ defined by $H_4(t,u) = (1-t)Tu + tL_1 u$ and let us prove existence of $R > 0$ large enough, such that for all $t \in [0,1]$ equation $H_4(t,u) = u$ has no solution in ∂P_R. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial K_n$ such that

$$u_n = (1-t_n)Tu_n + t_n L_1 u_n.$$

Note that $v_n = u_n/\|u_n\| \in \partial P_1$ and satisfies

$$v_n = (1-t_n)(Tu_n/\|u_n\|) + t_n L_1 v_n.$$

Thus, if χ_L is the bilinear form making of L_1 an operator in Π^P_{co}, we have

$$\begin{aligned}
\chi_{L_1}(v_n,\phi) &= (1-t_n)\chi_{L_1}(Tu_n/\|u_n\|,\phi) + t_n\chi_{L_1}(L_1 v_n,\phi) \\
&\geq \chi_{L_1}(L_1 v_n,\phi) - t_n\chi_{L_1}(F_1 u_n/\|u_n\|,\phi) \\
&= \lambda_{L_1}\chi_{L_1}(v_n,\phi) - t_n\chi_{L_1}(F_1 u_n/\|u_n\|,\phi)
\end{aligned}$$

leading to

$$(\lambda_{L_1} - 1)\chi_{L_1}(v_n,\phi) \leq t_n\chi_{L_1}(F_1 u_n/\|u_n\|,\phi).$$

The above estimate together with the continuity and the coercivity of χ_{L_1} leads to

$$0 < c^{co}_{L_1} (\lambda_{L_1} - 1) \leq c^{c}_{L_1} t_n \|F_1 (u_n)\| / \|u_n\| \tag{3.13}$$

where $c^{co}_{L_1}$ and $c^{c}_{L_1}$ are respectively constants of coercivity and continuity of χ_{L_1}. Letting $n \to \infty$ in (3.13) we get from Hypothesis (3.11) the contradiction

$$0 < (\lambda_{L_1} - 1) \leq 0.$$

For such a radius $R > 0$, homotopy property of the fixed point index leads to

$$i(T, P_R, P) = i(H_4(0, \cdot), P_R, P) = i(H_4(1, \cdot), P_R, P) = i(L_1, P_R, P) = 0.$$

Arguing as in proof of Theorem 3.24, we prove existence of $r > 0$ small enough, such that $i(T, P_r, P) = 1$ and this completes the proof

3.4. **The particular case of the abstract Hammerstein equation.** In this subsection, we consider the fixed point equation

$$u = LFu \tag{3.14}$$

where $L \in L^P_c(E)$ and the mapping $F : K \to K$ is continuous and bounded (maps bounded sets into bounded sets). Equation (3.14) is known as the Abstarct Hammerstein equation (see Chapter 7 in [15]).

We deduce immediately from Subsection 3.3 the following corollaries.

Corollary 3.27. Assume that the cone K is normal, $L \in SIJP(E)$ and there exists three positive constants α, β, γ and three continuous functions $F_1, F_2, F_3 : K \to K$ such that $\alpha\theta^{-}_{L,P} < 1 < \beta\theta^{+}_{L,P}$ and for all $u \in K$

$$\begin{array}{c} Fu \preceq \alpha u + F_1 u \text{ and} \\ \beta u - F_2 u \preceq Fu \preceq \gamma u + F_3 u. \end{array} .$$

If either

$$F_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_i u = \circ(\|u\|) \text{ as } u \to \infty,\ i = 2, 3 \tag{3.15}$$

or

$$F_1 u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } F_i u = \circ(\|u\|) \text{ as } u \to 0,\ i = 2, 3, \tag{3.16}$$

then Equation (3.14) has a positive solution.

Corollary 3.28. Assume that L is lower bounded on the cone P and there exists two positive constants α, β and two continuous functions $F_1, F_2 : K \to K$ such that $\alpha\theta^{-}_{L,P} < 1 < \beta\theta^{+}_{L,P}$ and

$$\alpha u + F_1 u \preceq Fu \preceq \beta u + F_2 u \text{ for all } u \in K.$$

If either

$$F_1 u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to 0 \tag{3.17}$$

or

$$F_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to \infty, \tag{3.18}$$

then Equation (3.14) has a positive solution.

Corollary 3.29. Assume that $L \in \Pi_{co}^{P}$ and there exists two positive constants α, β and two continuous functions $F_1, F_2 : K \to K$ such that $\alpha\lambda_L < 1 < \beta\lambda_L$ and

$$\alpha u - F_1 u \preceq Fu \preceq \beta u + F_2 u \text{ for all } u \in K.$$

If either

$$F_1 u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to 0 \tag{3.19}$$

or

$$F_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to \infty, \tag{3.20}$$

then Equation (3.14) admits a positive solution.

4. Applications to a fourth order bvp

The purpose of this section is to show how we can use the results given in Section 3. So, we consider here the singular fourth-order bvp

$$\begin{cases} \left(a(t)u'(t)\right)''' = b(t)f(u(t)), \; t \in (0,1) \\ u'(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{4.1}$$

where $a : (0,1) \to [0,+\infty)$ is a measurable function such that $a(t) > 0$ a.e. $t \in (0,1)$ and $\dfrac{1}{a}$ is integrable on any compact of $(0,1]$, $b : (0,1) \to [0,+\infty)$ is a measurable function such that $mes\{t \in (0,1) : b(t) > 0\} > 0$ and b is integrable on any compact of $[0,1)$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

In some results of this section, we will assume that the weights a and b satisfy the following conditions,

$$\lim_{x \to 0} (1/a(x)) \int_0^x b(t)dt = 0 \tag{4.2}$$

and

$$\int_0^1 \left((1/a(t)) \int_0^t b(s)ds\right) dt < \infty. \tag{4.3}$$

Let us introduce some spaces and operators needed for the proof of the main results of this section. In all the following, E denotes the Banach space of all real valued continuous functions defined on $[0,1]$, equipped with the sup-norm $\|u\| = \sup\{|u(t)|, \; 0 \le t \le 1\}$ and K is the cone of nonnegative functions in E. Let $L : E \to E$ and $F : K \to K$ be defined by

$$Lu(x) = \int_x^1 (1/a(t)) \int_0^t \left(b(s) \int_0^1 G(s,\tau)u(\tau)\,d\tau\right) dsdt \text{ and } Fu(x) = f(u(x))$$

where G is the Green's function associated with the operator $-\frac{\partial^2}{\partial x^2}$ and Dirichlet boundary conditions.

It is easy to see that $L \in L_c^K(E)$, F is continuous and bounded and u is a positive solution to bvp (4.1) whenever u is a positive solution to the Hammerstein equation

$$v = LFv. \tag{4.4}$$

We need to introduce the sequence of operators $L_n : E \to E$ defined by

$$L_n u(x) = \int_x^1 \left((1/a_n(t)) \int_0^t \left(b_n(s) \int_0^1 G(s,\tau) u(\tau)\, d\tau \right) ds \right) dt \tag{4.5}$$

where the sequences (a_n) and (b_n) are as follows:

$$a_n(t) = \begin{cases} a(t) \ \text{ if } t \in (\omega_n, 1) \\ \sup(a(t), a(\omega_n)) \ \text{ if } t \in (0, \omega_n), \end{cases}$$

$$b_n(t) = \begin{cases} b(t) \ \text{ if } t \in (0, 1-\omega_n) \\ \inf(b(t), b(1-\omega_n)) \ \text{ if } t \in (1-\omega_n, 1) \end{cases}$$

and the sequence (ω_n) decreases to 0 with $mes\,\{t \in (0, 1-\omega_0) : b(t) > 0\} > 0$.

Define for all $n \in \mathbb{N}$ the cone

$$P^n = \{u \in K : u(t) \geq p_n(t)\|u\| \text{ for all } t \in [0,1]\}$$

where

$$p_n(t) = \frac{1}{\rho_n} \int_t^1 \frac{ds}{a_n(s)} \text{ and } \rho_n = \int_0^1 \frac{ds}{a_n(s)}.$$

Lemma 4.1. The operator $L_n \in \Pi_{co}^{P^n}$ and has the SIJP at $r(L_n)$.

Proof. First let us prove that $L_n(K) \subset P^n$. Let $u \in K$, integrating by parts in (4.5) we get

$$L_n u(x) = \int_0^1 g_n(x,t)\, b_n(t) \left(\int_0^1 G(t,s)\, u(s)\, ds \right) dt \tag{4.6}$$

where

$$g_n(x,t) = \begin{cases} \int_x^1 \ \text{ if } 0 \leq t \leq x \leq 1 \\ \int_t^1 (1/a_n(s))\, ds \text{ if } 0 \leq x \leq t \leq 1. \end{cases}$$

It is easy to see that for all $x, t \in [0,1]$

$$g_n(x,t) \leq g_n(t,t) \tag{4.7}$$

$$g_n(x,t)/g_n(t,t) \geq p_n(x). \tag{4.8}$$

So, inserting (4.7) and (4.8) in (4.6) we get

$$L_n u(x) \geq p_n(x) \int_0^1 g_n(t,t)\, b_n(t) \left(\int_0^1 G(t,s)\, u(s)\, ds \right) dt \geq p_n(x)\, \|L_n u\|$$

and $L_n u \in P^n$.

Let for all $n \in \mathbb{N}$, χ_n be the bilinear form defined on E by

$$\chi_n(u,v) = \int_0^1 b_n(s)\, u(s) v(s) ds.$$

We have for all $u, v \in P^n$,

$$\chi_n(u,v) \geq \int_0^1 b_n(s)\, p_n^2(s)\, \|u\|\, \|v\|\, ds.$$

This shows that χ_n is positive and coercive on P^n. Also, we have for all $u, v \in E$,

$$|\chi_n(u,v)| \leq \int_0^1 b_n(s)\, ds\, \|u\|\, \|v\|$$

and this shows that χ_n is continuous on E.

Now, let $u, v \in P$ and set $U = L_n u,\ V = L_n v$. Integrating by parts, we get from the boundary conditions

$$U'(0) = U(1) = U''(0) = U''(1) = V'(0) = V(1) = V''(0) = V''(1) = 0,$$

that

$$\begin{aligned}
\chi_n (L_n u, v) &= \int_0^1 b_n(t)\, U(t) v(t) dt = \int_0^1 U(t) \left(a_n(t) V'(t)\right)''' dt \\
&= \int_0^1 \left(a_n(t) U'(t)\right)''' V(t) dt = \int_0^1 b_n(t) u(t) V(t) dt \\
&= \chi_n (u, L_n v).
\end{aligned}$$

At the end, since the cone K is total, $r(L_n)$ is the unique positive eigenvalue at which L_n has the SIJP.

Theorem 4.2. Assume that (4.2) and (4.3) hold. Then $r(L)$ is the unique positive eigenvalue of L at which it has the SIJP.

Proof. Since the sequence (L_n) is an increasing sequence of operators having the SIJP, we have to show that $L_n \to L$ in opeartor norm. Let $A : E \to E$ be the operator defined by

$$Au(t) = \int_0^1 G(t, s) u(s) ds$$

and observe that $L_n = B_n A$ and $L = BA$ where $B, B_n : E \to E$ are given by

$$B_n u(x) = \int_x^1 \tfrac{1}{a_n(t)} \left(\int_0^t b_n(s) u(s)\, ds\right) dt$$

and

$$Bu(x) = \int_x^1 (1/a(t)) \left(\int_0^t b(s) u(s)\, ds\right) dt.$$

Thus, we have to show that $B_n \to B$ in operator norm. We have for all $u \in E$ with $\|u\| = 1$

$$\|Bu - B_n u\| = \sup_{x \in [0,1]} |Bu(x) - B_n u(x)|$$

$$\le \int_0^1 (1/a(t)) \left(\int_0^t |b(s) - b_n(s)|\, ds\right) dt + \int_0^1 |(1/a(t)) - (1/a_n(t))| \left(\int_0^t b_n(s) ds\right) dt.$$

Leading to

$$\|B - B_n\| = \sup_{\|u\|=1} \|Bu - B_n u\|$$

$$\le \int_0^1 (1/a(t)) \left(\int_0^t |b(s) - b_n(s)|\, ds\right) dt + \int_0^1 |(1/a(t)) - (1/a_n(t))| \left(\int_0^t b_n(s) ds\right) dt.$$

Thus, taking in consideration Hypotheses (4.2) and (4.3) and definitions of the sequences (a_n) and (b_n), we obtain by Lebesgue dominated convergence theorem that $\|B - B_n\| \to 0$ as $n \to \infty$ and so $L_n \to L$ in operator norm.

At this stage we have proved that the operator L belongs to the class $\Gamma(E)$. Since the cone K is total in E, we deduce from Theorem 3.19 that $r(L)$ is the unique positive eigenvalue of L at which it has the SIJP. This ends the proof

In order to present an existence result for positive solutions to bvp (4.1) let us introduce the following notations. Let for $\nu = 0, +\infty$,

$$f_\nu = \liminf_{u\to\nu} (f(u)/u) \quad f^\nu = \limsup_{u\to\nu} (f(u)/u).$$

Theorem 4.3. Assume that $1/a, b \in L^1[0,1]$. Then Problem (4.1) admits a positive solution if either

$$f^0 r(L) < 1 < f_{+\infty} r(L)$$

or

$$f^{+\infty} r(L) < 1 < f_0 r(L).$$

Proof. First, note that since $1/a, b \in L^1[0,1]$, we understand from Lemma 4 and its proof, that the operator L has the SIJP at $r(L)$.

Now, suppose that $f^0 r(L) < 1 < f_\infty r(L)$ (the other case is checked similarly) and let $\epsilon > 0$ be such that

$$\left(f^0 + \epsilon\right) r(L) < 1 < (f_\infty - \epsilon) r(L).$$

There exists a positive constant C such that for all $x \geq 0$

$$(f_\infty - \epsilon) x - C \leq f(x) \leq \left(f^0 + \epsilon\right) x + f_1(x)$$

where $f_1(x) = \sup\left\{f(x) - \left(f^0 + \epsilon\right) x, 0\right\}$.

Therefore, we have for all $u \in K$

$$\alpha u - F_1 u \preceq Fu \preceq \beta u + F_2 u$$

where $\alpha = (f_\infty - \epsilon)$, $\beta = \left(f^0 + \epsilon\right)$, $F_1 u(t) = C$ and $F_2 u(t) = f_1(u(t))$.

Thus, Theorem 4.3 follows from Corollary 3.29

Remark 4.4. Note that Theorem 4 covers the cases

$$f^0 r(L) < 1 \text{ and } f_\infty = +\infty$$

and

$$f^\infty r(L) < 1 \text{ and } f_0 = +\infty.$$

By similar arguments, we obtain from Corollary 3.27 the following existence results.

Theorem 4.5. Assume that (4.2) and (4.3) hold. Then Problem (4.1) admits a positive solution if either

$$f^0 r(L) < 1 < f_{+\infty} r(L) \leq f^{+\infty} r(L) < \infty$$

or

$$f^{+\infty} r(L) < 1 < f_0 r(L) \leq f^0 r(L) < \infty.$$

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