Fixed Point Theory, 18(2017), No. 1, 69-84
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

APPROXIMATELY p-WRIGHT AFFINE FUNCTIONS, INNER PRODUCT SPACES AND DERIVATIONS

ANNA BAHYRYCZ**, JANUSZ BRZDĘK* AND MAGDALENA PISZCZEK*

*Department of Mathematics, Pedagogical University
Podchorążych 2, 30-084 Kraków, Poland
E-mail: bahyrycz@agh.edu.pl, jbrzdek@up.krakow.pl, magdap@up.krakow.pl

**AGH University of Science and Technology, Faculty of Applied Mathematics
Mickiewicza 30, 30-059 Krakow, Poland

Abstract. We prove a result on hyperstability (in normed spaces) of the equation that defines the p-Wright affine functions and show that it yields a simple characterization of complex inner product spaces. We also obtain in this way some inequalities describing derivations, Lie derivations and Lie homomorphisms.

Key Words and Phrases: Hyperstability, p-Wright affine function, inner product space, derivation, Lie derivation, Lie homomorphism, fixed point theorem.
2010 Mathematics Subject Classification: 16W25, 39B52, 39B62, 39B82, 46C99, 47J99.

1. Introduction

The subsequent theorem seems to be the most classical result concerning stability of the Cauchy equation

$$T(x+y) = T(x) + T(y). \tag{1.1}$$

Theorem 1.1. *Let E_1 and E_2 be two normed spaces, E_2 be complete, $c \geqslant 0$, $s \in \mathbb{R} \setminus \{1\}$, and $f\colon E_1 \to E_2$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant c(\|x\|^s + \|y\|^s), \qquad x, y \in E_1 \setminus \{0\}. \tag{1.2}$$

Then there exists a unique solution $T\colon E_1 \to E_2$ of equation (1.1) *with*

$$\|f(x) - T(x)\| \leqslant \frac{c\|x\|^s}{|1 - 2^{s-1}|}, \qquad x \in E_1 \setminus \{0\}. \tag{1.3}$$

That result is due to D.H. Hyers [24] ($s = 0$), T. Aoki [3] ($0 < s < 1$; cf. [46]), Z. Gajda [21] ($s > 1$) and Th.M. Rassias [47] ($s < 0$). For more information on stability of functional equations we refer to [8, 26, 34, 35, 42]. Let us only mention that the main motivation for the investigation of this issue was given by a problem raised by S.M. Ulam in 1940 and several papers inspired by it that were published in the next few years (see [3, 4, 5, 6, 7, 24, 25, 27, 28, 29]).

Moreover, recently, the following result (improving Theorem 1.1 for $s < 0$) has been proved in [11] (see also [45]).

Theorem 1.2. *Let E_1 and E_2 be two normed spaces, $c \geqslant 0$, $s \in (-\infty, 0)$, and $f\colon E_1 \to E_2$ satisfy* (1.2). *Then f is additive, i.e., it is a solution of equation* (1.1).

A result analogous to Theorem 1.1 has been obtained in [9] for the subsequent functional equation

$$f(px + (1-p)y) + f((1-p)x + py) = f(x) + f(y), \tag{1.4}$$

with a fixed $p \in \mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ ($\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, resp.), for functions f mapping a normed space over $\mathbb{F}$ into a normed space. It reads as follows.

Theorem 1.3. *Let E_1 be a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, E_2 be a Banach space, $p \in \mathbb{F}$, $A, k \in (0, \infty)$, $|p|^k + |1-p|^k < 1$, $g\colon E_1 \to E_2$, and*

$$\begin{aligned} \|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \\ \leqslant A(\|x\|^k + \|y\|^k), \qquad x, y \in E_1. \end{aligned}$$

Then there exists a unique solution $G\colon E_1 \to E_2$ of equation (1.4) *such that*

$$\|g(x) - G(x)\| \leqslant \frac{A\|x\|^k}{1 - |p|^k - |1-p|^k}, \qquad x \in E_1.$$

In this paper we prove a result (see Theorem 3.1) that complements Theorem 1.3 (analogously as Theorem 1.2 improves Theorem 1.1) and show that it yields a simple characterization of complex inner product spaces (see Corollary 3.1). We also show that from Theorem 3.1 we can derive inequalities characterizing derivations, Lie derivations and Lie homomorphisms in algebras and Lie algebras, respectively.

Let us recall (cf. [20]) that, for a fixed number $p \in \mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, a function f mapping a linear space E over $\mathbb{F}$ into a semigroup $(S, +)$ is p-Wright affine if it satisfies functional equation (1.4) (for all $x, y \in E$). This definition of p-Wright affine functions is connected to the notions of p-Wright convexity and p-Wright concavity (see, e.g., [20, 22, 38, 44, 49]) for $S = \mathbb{R} = \mathbb{F}$. Clearly, for $p = 1/2$, equation (1.4) is just the well known Jensen's equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

For

$$p = \frac{e^{i\alpha} + 1}{2} \tag{1.5}$$

(with $\alpha \in \mathbb{R}$) equation (1.4) characterizes norms in the complex inner product spaces (see Theorem 4.1).

2. Auxiliary result

To present an auxiliary (fixed point) result we need to introduce some necessary hypotheses ($\mathbb{R}_+$ stands for the set of nonnegative reals and A^B denotes the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$).

(H1) X is a nonempty set, E_2 is a Banach space, $f_1, \dots, f_k \colon X \to X$ and $L_1, \dots, L_k \colon X \to \mathbb{R}_+$ are given, and $\mathcal{T} \colon {E_2}^X \to {E_2}^X$ is an operator satisfying the inequality

$$\big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\big\| \leqslant \sum_{i=1}^{k} L_i(x)\big\|\xi(f_i(x)) - \mu(f_i(x))\big\|,$$
$$\xi, \mu \in {E_2}^X, x \in X.$$

(H2) $\Lambda \colon {\mathbb{R}_+}^X \to {\mathbb{R}_+}^X$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \qquad \delta \in {\mathbb{R}_+}^X, x \in X.$$

Now we are in a position to present the above mentioned fixed point theorem proved in [12, Theorem 1] (see also [13, Theorem 2] and [17]).

Theorem 2.1. *Let hypotheses* (H1), (H2) *be valid and functions* $\varepsilon \colon X \to \mathbb{R}_+$ *and* $\varphi \colon X \to E_2$ *fulfil the following two conditions*

$$\big\|\mathcal{T}\varphi(x) - \varphi(x)\big\| \leqslant \varepsilon(x), \qquad x \in X,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \qquad x \in X.$$

Then there exists a unique fixed point ψ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \leqslant \varepsilon^*(x), \qquad x \in X.$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n \varphi(x), \qquad x \in X.$$

From now on we assume that $(W, +)$ is a group uniquely divisible by 2, V is a linear space over a field $\mathbb{F}$, $p \in \mathbb{F} \setminus \{0, 1\}$, $\widehat{p} := 1 - p$. So, equation (1.4) can be written in the form

$$g(px + \widehat{p}y) + g(\widehat{p}x + py) = g(x) + g(y). \tag{2.1}$$

First, we prove an auxiliary proposition which describes functions $h \colon V \to W$ satisfying (1.4) for $x, y \in V \setminus \{0\}$. To this end let us recall that a function $h \colon V \to W$ is quadratic provided it is a solution to the functional equation

$$h(x + y) + h(x - y) = 2h(x) + 2h(y).$$

Proposition 2.2. *If a function* $g \colon V \to W$ *satisfies*

$$g(px + \widehat{p}y) + g(\widehat{p}x + py) = g(x) + g(y), \qquad x, y \in V \setminus \{0\}, \tag{2.2}$$

then

$$g(x) = A(x) + B(x) + c, \qquad x \in V, \tag{2.3}$$

with some $c \in W$*, an additive* $A \colon V \to W$ *and a quadratic* $B \colon V \to W$ *fulfilling the condition*

$$B(x) = B((2p - 1)x), \qquad x \in V. \tag{2.4}$$

Conversely, if a function $g\colon V \to W$ has the form (2.3) *with some $c \in W$, an additive $A\colon V \to W$ and a quadratic $B\colon V \to W$ such that* (2.4) *holds, then it satisfies the equation* (2.1) *(for all $x, y \in V$).*

Proof. Assume that g fulfils (2.2). Let $g_e, g_o\colon V \to W$ denote the even and the odd parts of g, respectively, i.e.,

$$g_e(x) := \frac{g(x)+g(-x)}{2}, \qquad g_o(x) := \frac{g(x)-g(-x)}{2}, \qquad x \in V. \tag{2.5}$$

Obviously, g_e and g_o are solution of (2.2), too.

Let $A := g_o$. We show that A is additive. Take $t \in V$ and write

$$s_1 = t - \frac{t}{p}.$$

Then $ps_1 + \widehat{p}t = 0$, whence (2.2) yields

$$\begin{aligned} &A(p(x+t)+\widehat{p}(y+s_1)) + A(\widehat{p}x+py) \\ &\quad = A(x+t)+A(y+s_1), \qquad x,y \in V,\ x \neq -t,\ y \neq -s_1. \end{aligned}$$

Subtracting this and (2.2) gives

$$\begin{aligned} &A(x+t)-A(x)+A(y+s_1)-A(y) \\ &\quad = A(p(x+t)+\widehat{p}(y+s_1)) - A(px+\widehat{p}y), \\ &\qquad x,y \in V\setminus\{0\},\ x \neq -t,\ y \neq -s_1. \end{aligned} \tag{2.6}$$

Next, write $s_2 = pt/\widehat{p}$. Then $\widehat{p}s_2 - pt = 0$, whence (2.6) (with x replaced by $x-t$ and y replaced by $y+s_2$) gives

$$\begin{aligned} &A(x)-A(x-t)+A(y+s_1+s_2)-A(y+s_2) \\ &\quad = A(p(x+t)+\widehat{p}(y+s_1)) - A(px+\widehat{p}y), \\ &\qquad x,y \in V,\ x \neq t,\ x \neq 0,\ y \neq -s_2,\ y \neq -s_1-s_2. \end{aligned}$$

Subtracting this and (2.6) we get

$$\begin{aligned} &A(x)-A(x-t)-A(x+t)+A(x) \\ &\quad = -A(y+s_1+s_2)+A(y+s_2)+A(y+s_1)-A(y) \\ &\qquad x,y \in V\setminus\{0\},\ x \notin \{t,-t\},\ y \notin \{-s_1,-s_2,-s_1-s_2\}. \end{aligned} \tag{2.7}$$

Replacing x by $-x$ in (2.7) we obtain

$$\begin{aligned} &A(-x)-A(-x-t)-A(-x+t)+A(-x) \\ &\quad = -A(y+s_1+s_2)+A(y+s_2)+A(y+s_1)-A(y), \\ &\qquad x,y \in V\setminus\{0\},\ x \notin \{t,-t\},\ y \notin \{-s_1,-s_2,-s_1-s_2\}. \end{aligned}$$

Subtracting this and (2.7) we finally have

$$4A(x)-2A(x-t)-2A(x+t) = 0, \qquad x \in V\setminus\{0,t,-t\}.$$

Thus we have proved that

$$2A(x) = A(x+t)+A(x-t), \qquad x,t \in V,\ x \notin \{0,t,-t\}. \tag{2.8}$$

Further, A is odd, so $A(0) = 0$ and

$$2A(0) = 0 = A(t) - A(t) = A(t) + A(-t), \qquad t \in V.$$

This and (2.8) imply that

$$2A(x) = A(x+t) + A(x-t), \qquad x, t \in V,\ x \notin \{t, -t\}. \tag{2.9}$$

Take $z, w \in V \setminus \{0\}$ and write

$$x = \frac{z+w}{2}, \qquad t = \frac{w-z}{2}.$$

Then $x \neq t$ and $x \neq -t$, whence (2.9) implies that

$$2A\Big(\frac{z+w}{2}\Big) = A(w) + A(z).$$

In this way we have proved that

$$A\Big(\frac{z+w}{2}\Big) = \frac{A(z)+A(w)}{2}, \qquad z, w \in V \setminus \{0\}. \tag{2.10}$$

Fix $z \in V \setminus \{0\}$ and write $V_z := \{az :\ a \in (0, \infty)\}$. Then V_z is a convex set and consequently there exist an additive mapping $A_z \colon V_z \to W$ and a constant $w_z \in W$ such that

$$A(x) = A_z(x) + w_z, \qquad x \in V_z.$$

Take $a \in (0, \infty)$. Then

$$\begin{aligned} A_z(az) + w_z = A(az) = A\Big(\frac{3az - az}{2}\Big) &= \frac{A(3az) - A(az)}{2} \\ &= \frac{A_z(3az) - A_z(az)}{2} = A_z(az), \end{aligned}$$

which means that $w_z = 0$. Hence

$$2A\Big(\frac{1}{2}z\Big) = 2A_z\Big(\frac{1}{2}z\Big) = A_z(z) = A(z).$$

Therefore, in view of (2.10), we obtain that

$$A\Big(\frac{z+w}{2}\Big) = \frac{A(z)+A(w)}{2}, \qquad z, w \in V.$$

This implies that A is additive.

Now we prove that the function

$$B(x) := g_e(x) - g(0), \qquad x \in V$$

is quadratic. First we show that

$$B(px + \widehat{p}y) + B(\widehat{p}x + py) = B(x) + B(y), \qquad x, y \in V. \tag{2.11}$$

Replacing x by $(p-1)x$ and y by px in (2.2) we get

$$g(0) + g((2p-1)x) = g((p-1)x) + g(px), \qquad x \in V. \tag{2.12}$$

Next, setting $y = -x$ in (2.2) we obtain

$$g((2p-1)x) + g((1-2p)x) = g(x) + g(-x), \qquad x \in V.$$

Thus the even part of g satisfies

$$g_e((2p-1)x) = g_e(x), \qquad x \in V.$$

By (2.12) it follows that

$$g_e(0) + g_e(x) = g_e(\widehat{p}x) + g_e(px), \qquad x \in V,$$

whence

$$B(px) + B(\widehat{p}x) = B(x) + B(0),$$

which means that (2.11) holds.

Take $t_1 \in V$ and write $s_1 = t_1 - \frac{t_1}{p}$. Then $ps_1 + \widehat{p}t_1 = 0$, so (2.11) yields

$$\begin{aligned} B(p(x+t_1) + \widehat{p}(y+s_1)) + B(\widehat{p}x + py) \\ = \quad B(x+t_1) + B(y+s_1), \qquad x, y \in V. \end{aligned}$$

Subtracting this and (2.11) gives

$$\begin{aligned} B(x+t_1) - B(x) + B(y+s_1) - B(y) \\ = \quad B(p(x+t_1) + \widehat{p}(y+s_1)) - B(px + \widehat{p}y), \qquad x, y \in V. \end{aligned} \tag{2.13}$$

Next, take $t_2 \in V$ and write $s_2 = -t_2 p/\widehat{p}$. Then $\widehat{p}s_2 + pt_2 = 0$, whence (2.13) (with x replaced by $x + t_2$ and y replaced by $y + s_2$) gives

$$\begin{aligned} B(x+t_1+t_2) - B(x+t_2) + B(y+s_1+s_2) - B(y+s_2) \\ = \quad B(p(x+t_1) + \widehat{p}(y+s_1)) - B(\widehat{p}x + py), \qquad x, y \in V. \end{aligned}$$

Subtracting this and (2.13) we get

$$\begin{aligned} B(x+t_1+t_2) - B(x+t_2) - B(x+t_1) + B(x) \\ = \quad -B(y+s_1+s_2) + B(y+s_2) + B(y+s_1) - B(y), \qquad x, y \in V. \end{aligned} \tag{2.14}$$

Replacing x by $x - t_2$ in (2.14) we obtain

$$\begin{aligned} B(x+t_1) - B(x) - B(x+t_1-t_2) + B(x-t_2) \\ = \quad -B(y+s_1+s_2) + B(y+s_2) + B(y+s_1) - B(y), \qquad x, y \in V. \end{aligned}$$

Subtracting this and (2.14), and taking $x = 0$ we finally have

$$B(t_1) - B(t_1 - t_2) + B(-t_2) - B(t_1 + t_2) + B(t_2) + B(t_1) = 0.$$

Thus we have proved that

$$B(t_1 - t_2) + B(t_1 + t_2) = 2B(t_1) + 2B(t_2), \qquad t_1, t_2 \in V,$$

which means that B is quadratic. Consequently (2.3) holds with $c = g(0)$.

For the proof of (2.4) note that (2.11) holds, and this, with $y = -x$, gives

$$B(px - \widehat{p}x) = B(x), \qquad x \in V \setminus \{0\},$$

which actually is (2.4).

For the proof of the converse, assume that a function $g\colon V \to W$ has the form (2.3) with some $c \in W$, an additive $A\colon V \to W$ and a quadratic $B\colon V \to W$ such that

(2.4) holds. It is well known (see, e.g., [1]) that there exists a biadditive symmetric $L\colon V^2 \to W$ such that $B(x) = L(x,x)$ for $x \in V$. It is easy to check that (2.4) implies

$$L(px, \widehat{p}x) = 0, \qquad x \in V, \tag{2.15}$$

whence by simple calculations we obtain

$$\begin{aligned} B(px + \widehat{p}y) &+ B(\widehat{p}x + py) \\ &= L(px + \widehat{p}y, px + \widehat{p}y) + L(\widehat{p}x + py, \widehat{p}x + py) \\ &= 2L(p(x+y), \widehat{p}(x+y)) + L(px, \widehat{p}x) + L(px, \widehat{p}x) + L(x,x) + L(y,y) \\ &= B(x) + B(y), \qquad x, y \in V. \end{aligned}$$

This implies that g satisfies the equation (2.1) (for all $x, y \in V$). □

Remark 2.3. *In view of Proposition 2.2, it is easily seen that $g\colon V \to W$ fulfils* (2.2) *if and only if g satisfies the equation* (2.1) *(for all $x, y \in V$).*

Remark 2.4. *The proof of Proposition 2.2 is very long. This raises a natural question of a shorter proof of this proposition.*

3. Approximately p-Wright affine functions

In this section we show that a fixed point approach (for some information on this approach see [8, 14, 15, 19]) can be applied to prove a theorem on stability of the equation of p-Wright affine functions (according to the terminology used in [39] (see also [8, 10, 23, 37]), it can be actually called a hyperstability result). Namely, we have the following.

Theorem 3.1. *Let X be a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, Y be a normed space, $p \in \mathbb{F} \setminus \{0, 1, 1/2\}$, $c \geqslant 0$ and $k < 0$. Then every function $g\colon X \to Y$ with*

$$\begin{aligned} \|g(px + \widehat{p}y) + g(\widehat{p}x + py) - g(x) - g(y)\| \\ \leqslant \quad c(\|x\|^k + \|y\|^k), \qquad x, y \in X \setminus \{0\}, \end{aligned} \tag{3.1}$$

is p-Wright affine (i.e., is a solution to (1.4)*).*

Proof. First we notice that without loss of generality we can assume that Y is a Banach space, because otherwise we can replace it by its completion.

Replacing x by $(mp - m + 1)x$ and taking $y = (mp + 1)x$ in (3.1), for $m \in \mathbb{N} \setminus \{1/\widehat{p}, -1/p\}$, we get

$$\begin{aligned} \|g((mp - m + 1)x) + g((mp + 1)x) - g((2mp - m + 1)x) - g(x)\| \\ \leqslant \quad c(|mp - m + 1|^k + |mp + 1|^k)\|x\|^k, \qquad x \in X \setminus \{0\}. \end{aligned} \tag{3.2}$$

Write

$$A_m := c(|mp - m + 1|^k + |mp + 1|^k), \quad \varepsilon_m(x) := A_m\|x\|^k, \qquad x \in X \setminus \{0\},$$

and

$$\mathcal{T}_m\xi(x) := \xi((mp - m + 1)x) + \xi((mp + 1)x) - \xi((2mp - m + 1)x)$$

for $x \in X \setminus \{0\}$, $\xi \in Y^{X \setminus \{0\}}$. Then (3.2) takes the form

$$\|\mathcal{T}_m g(x) - g(x)\| \leqslant \varepsilon_m(x), \qquad x \in X \setminus \{0\}.$$

Let

$$\Lambda_m\eta(x) := \eta((mp-m+1)x) + \eta((mp+1)x) + \eta((2mp-m+1)x)$$

for $\eta \in \mathbb{R}_+{}^{X\setminus\{0\}}$, $x \in X \setminus \{0\}$. Then it is easily seen that Λ_m has the form described in (H2) with $k = 3$ and

$$f_1(x) = (mp-m+1)x, \qquad f_2(x) = (mp+1)x,$$

$$f_3(x) = (2mp-m+1)x, \qquad L_1(x) = L_2(x) = L_3(x) = 1$$

for $x \in X \setminus \{0\}$. Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$, $x \in X \setminus \{0\}$,

$$\begin{aligned}
&\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| \\
&\quad = \|\xi((mp-m+1)x) + \xi((mp+1)x) - \xi((2mp-m+1)x) \\
&\qquad - \mu((mp-m+1)x) - \mu((mp+1)x) + \mu((2mp-m+1)x)\| \\
&\quad \leqslant \|\xi((mp-m+1)x) - \mu((mp-m+1)x)\| \\
&\qquad + \|\xi((mp+1)x) - \mu((mp+1)x)\| \\
&\qquad + \|\xi((2mp-m+1)x) - \mu((2mp-m+1)x)\| \\
&\quad = \sum_{i=1}^{3} \|\xi(f_i(x)) - \mu(f_i(x))\|,
\end{aligned}$$

so (H1) is valid.

Let $m_0 \in \mathbb{N}$ be such that $m_0 > \max\{1/\widehat{p}, -1/p\}$ and

$$|mp-m+1|^{-1} + |mp+1|^{-1} + |2mp-m+1|^{-1} < 1, \qquad m \geqslant m_0.$$

Then

$$\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&= A_m \sum_{n=0}^{\infty} \big(|mp-m+1|^k + |mp+1|^k + |2mp-m+1|^k\big)^n \|x\|^k \\
&= \frac{A_m\|x\|^k}{1 - |mp-m+1|^k - |mp+1|^k - |2mp-m+1|^k}
\end{aligned}$$

for $m \geqslant m_0$ and $x \in X \setminus \{0\}$.

Thus, according to Theorem 2.1, for each $m \geqslant m_0$ there exists a unique solution $G_m\colon X \setminus \{0\} \to Y$ of the equation

$$G_m(x) = G_m((mp-m+1)x) + G_m((mp+1)x) - G_m((2mp-m+1)x)$$

such that

$$\|g(x) - G_m(x)\| \leqslant \frac{A_m\|x\|^k}{1 - |mp-m+1|^k - |mp+1|^k - |2mp-m+1|^k}$$

for $x \in X \setminus \{0\}$. Moreover,

$$G_m(x) := \lim_{n\to\infty} (\mathcal{T}_m^n g)(x), \qquad x \in X \setminus \{0\}.$$

We show that

$$\begin{aligned}&\|\mathcal{T}_m^n g(px+\widehat{p}y)+\mathcal{T}_m^n g(\widehat{p}x+py)-\mathcal{T}_m^n g(x)-\mathcal{T}_m^n g(y)\| \\ &\quad\leqslant\ c(|mp-m+1|^k+|mp+1|^k+|2mp-m+1|^k)^n(\|x\|^k+\|y\|^k)\end{aligned} \tag{3.3}$$

for every $x, y \in X \setminus \{0\}$, $n \in \mathbb{N}_0$.

If $n = 0$, then (3.3) is simply (3.1). So, take $l \in \mathbb{N}_0$ and suppose that (3.3) holds for $n = l$ and $x, y \in X \setminus \{0\}$. Then

$$\begin{aligned}&\big\|\mathcal{T}_m^{l+1} g(px+\widehat{p}y)+\mathcal{T}_m^{l+1} g(\widehat{p}x+py)-\mathcal{T}_m^{l+1} g(x)-\mathcal{T}_m^{l+1} g(y)\big\| \\ &\quad=\ \big\|\mathcal{T}_m^l g((mp-m+1)(px+\widehat{p}y))+\mathcal{T}_m^l g((mp+1)(px+\widehat{p}y)) \\ &\qquad-\mathcal{T}_m^l g((2mp-m+1)(px+\widehat{p}y)) \\ &\qquad+\mathcal{T}_m^l g((mp-m+1)(\widehat{p}x+py))+\mathcal{T}_m^l g((mp+1)(\widehat{p}x+py)) \\ &\qquad-\mathcal{T}_m^l g((2mp-m+1)((\widehat{p}x+py)) \\ &\qquad-\mathcal{T}_m^l g((mp-m+1)x)-\mathcal{T}_m^l g((mp+1)x)+\mathcal{T}_m^l g((2mp-m+1)x) \\ &\qquad-\mathcal{T}_m^l g((mp-m+1)y)-\mathcal{T}_m^l g((mp+1)y)+\mathcal{T}_m^l g((2mp-m+1)y)\big\| \\ &\quad\leqslant\ \big\|\mathcal{T}_m^l g((mp-m+1)(px+\widehat{p}y))+\mathcal{T}_m^l g((mp-m+1)(\widehat{p}x+py)) \\ &\qquad\quad-\mathcal{T}_m^l g((mp-m+1)x)-\mathcal{T}_m^l g((mp-m+1)y)\big\| \\ &\qquad+\big\|\mathcal{T}_m^l g((mp+1)(px+\widehat{p}y))+\mathcal{T}_m^l g((mp+1)(\widehat{p}x+py)) \\ &\qquad\quad-\mathcal{T}_m^l g((mp+1)x)-\mathcal{T}_m^l g((mp+1)y)\big\| \\ &\qquad+\|\mathcal{T}_m^l g((2mp-m+1)(px+\widehat{p}y))+\mathcal{T}_m^l g((2mp-m+1)(\widehat{p}x+py)) \\ &\qquad\quad-\mathcal{T}_m^l g((2mp-m+1)x)-\mathcal{T}_m^l g((2mp-m+1)y)\|\end{aligned}$$

and consequently

$$\begin{aligned}&\big\|\mathcal{T}_m^{l+1} g(px+\widehat{p}y)+\mathcal{T}_m^{l+1} g(\widehat{p}x+py)-\mathcal{T}_m^{l+1} g(x)-\mathcal{T}_m^{l+1} g(y)\big\| \\ &\quad\leqslant c(|mp-m+1|^k+|mp+1|^k+|2mp-m+1|^k)^l \\ &\qquad\times(\|(mp-m+1)x\|^k+\|(mp-m+1)y\|^k) \\ &\qquad+c(|mp-m+1|^k+|mp+1|^k+|2mp-m+1|^k)^l \\ &\qquad\times(\|(mp+1)x\|^k+\|(mp+1)y\|^k) \\ &\qquad+c(|mp-m+1|^k+|mp+1|^k+|2mp-m+1|^k)^l \\ &\qquad\times(\|(2mp-m+1)x\|^k+\|(2mp-m+1)y\|^k) \\ &\quad=c(|mp-m+1|^k+|mp+1|^k \\ &\qquad+|2mp-m+1|^k)^{l+1}(\|x\|^k+\|y\|^k)\end{aligned}$$

for $x, y \in X \setminus \{0\}$.

Letting $n \to \infty$ in (3.3), we obtain that

$$G_m(px+\widehat{p}y)+G_m(\widehat{p}x+py)=G_m(x)+G_m(y), \qquad x, y \in X \setminus \{0\};$$

moreover,

$$\|g(x) - G_m(x)\| \leqslant \frac{c(|mp-m+1|^k + |mp+1|^k)}{1 - |mp-m+1|^k - |mp+1|^k - |2mp-m+1|^k}\|x\|^k$$

for $x \in X \setminus \{0\}$. Hence, with $m \to \infty$, we obtain that (2.2) holds. So, according to Proposition 2.2 and Remark 2.3, g satisfies equation (2.1) (for all $x, y \in X$), which completes the proof. □

4. Characterization of complex inner product spaces

In this part we show that Theorem 3.1 yields a characterization of complex inner product spaces.

Theorem 4.1. *The following three statements are valid.*

(i) *Let V be a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then, for every $p \in \mathbb{F}$ with $|2p-1| \notin \{0,1\}$, $r > 0$ and $k \in (-\infty, 0)$, we have*

$$\sup_{x,y \in V\setminus\{0\}} \frac{\big|\|px + \widehat{p}y\|^r + \|\widehat{p}x + py\|^r - \|x\|^r - \|y\|^r\big|}{\|x\|^k + \|y\|^k} = \infty.$$

(ii) *Assume that V is a complex normed space and there exist $p \in \mathbb{C} \setminus \mathbb{R}$ and $k \in (-\infty, 0)$ such that*

$$\sup_{x,y \in V\setminus\{0\}} \frac{\big|\|px + \widehat{p}y\|^2 + \|\widehat{p}x + py\|^2 - \|x\|^2 - \|y\|^2\big|}{\|x\|^k + \|y\|^k} < \infty. \tag{4.1}$$

Then V is an inner product space and $|2p-1| = 1$.

(iii) *Let V be an inner product space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then*

$$\|px + \widehat{p}y\|^2 + \|\widehat{p}x + py\|^2 = \|x\|^2 + \|y\|^2, \qquad x, y \in V,$$

for every $p \in \mathbb{F}$ with $|2p-1| = 1$.

Proof. Take $p \in \mathbb{F}$ with $|2p-1| \notin \{0,1\}$, $r > 0$ and $k \in (-\infty, 0)$ and suppose that

$$\sup_{x,y \in V\setminus\{0\}} \frac{\big|\|px + \widehat{p}y\|^r + \|\widehat{p}x + py\|^r - \|x\|^r - \|y\|^r\big|}{\|x\|^k + \|y\|^k} < \infty. \tag{4.2}$$

This means that the function $g\colon V \to \mathbb{R}$, $g(x) = \|x\|^r$, satisfies

$$\big|g(px + \widehat{p}y) + g(\widehat{p}x + py) - g(x) - g(y)\big| \leqslant M(\|x\|^k + \|y\|^k), \qquad x, y \in V \setminus \{0\}, \tag{4.3}$$

with some $M > 0$. Next, it is easily seen that $p \notin \{0, 1, 1/2\}$. Consequently, in view of Theorem 3.1,

$$\|px + \widehat{p}y\|^r + \|\widehat{p}x + py\|^r = \|x\|^r + \|y\|^r, \qquad x, y \in V. \tag{4.4}$$

Setting $y = -x$ in (4.4) we get

$$|2p-1|^r \|x\|^r = \|x\|^r, \qquad x \in V,$$

hence $|2p-1| = 1$, which is a contraction.

For the proof of (ii) observe that (4.1) is just condition (4.2) with $r = 2$. Hence

$$\|px + \widehat{p}y\|^2 + \|\widehat{p}x + py\|^2 = \|x\|^2 + \|y\|^2, \qquad x, y \in V, \tag{4.5}$$

and $|2p - 1| = 1$. According to Proposition 2.2, (2.3) holds with some $c \in W$, an additive $A\colon V \to W$, and a quadratic $B\colon V \to W$. From the fact that g is even and $g(0) = 0$, we obtain

$$g(x) = B(x), \qquad x \in V, \tag{4.6}$$

which means that for every $x, y \in V$ we have the parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Consequently V is an inner product space.

It remains to show (iii). So fix $p \in \mathbb{F}$ with $|2p - 1| = 1$. Note that the case $\mathbb{F} = \mathbb{R}$ is trivial, because then $p = 1$ or $p = 0$. So assume that $\mathbb{F} = \mathbb{C}$. Let $\langle x, y\rangle$ denote the inner product of vectors $x, y \in V$. Write

$$b(x, y) = \frac{\langle x, y\rangle + \langle y, x\rangle}{2}, \qquad g(x) := \|x\|^2 = b(x, x), \qquad x, y \in V.$$

Then

$$b(x, y) = \frac{g(x + y) - g(x - y)}{4}, \qquad x, y \in V,$$

whence (with x replaced by px and y by $\widehat{p}x$) we get

$$\begin{aligned} 0 &= (1 - |2p - 1|^2)\|x\|^2 = g(x) - g((2p - 1)x) \\ &= 4b(px, \widehat{p}x), \qquad x \in V. \end{aligned}$$

Consequently, by simple calculations we get

$$\begin{aligned} &g(px + \widehat{p}y) + g(\widehat{p}x + py) - g(x) - g(y) \\ &\quad= 2\big(b(px, y) + b(x, py) - 2b(px, py)\big) \\ &\quad= 2\big(b(px, \widehat{p}y) + b(py, \widehat{p}x)\big) \\ &\quad= 2\big(b(p(x + y), \widehat{p}(x + y))\big) = 0, \qquad x, y \in V. \end{aligned}$$

□

Remark 4.2. *Note that if $p \in \mathbb{R}$, then the condition $|2p-1| = 1$ means that $p \in \{0, 1\}$. Moreover, every $p \in \mathbb{C}$ satisfying the condition $|2p - 1| = 1$ is of the form* (1.5) *with some $\alpha \in \mathbb{R}$.*

5. Some final observations

We end the paper with some examples of simple applications of Theorem 3.1 in characterizations of derivations, Lie derivations and Lie homomorphisms.

Let us start with some auxiliary results. The first one concerns the linearity of additive mappings.

Lemma 5.1. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{A}$ be a linear space over $\mathbb{F}$, Y be a normed space and $g\colon \mathcal{A} \to Y$ be additive. Assume that the following hypothesis holds.*

($\mathcal{H}_1$) *for each $x \in \mathcal{A}$ there is a set $D_x \subset \mathbb{F}$ such that $\operatorname{int}(D_x - D_x) \neq \emptyset$ and g is bounded on the set $D_x x := \{ax : a \in D_x\}$.*

Then g is $\mathbb{F}$-homogenous (i.e., $g(\alpha x) = \alpha g(x)$ for $\alpha \in \mathbb{F}$, $x \in \mathcal{A}$).

Proof. For the proof (which actually is a routine by now) it is enough to note that, for each $x \in \mathcal{A}$, the function $g_x\colon \mathbb{F} \to Y$ such that $g_x(a) = ax$ is additive and bounded on the set D_x, which means that it is continuous and consequently linear (see, e.g., [36]). □

It is well known that $\operatorname{int}(D - D) \neq \emptyset$ if a set $D \subset \mathbb{F}$ (with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$) has a positive inner Lebesgue measure or contains a subset of the second category and with the Baire property (see, e.g., [36]). For some information on related results see [30, 31, 36].

The subsequent lemma follows at once from [18, Theorem 1].

Lemma 5.2. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{A}$ and Y be normed spaces over $\mathbb{F}$, $g\colon \mathcal{A} \to Y$, $g(0) = 0$ and the following hypothesis be fulfilled.*

($\mathcal{H}_2$) *There exists $(u, v) \in \mathbb{R}^2 \setminus \big([1, \infty) \times [0, \infty)\big)$ such that*

$$\sup_{x \in \mathcal{A}\setminus\{0\},\ \alpha \in \mathbb{F}\setminus\{0\}} \frac{\|g(\alpha x) - \alpha g(x)\|}{|\alpha|^u + \|x\|^v} < \infty.$$

Then g is $\mathbb{F}$-homogenous.

The next lemma can be derived from [32, 48].

Lemma 5.3. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{A}$ and Y be normed spaces over $\mathbb{F}$, $g\colon \mathcal{A} \to Y$, $g(0) = 0$ and the following hypothesis be fulfilled.*

($\mathcal{H}_3$) *There exists $(u, v) \in \mathbb{R}^2 \setminus \{(1, 1)\}$ such that*

$$\sup_{x \in \mathcal{A}\setminus\{0\},\ \alpha \in \mathbb{F}\setminus\{0\}} \frac{\|g(\alpha x) - \alpha g(x)\|}{|\alpha|^u \|x\|^v} < \infty.$$

Then g is $\mathbb{F}$-homogenous.

Proof. If $u \neq v$, then the statement follows from [48, Corollary 3] (attention: the assumption $p \neq p_2$ in [48, Corollary 2] should be $p_1 \neq p_2$). The case $u = v \neq 1$ can be deduced from [32, Theorem 1] (with $K(x) \equiv \|x\|^u$, $\delta(\alpha) \equiv |\alpha|^v$, $\psi(\alpha) \equiv |\alpha|^v$ and all ideals being trivial, i.e., equal to $\{\emptyset\}$). □

Let us yet remind that an additive function h, mapping an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$, is a derivation provided that

$$h(xy) = xh(y) + yh(x), \qquad x, y \in \mathcal{A}.$$

Now we are in a position to present the subsequent corollary, which corresponds to some recent results in, e.g., [33, 40, 41, 43].

Corollary 5.4. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{A}$ be a normed algebra over $\mathbb{F}$, $\mathcal{M}$ be a normed $\mathcal{A}$-bimodule, $\mathcal{A}$ has an element e that is not a zero divisor, and $g\colon \mathcal{A} \to \mathcal{M}$. Assume that there exist $p \in \mathbb{F} \setminus \{0, 1, 1/2\}$ and $k, l \in (-\infty, 0)$ such that*

$$\sup_{x,y \in \mathcal{A}\setminus\{0\}} \frac{\|g(px + \widehat{p}y) + g(\widehat{p}x + py) - g(x) - g(y)\|}{\|x\|^k + \|y\|^k} < \infty, \tag{5.1}$$

$$\sup_{x,y\in\mathcal{A}\setminus\{0\}} \frac{\|g(xy)-g(x)y-xg(y)\|}{\|x\|^l\|y\|^l} < \infty. \tag{5.2}$$

Then g is a derivation. Moreover, if one of hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ is valid, then g is linear.

Proof. Clearly (5.1) and (5.2) mean that

$$\begin{aligned}\|g(px+\widehat{p}y)+&g(\widehat{p}x+py)-g(x)-g(y)\|\\ &\leqslant M(\|x\|^k+\|y\|^k), \qquad x,y\in\mathcal{A}\setminus\{0\},\end{aligned}$$

$$\|g(xy)-g(x)y-xg(y)\| \leqslant M\|x\|^l\|y\|^l, \qquad x,y\in\mathcal{A}\setminus\{0\}, \tag{5.3}$$

with some $M>0$. Hence, according to Theorem 3.1, g is p-Wright affine and consequently, by Proposition 2.2, there exist $c\in\mathcal{M}$, an additive $A\colon\mathcal{A}\to\mathcal{M}$ and an quadratic $B\colon\mathcal{A}\to\mathcal{M}$ such that

$$g(z)=B(z)+A(z)+c, \qquad z\in\mathcal{A}. \tag{5.4}$$

Next, (5.3) with x replaced by sx and y replaced by ty ($s,t\in\mathbb{Q}\setminus\{0\}$) and (5.4) give

$$\begin{aligned}\big\|st\,B(xy)+A(xy)-s\,B(x)y-A(x)y-t\,xB(y)-xA(y)-s^{-1}t^{-1}c\big\|\\ \leqslant M|s|^{l-1}|t|^{l-1}\|x\|^l\|y\|^l, \qquad s,t\in\mathbb{Q}\setminus\{0\},\ \ x,y\in\mathcal{A}\setminus\{0\},\end{aligned}$$

whence

$$A(xy)=A(x)y+xA(y), \qquad B(x)y=0, \quad x,y\in\mathcal{A}. \tag{5.5}$$

The second equality in (5.5), with $y=e$, means that $B(x)=0$ for $x\in\mathcal{A}$. Using this and (5.3)-(5.5) we have $\|c\|\leqslant M\|x\|^l\|y\|^l$ for $x,y\in\mathcal{A}\setminus\{0\}$, whence $c=0$, and consequently $g=A$. □

In the rest of this paper $\mathcal{A}$ is a real or complex Lie algebra and $\mathcal{M}$ is an $\mathcal{A}$-bimodule. For all $x\in\mathcal{A}$ and $u\in\mathcal{M}$, the symbols $[x,u]$ and $[u,x]$ denote in $\mathcal{M}$ the commutators $xu-ux$ and $ux-xu$, respectively. Moreover, we say that an additive mapping $d:\mathcal{A}\to\mathcal{M}$ is a Lie derivation provided

$$d([x,y])=[d(x),y]+[x,d(y)], \qquad x,y\in\mathcal{A}.$$

Corollary 5.5. *Let $\mathcal{A}$ be a normed Lie algebra over $\mathbb{F}$, $\mathcal{M}$ be a normed $\mathcal{A}$-bimodule, and $g\colon\mathcal{A}\to\mathcal{M}$. Assume that there exist $p\in\mathbb{Q}\setminus\{0,1,1/2\}$, $k\in(-\infty,0)$, and $l\in\mathbb{R}\setminus\{1\}$ such that (5.1) holds and*

$$\sup_{x,y\in\mathcal{A}\setminus\{0\}} \frac{\|g([x,y])-[g(x),y]-[x,g(y)]\|}{\|x\|^l\|y\|^l} < \infty. \tag{5.6}$$

Then g is a Lie derivation. Moreover, if one of hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ is valid, then g is linear.

Proof. Clearly (5.6) means that

$$\|g([x,y])-[g(x),y]-[x,g(y)]\| \leqslant M\|x\|^l\|y\|^l, \qquad x,y\in\mathcal{A}\setminus\{0\}, \tag{5.7}$$

with some $M > 0$. Next, as in the proof of Corollary 5.4, we obtain that there exist $c \in \mathcal{M}$, an additive $A\colon \mathcal{A} \to \mathcal{M}$ and a quadratic $B\colon \mathcal{A} \to \mathcal{M}$ such that (5.4) holds.

Note that condition (5.7) with $y = x$ yields $\|c\| \leqslant M\|x\|^{2l}$ for $x \in \mathcal{A} \setminus \{0\}$, whence $c = 0$. Hence

$$\begin{aligned} B(px+\widehat{p}y) &+ B(\widehat{p}x + py) \\ &= g(px + \widehat{p}y) + g(\widehat{p}x + py) - A(px + \widehat{p}y) - A(\widehat{p}x + py) \\ &= g(x) + g(y) - A(x) - A(y) = B(x) + B(y), \qquad x, y \in \mathcal{A}. \end{aligned}$$

This, with $y = 0$, implies that

$$p^2 B(x) + \widehat{p}^{\,2} B(x) = B(x), \qquad x \in \mathcal{A},$$

because p is a rational number. Hence $B(x) = 0$ for each $x \in X$, which means that $g = A$. Consequently, by (5.7) with x replaced by tx ($t \in \mathbb{Q} \setminus \{0\}$),

$$\|g([x, y]) - [g(x), y] - [x, g(y)]\| \leqslant M|t|^{l-1}\|x\|^l\|y\|^l, \qquad x, y \in \mathcal{A} \setminus \{0\},\ t \in \mathbb{Q} \setminus \{0\},$$

whence g is a Lie derivation. □

Let $\mathcal{B}$ be a Lie algebra. In what follows we say that an additive mapping $h : \mathcal{A} \to \mathcal{B}$ is a Lie homomorphism if

$$h([x, y]) = [h(x), h(y)], \qquad x, y \in \mathcal{A}.$$

We end this paper with a corollary corresponding to the results in [16].

Corollary 5.6. *Let $\mathcal{A}$ be a normed Lie algebra over $\mathbb{F}$, $\mathcal{B}$ be a normed Lie algebra, and $g\colon \mathcal{A} \to \mathcal{B}$. Assume that there exist $p \in \mathbb{Q} \setminus \{0, 1, 1/2\}$, $k \in (-\infty, 0)$, and $l \in \mathbb{R} \setminus \{1\}$ such that* (5.1) *holds and*

$$\sup_{x,y\in\mathcal{A}\setminus\{0\}} \frac{\|g([x, y]) - [g(x), g(y)]\|}{\|x\|^l\|y\|^l} < \infty. \tag{5.8}$$

Then g is a Lie homomorphism. Moreover, if one of hypotheses ($\mathcal{H}_1$)-($\mathcal{H}_3$) is valid and $\mathcal{B}$ is a complex linear space over $\mathbb{F}$ (i.e., $\mathcal{B}$ is a complex linear space when $\mathbb{F} = \mathbb{C}$), then g is linear.

Proof. Condition (5.8) implies that

$$\|g([x, y]) - [g(x), g(y)]\| \leqslant M\|x\|^l\|y\|^l, \qquad x, y \in \mathcal{A} \setminus \{0\}, \tag{5.9}$$

with some $M > 0$. Analogously as in the proof of Corollary 5.4, we obtain that there exist $c \in \mathcal{B}$, an additive $A\colon \mathcal{A} \to \mathcal{B}$ and a quadratic $B\colon \mathcal{A} \to \mathcal{B}$ such that $g(z) = B(z) + A(z) + c$ for $z \in \mathcal{A}$. Next, as in the proof of Corollary 5.5, we show that $c = 0$ and $B(x) = 0$ for $x \in \mathcal{A}$, which means that g is additive. Consequently, by (5.9) with x replaced by tx ($t \in \mathbb{Q} \setminus \{0\}$),

$$\|g([x, y]) - [g(x), g(y)]\| \leqslant M|t|^{l-1}\|x\|^l\|y\|^l, \qquad x, y \in \mathcal{A} \setminus \{0\},\ t \in \mathbb{Q} \setminus \{0\},$$

whence g is a Lie homomorphism. □

References

[1] J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 1989.

[2] M. Albert, J.A. Baker, *Functions with bounded m-th differences*, Ann. Polon. Math., **43**(1983), 93-103.

[3] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2**(1950), 64-66.

[4] A. Bahyrycz, J. Brzdęk, M. Piszczek, J. Sikorska, *Hyperstability of the Frechet equation and a characterization of inner product spaces*, J. Funct. Spaces Appl., **2013**(2013), Article ID 496361, 6 pages.

[5] A. Bahyrycz, M. Piszczek, *Hyperstability of the Jensen functional equation*, Acta Math. Hungar., **142**(2014), no. 2, 353-365.

[6] D.G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J., **16**(1949), 385-397.

[7] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., **57**(1951), 223-237.

[8] N. Brillouët-Belluot, J. Brzdęk, K. Ciepliński, *On some recent developments in Ulam's type stability*, Abstr. Appl. Anal., **2012**(2012), Article ID 716936, 41 pages.

[9] J. Brzdęk, *Stability of the equation of the p-Wright affine functions*, Aequationes Math., **85**(2013), no. 3, 497-503.

[10] J. Brzdęk, *Remarks on hyperstability of the Cauchy functional equation*, Aequationes Math., **86**(2013), no. 3, 255-267.

[11] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar., **141**(2013), no. 1-2, 58-67.

[12] J. Brzdęk, J. Chudziak, Z. Páles, *A fixed point approach to stability of functional equations*, Nonlinear Anal., **74**(2011), 6728-6732.

[13] J. Brzdęk, K. Ciepliński, *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces*, Nonlinear Anal., **74**(2011), 6861-6867.

[14] J. Brzdęk, L. Cădariu, K. Ciepliński, *Hyperstability and superstability*, Abst. Appl. Anal., **2013**(2013), Article ID 401756, 13 pages.

[15] J. Brzdęk, K. Ciepliński, *Fixed point Ttheory and the Ulam stability*, J. Funct. Spaces, **2014**(2014), Article ID 829419, 16 pages.

[16] J. Brzdęk, A. Fošner, *Remarks on the stability of Lie homomorphisms*, J. Math. Anal. Appl., **400**(2013), 585-596.

[17] L. Cădariu, L. Găvruţa, P. Găvruţa, *Fixed points and generalized Hyers-Ulam stability*, Abstr. Appl. Anal., **2012**(2012), Article ID 712743, 10 pages.

[18] J. Chudziak, *Stability of the homogeneous equation*, Demonstratio Math., **31**(1998), 765-772.

[19] K. Ciepliński, *Applications of fixed point theorems to the Hyers-Ulam stability of functional equations - a survey*, Ann. Funct. Anal., **3**(2012), 151-164.

[20] Z. Daróczy, K. Lajkó, R. L. Lovas, G. Maksa, Z. Páles, *Functional equations involving means*, Acta Math. Hungar., **116**(2007), 79-87.

[21] Z. Gajda, *On stability of additive mappings*, Int. J. Math. Math. Sci., **14**(1991), 431-434.

[22] A. Gilányi, Z. Páles, *On Dinghas-type derivatives and convex functions of higher order*, Real Anal. Exchange, **27**(2001/2002), 485-493.

[23] E. Gselmann, *Hyperstability of a functional equation*, Acta Math. Hungar., **124**(2009), 179-188.

[24] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A., **27**(1941), 222-224.

[25] D.H. Hyers, *Transformations with bounded mth differences*, Pacific J. Math., **11**(1961), 591-602.

[26] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.

[27] D.H. Hyers, S.M. Ulam, *On approximate isometries*, Bull. Amer. Math. Soc., **51**(1945), 288-292.

[28] D.H. Hyers, S.M. Ulam, *Approximate isometries of the space of continuous functions*, Ann. Math., **48**(1947), 285-289.

[29] D. H. Hyers, S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc., **3**(1952), 821-828.

[30] W. Jabłoński, *On a class of sets connected with a convex function*, Abh. Math. Sem. Univ. Hamburg, **69**(1999), 205-210.

[31] W. Jabłoński, *Sum of graphs of continuous functions and boundedness of additive operators*, J. Math. Anal. Appl., **312**(2005), 527-534.

[32] W. Jabłoński, *Stability of homogeneity almost everywhere*, Acta Math. Hungar., **117**(2007), 219-229.

[33] K.W. Jun, D.W. Park, *Almost derivations on the Banach algebra $C^n[0,1]$*, Bull. Korean Math. Soc., **33**(1996), 359-366.

[34] S.M. Jung, *Hyers-Ulam Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Inc., Palm Harbor, FL, 2001.

[35] S.M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Optimization and Its Applications, vol. 48, Springer, New York-Dordrecht-Heidelberg-London, 2011.

[36] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*, Birkhäuser, 2nd edition, 2009.

[37] G. Maksa, *The stability of the entropy of degree alpha*, J. Math. Anal. Appl., **346**(2008), 17-21.

[38] G. Maksa, K. Nikodem, Z. Páles, *Results on t-Wright convexity*, C.R. Math. Rep. Acad. Sci. Canada, **13**(1991), 274-278.

[39] G. Maksa, Z. Páles, *Hyperstability of a class of linear functional equations*, Acta Math. Acad. Pedag. Nyìregyháziensis, **17**(2001), 107-112.

[40] T. Miura, H. Oka, G. Hirasawa, S.E. Takahasi, *Superstability of multipliers and ring derivations on Banach algebras*, Banach J. Math. Anal., **1**(2007), 125-130.

[41] M.S. Moslehian, *Ternary derivations, stability and physical aspects*, Acta Appl. Math., **100**(2008), 187-199.

[42] Z. Moszner, *On the stability of functional equations*, Aequationes Math., **77**(2009), 33-88.

[43] A. Najati, C. Park, *Stability of homomorphisms and generalized derivations on Banach algebras*, J. Inequal. Appl., **2009**(2009), 1-12.

[44] K. Nikodem, Z. Páles, *On approximately Jensen-convex and Wright-convex functions*, C.R. Math. Rep. Acad. Sci. Canada, **23**(2001), 141-147.

[45] M. Piszczek, *Remark on hyperstability of the general linear equation*, Aequationes Math., **88**(2014), 163-168.

[46] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[47] Th.M. Rassias, *On a modified Hyers-Ulam sequence*, J. Math. Anal. Appl., **158**(1991), 106-113.

[48] J. Tabor, Tabor, *Homogeneity is superstable*, Publ. Math. Debrecen, **45**(1994), 123-130.

[49] E.M. Wright, *An inequality for convex functions*, Amer. Math. Monthly, **61**(1954), 620-622.

Received: May 29, 2014; Accepted: March 26, 2015.