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SPLIT FEASIBILITY AND FIXED POINT PROBLEMS FOR ASYMPTOTICALLY k-STRICT PSEUDO-CONTRACTIVE MAPPINGS IN INTERMEDIATE SENSE

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Abstract. This paper deals with the weak convergence of the relaxed extragradient method with regularization for computing a common element of the solution set of split feasibility problem and the fixed points set of a asymptotically k-strict pseudo-contractive mapping in intermediate sense. A numerical example is provided to illustrate the main result of this paper.
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1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator. The *split feasibility problem* (in short, SFP) is to find a point x such that

$$x \in C \quad \text{and} \quad Ax \in Q. \tag{1.1}$$

Throughout the paper, we denote by Γ the solution set of the split feasibility problem (SFP), that is,

$$\Gamma = \{x \in C : Ax \in Q\} = C \cap A^{-1}(Q).$$

During the last decade, the split feasibility problems (in short, SFP) are emerged as models of several problems, namely, signal processing, phase retrievals, image reconstruction, intensity-modulated radiation therapy, etc, see, for example, [1, 3, 4, 5, 6, 10]. Several iterative methods have appeared in the literature to compute the approximate solutions of such problems. For a comprehensive bibliography and survey on split feasibility problems, we refer to [3] and the references therein. Finding the common solution of a split feasibility problem and a fixed point problem is one of the core interests of many researchers, see, for example [3, 7, 8, 9, 11] and the references therein. Recently, Ceng et al. [8] introduced a relaxed extragradient method with regularization for finding a common element of the solution set of SFP and the set $\mathrm{Fix}(T)$ of the fixed points of a nonexpansive mapping T. Very recently, inspired by the work of Ceng et al. [8] and Xu [19], Deepho and Kumam [11] introduced and analyzed a relaxed extragradient method with regularization for finding a common element of the solution set Γ of the split feasibility problem and fixed points set $\mathrm{Fix}(T)$ of an uniformly Lipschitz continuous and asymptotically quasi-nonexpansive mappings in the setting of real Hilbert spaces.

Definition 1.1. Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ be a mapping whose fixed points set is denoted by $\mathrm{Fix}(T)$ and $R(T)$ denotes the range of T. The mapping T is said to be:

(a) *asymptotically quasi-nonexpansive* [16] if there exists a sequence $\{\nu_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \nu_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \nu_n)\|x - p\|, \quad \text{for all } x \in K,\ p \in \mathrm{Fix}(T) \text{ and } n \in \mathbb{N};$$

(b) *asymptotically k-strict pseudo-contractive mapping* [13] with sequence $\{\nu_n\}$ if there exist a constant $k \in [0, 1)$ and a sequence $\{\nu_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \nu_n = 0$ such that

$$\|T^n x - T^n y\|^2 \leq (1 + \nu_n)\|x - y\|^2 + k\|x - T^n x - (y - T^n y)\|^2,$$

for all $x, y \in K$ and $n \in \mathbb{N}$;

(c) *asymptotically k-strict pseudo-contractive mapping in the intermediate sense* [18] with sequence $\{\nu_n\}$ if there exist a constant $k \in [0, 1)$ and a sequence $\{\nu_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \nu_n = 0$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in K} \left(\|T^n x - T^n y\|^2 - (1 + \nu_n)\|x - y\|^2 - k\,\|x - T^n x - (y - T^n y)\|^2 \right) \leq 0. \tag{1.2}$$

Throughout this paper, we assume that

$$c_n := \max\left\{0, \sup_{x,y\in K} \left(\|T^n x - T^n y\|^2 - (1+\nu_n)\|x-y\|^2 - k\,\|x - T^n x - (y - T^n y)\|^2 \right)\right\}. \tag{1.3}$$

Then, $c_n \geq 0$ for all $n \in N$, $c_n \to 0$ as $n \to \infty$ and inequality (1.2) becomes

$$\|T^n x - T^n y\|^2 \leq (1 + \nu_n)\|x - y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 + c_n,$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

When $\mathrm{Fix}(T) \neq \emptyset$, and $\nu_n = 0$, then every asymptotically quasi-nonexpansive mapping T is asymptotically k-strict pseudo-contractive mapping in the intermediate, but the converse is not true. For example, let $K = [0, 1]$ and $T : K \to K$ be a mapping defined by

$$Tx = \frac{x}{2x + 1}, \quad \text{for all } x \in K.$$

Then,

$$T^n x = \frac{x}{2nx+1}, \quad \text{for all } x \in K, \ n \in \mathbb{N},$$

$\mathrm{Fix}(T) \neq \emptyset$ and T is asymptotically k-strict pseudo-contractive in intermediate sense but not asymptotically quasi-nonexpansive. The purpose of this paper is to consider and analyze the relaxed extragradient method with regularization proposed in [11] for finding a common element of Γ and $\mathrm{Fix}(T)$, where T is an asymptotically k-strict pseudo-contractive mapping in intermediate sense. We prove that the sequence generated by the considered algorithm converges weakly to an element of $\mathrm{Fix}(T) \cap \Gamma$. We provide an example to show that our result is applicable, but the result given in [11] is not.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively. We write $x_n \to x$ (respectively, $x_n \rightharpoonup x$) to indicate that the sequence $\{x_n\}$ converges strongly (respectively, weakly) to x. Let K be a nonempty closed convex subset of H. For every $x \in H$, there exists a unique nearest point in K, denoted by $P_K x$ such that

$$\|x - P_K x\| \leq \|x - y\|, \quad \text{for all } y \in K,$$

where P_K is called the *metric projection* of H onto K. It is well known that P_K is a nonexpansive mapping from H onto K. In the following proposition, we collect some known and useful properties of a metric projection which will be used in the sequel.

Proposition 2.1. [12] *For a given $x \in H$ and $z \in K$, we have*

(i) *$z = P_K x$ if and only if $\langle x - z, y - z \rangle \geq 0$, for all $y \in K$;*
(ii) *$z = P_K x$ if and only if $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, for all $y \in K$;*
(iii) *$\langle P_K x - P_K y, x - y \rangle \geq \|P_K x - P_K y\|^2$, for all $y \in H$.*

Let K be a nonempty closed convex subset of a real Hilbert space H and $F : K \to H$ be a mapping. *The variational inequality problem* (VIP) is to find $x \in K$ such that

$$\langle Fx, y - x \rangle \geq 0, \quad \text{for all } y \in K. \tag{2.4}$$

The solution set of VIP is denoted by $\mathrm{VI}(K, F)$. For further details on variational inequalities, we refer to [2] and the references therein. It is well known that

$$x \in \mathrm{VI}(K, F) \quad \Leftrightarrow \quad x = P_K(x - \lambda F x), \quad \text{for all } \lambda > 0.$$

A set-valued mapping $T : H \to 2^H$ is called *monotone* if

$$\langle x - y, f - g \rangle \geq 0, \quad \text{whenever } f \in Tx, \ g \in Ty.$$

It is said to be *maximal monotone* if, in addition, the graph

$$G(T) = \{(x, f) \in H \times H \ : \ f \in Tx\}$$

of T is not properly contained in the graph of any other monotone operator. It is well known that a monotone mapping T is *maximal* if and only if, for $(x, f) \in H \times H$,

$$\langle x - y, f - g \rangle \geq 0 \text{ for every } (y, g) \in G(T) \text{ implies } f \in Tx.$$

Let $F : K \to H$ be a monotone, that is, $\langle Fx - Fy, x - y\rangle \geq 0, \quad \text{for all } x,\ y \in K$, and k-Lipshitz continuous mapping. Let $N_K v$ be the normal cone to K at $v \in K$, that is,

$$N_K v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \text{for all } u \in K\}.$$

Define

$$Tv = \begin{cases} Fv + N_K v, & \text{if } v \in K, \\ \emptyset, & \text{if } v \notin K. \end{cases} \tag{2.5}$$

Then, T is maximal monotone set-valued mapping as proved by Rockafellar [17]. It is well-konwn that if $0 \in T(v)$, then $-Fv \in N_K(v)$, which is further equivalent to the variational inequality (2.4).

Proposition 2.2. [7] *Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator. For given $x^* \in H_1$, the following statement are equivalent.*

(i) *x^* solves* SFP*;*
(ii) *x^* solves fixed point equation $P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*$;*
(iii) *x^* solves variational inequality problem* (VIP) of finding $x^* \in C$ such that

$$\langle \nabla f(x^*), x - x^*\rangle \geq 0, \quad \text{for all } x \in C,$$

where $\nabla f = A^*(I - P_Q)A$, and A^* is the adjoint of A.

We now mention some known results which will be used in the sequel.

Lemma 2.1. *Let H be a real Hilbert space. Then, for all $x,\ y \in H$, we have*

(i) $\|x - y\|^2 \leq \|x\|^2 + \|y\|^2$*;*
(ii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \quad$ *for all* $\lambda \in [0, 1]$.

Lemma 2.2. [18, Demiclosedness Principle] *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a continuous asymptotically k-strict pseudo-contractive mapping in the intermediate sense. Then, $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup\limits_{m\to\infty} \limsup\limits_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$.*

Sahu et al.[18] extended Lemma 2.2 for uniformly continuous mappings and established the following result.

Lemma 2.3. [18, Lemma 2.7] *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically k-strict pseudo-contractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then, $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Lemma 2.4. [14, Lemma 1] *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be the sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \textit{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim\limits_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim\limits_{n\to\infty} a_n = 0$.

3. An Algorithm and a Convergence Result

Very recently, Deepho and Kumam [11] proposed the following algorithm for finding the common element of the solution set Γ of the split feasibility problem and set $\mathrm{Fix}(T)$ of all fixed points of an asymptotically quasi-nonexpansive and Lipschitz continuous mapping in a real Hilbert space.

Algorithm 3.1. Initialization: Take arbitrary $x_0 \in H_1$.
First Step: For a given current $x_n \in H_1$, compute

$$\begin{aligned} y_n &= P_C(I - \lambda_n \nabla f_{\alpha_n})(x_n), \\ x_{n+1} &= (1-\beta_n)x_n + \beta_n T^n(y_n), \quad \text{for all } n \geq 0, \end{aligned} \tag{3.6}$$

where $\nabla f_{\alpha_n} = \nabla f + \alpha_n I = A^*(I - P_Q)A + \alpha_n I$, where I is an identity map and three sequences of parameters $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ satisfies the following conditions:

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$;
(ii) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;
(iii) $\{\beta_n\} \subset [c, d]$ for some $c, d \in (0, 1)$.

Second Step: Update $n := n + 1$.

We establish the following weak convergence result for Algorithm 3.1, where $T : C \to C$ is an uniformly continuous and asymptotically k-strict pseudo-contractive mapping in intermediate sense.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H_1 and $T : C \to C$ be an uniformly continuous and asymptotically k-strict pseudo-contractive mapping in intermediate sense with sequence $\{\nu_n\}$ such that* $\mathrm{Fix}(T) \cap \Gamma \neq \emptyset$ *and $R(T) = C$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by Algorithm 3.1. Assume that the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{\nu_n\}$ satisfy the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$*;*
(ii) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$*;*
(iii) $\{\beta_n\} \subset [d, e]$ *for some* $d, e \in (0, 1)$, $0 < \beta_n < 1 - k < 1$ *and* $\sum_{n=1}^{\infty} \beta_n c_n < \infty$, *where c_n is defined by (1.3);*
(vi) $\sum_{n=1}^{\infty} \nu_n < \infty$*;*
(v) $\{\nabla f_{\alpha_n}(x_n)\}_{n=1}^{\infty}$ *is a bounded sequence.*

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $x^* \in Fix(T) \cap \Gamma$.

Proof. Let $p \in \mathrm{Fix}(T) \cap \Gamma$ be arbitrarily chosen. Then, we have $T(p) = p \in C$ and $Ap \in Q$. Therefore, $P_C(p) = p$ and $P_Q(Ap) = Ap$. Since P_C is nonexpansive, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})(x_n) - P_C(p)\|^2 \\ &\leq \|(x_n - p) - \lambda_n \nabla f_{\alpha_n}(x_n)\|^2 \\ &\leq \|x_n - p\|^2 + \lambda_n^2 \|\nabla f_{\alpha_n}(x_n)\|^2. \end{aligned} \tag{3.7}$$

Since $y_n \in C$ and $T^n y_n \in C$, we have

$$\begin{aligned}\|y_n - T^n y_n\|^2 &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})(x_n) - P_C(T^n y_n)\|^2 \\ &\leq \|(x_n - T^n y_n) - \lambda_n \nabla f_{\alpha_n}(x_n)\|^2 \\ &\leq \|x_n - T^n y_n\|^2 + \lambda_n^2 \|\nabla f_{\alpha_n}(x_n)\|^2. \end{aligned} \tag{3.8}$$

By asymptotically k-strict pseudo-contractiveness in intermediate sense with sequence $\{\nu_n\}$ of T, Lemma 2.1 (ii) and inequalities (3.7) and (3.8), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n y_n - p)\|^2 \\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T^n y_n - p\|^2 - \beta_n(1-\beta_n)\|T^n y_n - x_n\|^2 \\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\{(1+\nu_n)\|y_n - p\|^2 + k\|T^n y_n - y_n\|^2 + c_n\} \\ &\quad - \beta_n(1-\beta_n)\|T^n y_n - x_n\|^2 \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n(1+\nu_n)\{\|x_n - p\|^2 + \lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2\} \\ &\quad + \beta_n k\|x_n - T^n y_n\|^2 + \beta_n k \lambda_n^2 \|\nabla f_{\alpha_n}(x_n)\|^2 \\ &\quad + \beta_n c_n - \beta_n(1-\beta_n)\|T^n y_n - x_n\|^2 \\ &\leq (1+\beta_n\nu_n)\|x_n - p\|^2 + \beta_n c_n - \beta_n(1-\beta_n-k)\|x_n - T^n y\|^2 \\ &\quad + \beta_n\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2 + \beta_n\nu_n\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2 + k\beta_n\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2 \\ &\leq (1+\beta_n\nu_n)\|x_n - p\|^2 - \beta_n(1-\beta_n-k)\|x_n - T^n y\|^2 \\ &\quad + \beta_n\{c_n + (1+\nu_n)\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2 + k\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2\} \\ &\leq (1+\beta_n\nu_n)\|x_n - p\|^2 - \beta_n(1-\beta_n-k)\|x_n - T^n y\|^2 \qquad (3.9) \\ &\quad + \beta_n\{c_n + (1+\nu_n)\lambda_n^2 M + k\lambda_n^2 M\} \\ &\leq (1+\nu_n)\|x_n - p\|^2 + \beta_n\{c_n + (1+\nu_n)\lambda_n^2 M + k\lambda_n^2 M\}. \end{aligned}$$

Thus, we have

$$\|x_{n+1} - p\|^2 \leq (1+\nu_n)\|x_n - p\|^2 + b_n,$$

where $b_n = \beta_n\{c_n + (1+\nu_n)\lambda_n^2 M + \lambda_n^2 M\}$. Since $\sum_{n=1}^{\infty} \nu_n < \infty$, $0 < \beta_n < 1$, $\sum_{n=1}^{\infty} \beta_n c_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, $0 \leq k < 1$ and M is a constant, we conclude that $\sum_{n=1}^{\infty} b_n < \infty$. Therefore, by Lemma 2.4, we have

$$\lim_{n\to\infty} \|x_n - p\| \quad \text{exists.}$$

Also, from (3.7), we have

$$\lim_{n\to\infty} \|y_n - p\| \quad \text{exists.}$$

Thus, from (3.9), we obtain

$$\begin{aligned}\beta_n(1-\beta_n-k)\|T^n y_n - x_n\|^2 &\leq (1+\beta_n\nu_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\qquad + \beta_n\{c_n + (1+\nu_n)\lambda_n^2 M + k\lambda_n^2 M\}.\end{aligned}$$

Since T is asymptotically k-strict pseudo-contractive mapping in the intermediate with sequence $\{\nu_n\}$, then $\lim_{n\to\infty} \nu_n = 0$, and by the conditions (ii) and (iii), we have

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0. \tag{3.10}$$

By using condition (ii), (3.10) and (3.8), we obtain

$$\lim_{n\to\infty} \|T^n y_n - y_n\| = 0. \tag{3.11}$$

From Algorithm 3.1 and (3.10), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \beta_n \|T^n y_n - x_n\| \to 0. \tag{3.12}$$

Since $y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n))$ and by Proposition 2.1 (ii), as in [11], we have

$$\begin{aligned}\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n \alpha_n \|p\|\|p - x_n\| \\ + 2\lambda_n \|\nabla f_{\alpha_n}(x_n)\|\|y_n - p\| + 2\lambda_n \|\nabla f_{\alpha_n}(x_n)\|\|x_n - p\|.\end{aligned} \tag{3.13}$$

Consequently, by asymptotically k-strict pseudo-contractiveness in intermediate sense with sequence $\{\nu_n\}$ of T, utilizing Lemma 2.1 (ii) and inequality (3.13), we conclude that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n y_n - p)\|^2 \\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n \|T^n y_n - p\|^2 - \beta_n(1-\beta_n)\|T^n y_n - x_n\|^2 \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\{(1+\nu_n)\|y_n - p\|^2 + k\|y_n - T^n y_n\| + c_n\} \\ &- \beta_n(1-\beta_n)\|T^n y_n - x_n\|^2. \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n(1+\nu_n)\{\|x_n - p\|^2 - \|x_n - y_n\|^2 \\ &+ 2\lambda_n \alpha_n \|p\|\|p - x_n\| + 2\lambda_n \|\nabla f_{\alpha_n}(x_n)\|\|y_n - p\| \\ &+ 2\lambda_n \|\nabla f_{\alpha_n}(x_n)\|\|x_n - p\|\} + \beta_n k \|y_n - T^n y_n\|^2 + \beta_n c_n \\ &- \beta_n(1-\beta_n)\|T^n y_n - x_n\|^2.\end{aligned}$$

Taking limits both the sides and using the conditions (i)-(iv) and equation (3.10), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.14}$$

From Algorithm 3.1, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|P_C(I - \lambda_{n+1}\nabla f_{\alpha_{n+1}})(x_{n+1}) - P_C(I - \lambda_n \nabla f_{\alpha_n})(x_n)\| \\ &\leq \|x_{n+1} - x_n\| + \lambda_{n+1}\|\nabla f_{\alpha_{n+1}}(x_{n+1})\|^2 + \lambda_n \|\nabla f_{\alpha_n}(x_n)\|^2.\end{aligned}$$

Taking limits both the sides and using condition (ii) and (3.12), we have

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.15}$$

Since $\|y_{n+1} - y_n\| \to 0$, $\|T^n y_n - y_n\| \to 0$ as $n \to \infty$ and T is uniformly continuous, we obtain from Lemma 2.3 that $\|Ty_n - y_n\| \to 0$ as $n \to \infty$. Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some x^*. In fact, $x_n \rightharpoonup x^*$.

Indeed, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$. Assume $x^* \neq \bar{x}$. From Opial condition [15], we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n - x^*\| &= \liminf_{i\to\infty}\|x_{n_i} - x^*\| < \liminf_{n\to\infty}\|x_{n_i} - \bar{x}\| \\ &= \lim_{n\to\infty}\|x_n - \bar{x}\| = \lim_{j\to\infty}\|x_{n_j} - \bar{x}\| \\ &< \lim_{j\to\infty}\|x_{n_j} - x^*\| = \lim_{n\to\infty}\|x_n - x^*\|.\end{aligned}$$

This contradict to our assumption $x^* \neq \bar{x}$. Hence, $x_{n_j} \rightharpoonup x^*$. This shows that every subsequence of $\{x_n\}$ converges weakly to x^*. This implies that $x_n \rightharpoonup x^*$, and for all $f \in H$, we have $f(x_n) \to f(x^*)$. Next we show that $y_n \rightharpoonup x^*$. For all $f \in H$, we consider

$$\begin{aligned}\|f(y_n) - f(x^*)\| &= \|f(y_n) - f(x_n) + f(x_n) - f(x^*)\| \\ &\leq \|f(y_n) - f(x_n)\| + \|f(x_n) - f(x^*)\| \\ &\leq \|f\|\|y_n - x_n\| + \|f(x_n) - f(x^*)\|.\end{aligned}$$

From (3.14), we conclude that $\lim_{n\to\infty}\|f(y_n) - f(x^*)\| = 0$, for all $f \in H$. Hence, $y_n \rightharpoonup x^*$. Note that T is uniformly continuous and $\|Ty_n - y_n\| \to 0$, we see that $\|y_n - T^m y_n\| \to 0$ for all $m \in \mathbb{N}$. Thus, by Lemma 2.2, we obtain that $x^* \in \mathrm{Fix}(T)$. Now we show that $x^* \in \Gamma$. Let

$$Sw_1 = \begin{cases}\lambda_n \nabla f w_1 + N_C w_1, & \text{if } w_1 \in C, \\ \emptyset, & \text{if } w_1 \notin C,\end{cases} \tag{3.16}$$

where $N_C w_1 = \{z \in H_1 : \langle w_1 - u, z\rangle \geq 0 \text{ for all } u \in C\}$. To show that $x^* \in \Gamma$, it is sufficient to show that $0 \in Sx^*$. Let $(w_1, z) \in G(S)$, we have $z \in Sw_1 = \lambda_n \nabla f w_1 + N_C w_1$, and hence, $z - \lambda_n \nabla f w_1 \in N_C w_1$. So, we have $\langle w_1 - u, z - \lambda_n \nabla f w_1\rangle \geq 0$ for all $u \in C$. Since $w_1 \in C$, from Algorithm 3.1, we have $y_n = P_C(I - \lambda_n \nabla f \alpha_n)x_n$, and then from Proposition 2.2 (i), we have

$$\langle x_n - \lambda_n \nabla f \alpha_n x_n - y_n, y_n - w_1\rangle \geq 0,$$

and

$$\langle w_1 - y_n, y_n - x_n + \lambda_n \nabla f \alpha_n x_n\rangle \geq 0.$$

Since $z - \lambda_n \nabla f w_1 \in N_C w_1$ and $y_{n_i} \in C$, it follows that

$$\begin{aligned}\langle w_1 - y_{n_i}, z\rangle &\geq \langle w_1 - y_{n_i}, \lambda_{n_i} \nabla f w_1\rangle \\ &\geq \langle w_1 - y_{n_i}, \lambda_{n_i} \nabla f w_1\rangle - \langle w_1 - y_{n_i}, y_{n_i} - x_{n_i} + \lambda_{n_i} \nabla f \alpha_{n_i} x_{n_i}\rangle \\ &\geq \langle w_1 - y_{n_i}, \lambda_{n_i} \nabla f w_1\rangle - \langle w_1 - y_{n_i}, y_{n_i} - x_{n_i} + \lambda_{n_i} \nabla f x_{n_i}\rangle \\ &- \lambda_{n_i}\alpha_{n_i}\langle w_1 - y_{n_i}, x_{n_i}\rangle \\ &= \langle w_1 - y_{n_i}, \lambda_{n_i} \nabla f w_1 - \lambda_{n_i} \nabla f y_{n_i}\rangle \\ &+ \langle w_1 - y_{n_i}, \lambda_{n_i} \nabla f y_{n_i} - \lambda_{n_i} \nabla f x_{n_i}\rangle \\ &- \langle w_1 - y_{n_i}, y_{n_i} - x_{n_i}\rangle - \lambda_{n_i}\alpha_{n_i}\langle w_1 - y_{n_i}, x_{n_i}\rangle \\ &\geq \langle w_1 - y_{n_i}, \lambda_{n_i} \nabla f y_{n_i} - \lambda_{n_i} \nabla f x_{n_i}\rangle - \langle w_1 - y_{n_i}, y_{n_i} - x_{n_i}\rangle \\ &- \lambda_{n_i}\alpha_{n_i}\langle w_1 - y_{n_i}, x_{n_i}\rangle.\end{aligned}$$

Taking limit as $i \to \infty$, we obtain

$$\langle w_1 - x^*, z\rangle \geq 0, \quad \text{as} i \to \infty.$$

Since $\langle w_1 - x^*, z - 0\rangle \geq 0$ for every $(w_1, z) \in G(S)$, therefore the maximality of S implies that $0 \in Sx^*$. Thus, we have $x^* \in \mathrm{VI}(C, \nabla f)$. Finally, Proposition 2.2 implies that $x^* \in \Gamma$.

4. A Numerical Example

To illustrate Algorithm 3.1 and Theorem 3.1, we present the following example.

Example 4.1. Let $C = Q = [0, 1]$ be a closed convex subset of $\mathbb{R}$ and $T : C \to C$ be defined by

$$Tx = \frac{x}{2x+1}, \quad \text{for all } x \in C.$$

Then, T is asymptotically k-strict pseudo-contractive mapping in intermediate sense, and

$$T^n x = \frac{x}{2xn+1}, \text{ for all } n \in \mathbb{N}.$$

Let $A(x) = 2x$, for all $x \in C$, be a bounded linear operator. Let $\alpha_n = \frac{1}{n^2}$, $\beta_n = \frac{1}{9n}$ and $\lambda_n = \frac{1}{9n^2}$ be the sequences of parameters. Now, we show that T is asymptotically k-strict pseudo-contractive mapping in intermediate sense with sequence $\{\nu_n\}$.

By the definition of asymptotically k-strict pseudo-contractive mapping in intermediate sense with sequence $\{\nu_n\}$, we have

$$|T^n x - T^n y|^2 \leq (1 + \nu_n)|x - y|^2 + k|(x - T^n x) - (y - T^n y)|^2 + c_n, \tag{4.17}$$

where $\{\nu_n\} \subseteq [0, \infty)$ with $\lim_{n\to\infty} \nu_n = 0$, $k \in [0, 1)$. If we take $1 + \nu_n = k_n$, then $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$, and thus (4.17) becomes

$$|T^n x - T^n y|^2 \leq k_n|x - y|^2 + k|(x - T^n x) - (y - T^n y)|^2 + c_n, \tag{4.18}$$

where

$$c_n = \max\left\{0, \sup_{x,y\in[0,1]} \left\{|T^n x - T^n y|^2 - k_n|x - y|^2 - k|(x - T^n x) - (y - T^n y)|^2\right\}\right\}.$$

Indeed

$$\begin{aligned}
|T^n x - T^n y|^2 &= \left|\frac{x}{2nx+1} - \frac{y}{2ny+1}\right|^2 \\
&= \frac{|2nxy + x - 2nxy - y|^2}{(2nx+1)^2(2ny+1)^2} \\
&= \frac{|x - y|^2}{(2nx+1)^2(2ny+1)^2}.
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
k|(x-T^nx)-(y-T^ny)|^2 &= k\left|(x-\frac{x}{2nx+1})-(y-\frac{y}{2ny+1})\right|^2\\
&= k\frac{|2nx^2-2ny^2|^2}{(2nx+1)^2(2ny+1)^2}\\
&= \frac{k4n^2|x-y|^2|x+y|^2}{(2nx+1)^2(2ny+1)^2}\\
&\leq \frac{16n^2k|x-y|^2}{(2nx+1)^2(2ny+1)^2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&k_n|x-y|^2+k|(x-T^nx)-(y-T^ny)|^2+c_n\\
&= k_n|x-y|^2+\frac{16n^2k|x-y|^2}{(2nx+1)^2(2ny+1)^2}\\
&= \left\{k_n+\frac{16n^2k}{(2nx+1)^2(2ny+1)^2}\right\}|x-y|^2\\
&= \left\{(2nx+1)^2(2ny+1)^2k_n+16n^2k\right\}\frac{|x-y|^2}{(2nx+1)^2(2ny+1)^2}.
\end{aligned} \tag{4.20}$$

For all $x\in[0,1]$ and all $n\in\mathbb{N}$, we have

$$\begin{aligned}
(2nx+1)^2(2ny+1)^2\geq 1 &\Leftrightarrow (2nx+1)^2(2ny+1)^2k_n\geq 1\\
&\Leftrightarrow (2nx+1)^2(2ny+1)^2k_n+16n^2k\geq 1.
\end{aligned}$$

Thus we have,

$$(2nx+1)^2(2ny+1)^2k_n+16n^2k\geq 1. \tag{4.21}$$

By combining (4.19) and (4.19), and using (4.21), we have,

$$\begin{aligned}
&|T^nx-T^ny|^2-k_n|x-y|^2-k|(x-T^nx)-(y-T^ny)|^2\\
&= \frac{|x-y|^2}{(2nx+1)^2(2ny+1)^2}-\left\{(2nx+1)^2(2ny+1)^2k_n+16n^2k\right\}\frac{|x-y|^2}{(2nx+1)^2(2ny+1)^2}\\
&= \left(1-\{(2nx+1)^2(2ny+1)^2k_n+16n^2k\}\right)\frac{|x-y|^2}{(2nx+1)^2(2ny+1)^2}\leq 0.
\end{aligned}$$

This implies that

$$\sup_{x\in[0,1]}\{|T^nx-T^ny|^2-k_n|x-y|^2-k|(x-T^nx)-(y-T^ny)|^2\}=0.$$

So from the definition of c_n, we have $c_n=0$. Also from (4.21), we have,

$$\frac{|x-y|^2}{(2nx+1)^2(2ny+1)^2}\leq\{(2nx+1)^2(2ny+1)^2k_n+16n^2k\}\frac{|x-y|^2}{(2nx+1)^2(2ny+1)^2}.$$

Thus,

$$|T^nx-T^ny|^2\leq k_n|x-y|^2+k|(x-T^nx)-(y-T^ny)|^2+c_n.$$

This implies that T is asymptotically k-strict pseudo-contractive mapping in intermediate sense with sequence $\{k_n\}$. The sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 3.1 starting with $x_1 = 1$ converges to $0 \in \mathrm{Fix}(T) \cap \Gamma$.

Next, we show that T is not asymptotically quasi-nonexpansive. By the definition of asymptotically quasi-nonexpansive, we have

$$|T^n x - p| \leq k_n |x - p|, \quad \text{for all } x \in [0,1] \text{ and all } p \in \mathrm{Fix}(T), \tag{4.22}$$

where $k_n \geq 1$ and $\lim\limits_{n\to\infty} k_n = 1$. For all $x \in [0,1]$ and all $n \in \mathbb{N}$, we have

$$\begin{aligned} x < 2nx + 1 &\Rightarrow \frac{x}{2nx+1} < 1 \\ &\Rightarrow \left|\frac{x}{2nx+1}\right| < 1, \quad \text{for all } \ x \in [0,1] \\ &\Rightarrow |T^n x| < 1 \\ &\Rightarrow |T^n x| \not< k_n |x|. \end{aligned}$$

Since $0 \in \mathrm{Fix}(T)$, we have $|T^n x - 0| \not< k_n |x - 0|$, that is, (4.22) does not hold. Hence, T is not asymptotically quasi-nonexpansive.

Remark 4.1. Since T is not asymptotically quasi-nonexpansive, [11, Theorem 3.1] is not applicable in this case.

Now we show the convergence of the Algorithm 3.1 with the help of Matlab Programming.

We did the computation in Matlab R2010 and got the solution 0.

Convergence Table

No. of Iterations (n)	y_n	x_n	No. of Iterations (n)	y_n	x_n
1	.8889	1.0000	18	.0083	.0093
2	.8217	.9244	19	.0062	.0070
3	.7005	.7881	20	.0046	.0052
4	.5535	.6226	21	.0034	.0038
5	.4113	.4627	22	.0025	.0028
6	.2942	.3310	23	.0018	.0020
7	.2078	.2338	24	.0013	.0014
8	.1477	.1661	25	.0009	.0010
9	.1065	.1198	26	.0006	.0007
10	.0780	.0878	27	.0004	.0005
11	.0580	.0652	28	.0003	.0003
12	.0436	.0490	29	.0002	.0002
13	.0330	.0371	30	.0001	.0001
14	.0251	.0282	31	.0001	.0001
15	.0191	.0215	32	.0000	.0000
16	.0145	.0163	33	.0000	.0000
17	.0110	.0124	34	.0000	.0000

References

[1] Q.H. Ansari, *Topics in Nonlinear Analysis and Optimization*, World Education, Delhi, 2012.

[2] Q.H. Ansari, C.S. Lalitha, M. Mehta, *Generalized Convexity, Nonsmooth Variational Inequalities, and Nonsmooth Optimization*, CRC Press, Taylor & Francis Group, Boca Raton, London, New York, 2014.

[3] Q.H. Ansari, A. Rehan, *Split feasibility and fixed point problems*, Nonlinear Analysis: Approximation Theory, Optimization and Applications, (Ed. Q.H. Ansari), Birkhäuser, Springer, New Delhi, Heidelberg, New York, London, 2014, 281-322.

[4] C. Byrne, *Block iterative methods for image reconstruction form projections*, IEEE Trans. Image. Process., **5**(1996), 96-103.

[5] C. Byrne, *Iterative oblique projection onto convex subsets and the split feasibility problem*, Inverse Probl., **18**(2002), 441-453.

[6] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl., **20**(2004), 103-120.

[7] L.-C. Ceng, Q.H. Ansari, J.-C. Yao, *An extragradient method for solving split feasibility and fixed point problems*, Computat. Math. Appl., **64**(2012), 633-642.

[8] L.-C. Ceng, Q.H. Ansari, J.-C. Yao, *Relaxed extragradient method for solving split feasibility and fixed point problem*, Nonlinear Anal., **75**(2012), 2116-2125.

[9] L.-C. Ceng, Q.H. Ansari, J.-C. Yao, *Mann type iterative methods for finding a common solution of split feasibility and fixed point problems*, Positivity, **16**(2012), 471-495.

[10] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol., **51**(2006), 2353-2365.

[11] J. Deepho, P. Kumam, *Split feasibility and fixed-point problems for asymptotically quasi-nonexpansive*, Fixed Point Theory Appl., **2013**(2013), Article ID 322.

[12] K. Geobel, W.A. Kirk, *Topics in Metric Fixed Point Theory*. Cambridge Studies in Advanced Mathematics, **28**, Cambridge University Press, Cambridge, 1990.

[13] T.H. Kim, H.-K. Xu, *Strong convergence of modified Mann iterations for asymptotically strict pseudo-contraction*, Nonlinear Anal., **68**(2008), 2828-2836.

[14] M.O. Osilike, S.C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Math. Computat. Model., **32**(2000), 1181-1191.

[15] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73**(1967), 591-597.

[16] L. Qihou, *Iterative sequence for asymptotically quasi-nonexpanslve mappings*, J. Math. Anal. Appl., **259**(2001), 1-7.

[17] R.T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., **149**(1970), 75-88.

[18] D.R. Sahu, H.-K. Xu, J.-C. Yao, *Asymptotically strict pseudo-contractive mappings in the intermediate sense*, Nonlinear Anal., **70**(2009), 3502-3511,

[19] H.-K. Xu, *Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces*, Inverse Probl., **26**(2010), Article ID 105018, 17 pp.

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