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ON A FIXED POINT THEOREM IN UNIFORM SPACES AND ITS APPLICATION TO NONLINEAR VOLTERRA TYPE OPERATORS

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Abstract. In the present work, we prove a fixed point theorem for nonlinear operators acting in Hausdorff sequentially complete uniform spaces whose uniformity is generated by a saturated family of pseudometrics. As an application we consider nonlinear abstract Volterra type integral equations of second kind in the case when the independent variable belongs to arbitrary completely regular Hausdorff space. Existence and uniqueness of the solutions of these equations in nonhomogeneous case are also proved.
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1. Introduction

A lot of results are obtained in the last few decades devoted to integral equations and its applications (cf. [2], [3], [5], [12] and the references there in). Consumer of this results are several applied fields, such as population dynamics, spread of epidemics, automatic control theory, network theory and the dynamics of nuclear reactors. In view of the applications it is important to study equations and inequalities in the case when the domains of integration are not necessary Cartesian product of bounded and closed intervals. Following this concept it is obviously that the first step is the problem of the solvability of the integral equations ([6], [17]).

The paper is structured as follows:

In Section 2 we consider $\mu-$equivalence of sets with respect to nonatomic $\sigma-$finite Borel measure and prove some auxiliary assertions.

In Section 3 we modify some results from [4] in order to apply them to integral equations considered. We mention some basic results devoted to fixed point theory [1], [8], [13]–[16].

Section 4 is devoted to apply the results obtained for the study of a Volterra type integral equation (2.1) introduced below.

2. Preliminaries

Let Ω be a completely regular Hausdorff topological space, $B_\Omega \subset 2^\Omega$ denotes the σ-algebra of the Borel subsets of Ω and let $\mu : B_\Omega \to [0,\infty)$ be a nontrivial, nonatomic σ-finite Borel measure. Let B be a real Banach space with norm $||.||_B$.

Definition 2.1. [10] *The sets $G, H \in B_\Omega$ will be called μ-equivalent $G \sim^\mu H$, if $\mu(G\Delta H) = 0$, where $G\Delta H = (G \setminus H) \cup (H \setminus G)$.*

Definition 2.2. [10] *The set $G \in B_\Omega$ is called μ-dense (in B_Ω), if $\forall x \in G$ and for every open neighborhood $O(x)$ of x, $\mu(O(x)) > 0$, the inequality $\mu(G\cap O(x)) > 0$ holds.*

Following [7] we define the map $M : \Omega \to 2^\Omega$, which assigns to every point $x \in \Omega$ a closed subset $M(x) = M_x \subset \Omega$.

We say that the conditions **(A)** hold if for the map $M : \Omega \to 2^\Omega$ the following conditions are fulfilled:
A1. For every point $x \in \Omega$ the set M_x is compact.
A2. For each $\epsilon > 0$ and every $x \in \Omega$, there exists an open neighborhood $O(x,\epsilon)$ of x, such that for each $y \in O(x,\epsilon)$ we have that $\mu(M_x\Delta M_y) < \epsilon$.
A3. For every $x \in \Omega$ the inclusion $M_y \subset M_x$ holds for each $y \in M_x$.
A4. There exists $x_0 \in \Omega$ such that $\mu(M_{x_0}) = 0$.

For every map $M : \Omega \to 2^\Omega$ for which the conditions (A) hold, we will denote by $\mathbf{M} = \{M_x : x \in \Omega\}$ and $KerM = \{x \in \Omega : \mu(M_x) = 0\}$.

Remark 2.3. *In general, the conditions* **(A)** *can be fulfilled for the map $M : \Omega \to 2^\Omega$, but $x \notin M_x$. A simple example when this is true is $\Omega = [0,\infty)$ and $M_x = [0, \frac{x}{2}]$.*

Definition 2.4. *We say that the set $G \subset \Omega$ is M-star if for every $x \in G$ the inclusion $M_x \subseteq G$ holds.*

Remark 2.5. *It is easy to see that condition $A3$ implies that for each $x \in \Omega$ the set M_x is M-star set. Moreover, the union and the intersection of an arbitrary family of M-star sets are M-star set. The sets $KerM$ and $M_\Omega = \bigcup_{x\in\Omega} M_x$ are M-star sets ones.*

Let $\Omega^* \subseteq \Omega$ be an arbitrary compact M-star set and denote by $C(\Omega^*, B)$ the Banach space of all continuous maps $\varphi : \Omega^* \to B$. Introduce the linear spaces

$$C_M(\Omega) = \{\varphi : \Omega \to B \ : \ \varphi \in C(M_s, B), \forall s \in \Omega\}$$
$$C_M = \{f : M_\Omega \to B \ : \ f = \varphi|_{M_\Omega}, \varphi \in C_M(\Omega)\}$$

and consider the equations

$$f(x) = p(x) + \int_{M_x} Q(x, y, f(y))\mathrm{d}\mu_y \tag{2.1}$$

where the unknown function is $f \in C_M(\Omega)$, with known functions $p \in C_M(\Omega)$ and the operator $Q : \Omega \times \Omega \times B \to B$.

Remark 2.6. *If the conditions* **(A)** *hold and the operator Q is continuous for every $s \in \Omega$ in the set $M_s \times M_s \times B$, $M_s \subset \Omega$ then the Bochner integral in* (2.1) *exists $\forall x \in M_s$ (see* [11]*, Chapter 3 and* [9]*, Chapter V).*

It is well known that even for finite dimensional metric spaces Ω, two sets M_x and M_y can be very close (even equal) in measure sense but very different in metric sense. The next theorem solves this problem when the sets M_x are compact.

Let $G \in B_\Omega$ be an arbitrary set and the $(A, \succ)$ be a directed index set.

Definition 2.7. *The point $x \in \Omega$ will be called essential for the set G, if for every open neighborhood $O(x)$ of x, $\mu(O(x)) > 0$, we have $\mu(G \cap O(x)) > 0$.*

Definition 2.8. *The point $x \in \Omega$ will be called unessential for the set G, if there exists an open neighborhood $O(x)$ of x, $\mu(O(x)) > 0$, such that $\mu(G \cap O(x)) = 0$.*

For arbitrary closed subset $G \subset \Omega$ we denote by G^μ the set of all points $x \in G$, which are essential for G and by G^ν the set $G^\nu = G \setminus G^\mu$.

Theorem 2.9. *Let G be a compact subset of Ω and $\mu(G) > 0$.*

Then the set G^μ is a nonempty compact set and the sets G and G^μ are μ-equivalent.

Proof. Suppose that $G^\mu = \emptyset$. Then $G^\nu = G$ and for each $x \in G^\nu$ there exists an open neighborhood $O(x)$ of x, $\mu(O(x)) > 0$, such that $\mu(O(x) \cap G) = 0$.

Since G is a compact subset of Ω and $\bigcup\limits_{x \in G} O(x)$ is an open cover of G, then there exist points $x_1, \dots, x_k \in G$, and open neighborhoods $O(x_1), \dots, O(x_k)$, such that $\mu(G) \le \mu(\bigcup\limits_{i=1}^{k} (O(x_i) \cap G)) = 0$ which is impossible. Therefore $G^\mu \neq \emptyset$.

Obviously G^μ includes all internal points of G. Let $\{x_\alpha\}_{\alpha \in A} \subset int\, G$ be an arbitrary Cauchy sequence. Assume that $\lim\limits_{\alpha \in A} x_\alpha = y$ and $y \in \partial G$ is unessential. Then there exists an open neighborhood $O(y)$ of y and $\alpha_0 \in A$ such that $\{x_\alpha\}_{\alpha \succ \alpha_0} \cap O(y) = \emptyset$, which is impossible. Therefore the set G^μ is a closed subset of G and we can conclude that G^μ is compact.

Since every point $x \in G^\nu$ is unessential, there exists an open neighborhood $O(x)$ of x, such that $\mu(G \cap O(x)) = 0$ and $G^\nu \subset \bigcup\limits_{x \in G^\nu} O(x)$. Let $V_1, V_2, \dots, V_n$ are a finite open cover of G^μ, i.e. $G^\mu \subset \bigcup\limits_{i=1}^{n} V_i$. Therefore $G \subset (\bigcup\limits_{i=1}^{n} W_i) \cup (\bigcup\limits_{x \in G^\nu} O(x))$, where $W_i = V_i \setminus (\bigcup\limits_{x \in G^\nu} O(x))$ are open subsets of Ω. Since G is a compact set then there exist points $x_1, \dots, x_k \in G^\nu$ and open neighborhoods $O(x_1), \dots, O(x_k)$, such that $G \subset (\bigcup\limits_{i=1}^{n} W_i) \cup (\bigcup\limits_{j=1}^{k} O(x_j))$ and therefore $G^\nu \subset \bigcup\limits_{j=1}^{k} O(x_j)$.

This implies that $\mu(G^\nu) = 0$ and $G \sim^\mu G^\mu$. □

Remark 2.10. *The statement of Theorem 2.9 is proved in* [17] *in the case when Ω is arbitrary complete metric space. Since the compact sets M_x will be used as domains of integration then without loss of generality we can assume that all M_x are μ-dense.*

Lemma 2.11. *Let for the map $M : \Omega \to 2^{\Omega}$ the conditions (A) hold.*

Then for every $x \in \Omega$ for which $\mu(M_x) > 0$ there exists an open neighborhood $O(x)$ of x, such that for each $s \in O(x)$ we have that $M_x \cap M_s \neq \emptyset$.

Proof. Suppose that the statement of the Lemma 2.11 is not true. Denote by A the set of all open neighborhoods $\alpha = \alpha(x) = O(x, \alpha)$ of x directed by inclusions. Then there exists a sequence $\{y_\alpha\}_{\alpha \in A} \subset \Omega$ such that $M_x \cap M_{y_\alpha} = \emptyset$ and $\lim_{\alpha \in A} y_\alpha = x$. Condition A2 implies that there exits $\alpha_0(x) \in A$ such that for every $\alpha \succ \alpha_0(x)$ we have that $\mu(M_x \Delta M_{y_\alpha}) < 2^{-1}\mu(M_x)$. Therefore if $\alpha \succ \alpha_0(x)$, then we have that $\mu(M_x) > 2\mu(M_x \Delta M_{y_\alpha}) = 2\mu(M_x \cup M_{y_\alpha}) \geq 2\mu(M_x) > 0$ which is impossible. □

Corollary 2.12. *Let the conditions of Lemma 2.11 hold and all sets M_s are μ-dense.*

Then $\forall x \in \Omega$ there exists an open neighborhood $O(x)$ of x, such that for each $s \in O(x)$ we have that $\mu(M_x \cap M_s) > 0$.

Corollary 2.13. *Let the conditions of Lemma 2.11 hold and all sets M_x are μ-dense and connected.*

Then if Ω is connected, the set $M_\Omega = \bigcup_{x \in \Omega} M_x$ is connected too.

Proof. Assume the contrary there exist two open sets $G, H \subset M_\Omega, G \cup H = M_\Omega$ and $G \cap H = \emptyset$. Since all sets M_x are connected then the sets $G^* = \{x \in \Omega : M_x \subset G\}$, $H^* = \{y \in \Omega : M_y \subset H\}$ are disjoint and obviously $G^* \cup H^* = \Omega$. Let $x \in G^*$ be an arbitrary point. From Corollary 2.12 it follows that there exists an open neighborhood $O(x)$ of x, such that for each $s \in O(x)$ we have that $\mu(M_x \cap M_s) > 0$ and therefore $M_s \subset G$. Thus we proved that G^* is open. Analogously we can prove that H^* is open too, which is impossible. □

Theorem 2.14. *Let for the map $M : \Omega \to 2^{\Omega}$ the following conditions be fulfilled:*

1. The conditions (A) hold.

2. Ω is a connected space and $0 < \mu(\Omega) \leq \infty$.

Then for every $x \in \Omega$ we have that $\mu(M_x) < \infty$.

Proof. Consider the set $\Omega_1 = \{x \in \Omega : \mu(M_x) < \infty\}$ and let $\Omega_2 = \Omega \setminus \Omega_1$.

Condition A4 implies that $\Omega_1 \neq \emptyset$. First we will prove that Ω_1 is a closed subset of Ω. Let $y \in \partial\Omega_1$ is arbitrary point. Then there exists a sequence $\{x_\alpha\}_{\alpha \in A} \subset int\, \Omega_1$ such that $\lim_{\alpha \in A} x_\alpha = y$. Let $\epsilon > 0$ is arbitrarily fixed.

From condition A2 it follows that there exists $\alpha_0(\epsilon) \in A$ such that for each $\alpha \succ \alpha_0$ we have $\mu(M_{x_\alpha} \Delta M_y) < \epsilon$.

Let $x_{\alpha^*} \in \{x_\alpha\}_{\alpha \in A}, \alpha^* \succ \alpha_0$. Since $\mu(M_{x_{\alpha^*}}) < \infty$ and $\mu(M_{x_{\alpha^*}} \Delta M_y) < \epsilon$, then we have that $\mu(M_y) \leq \mu(M_{x_{\alpha^*}}) + \mu(M_{x_{\alpha^*}} \Delta M_y) \leq \mu(M_{x_{\alpha^*}}) + \epsilon < \infty$.

Therefore $y \in \Omega_1$ and thus we proved that Ω_1 is a closed subset of Ω.

Let us assume that $\Omega_2 \neq \emptyset$ too. It is easy to see that for each $z \in \partial\Omega_2$ from condition A2 it follows that $\mu(M_z) = +\infty$. Then Ω_2 is a closed set too.

Consequently we obtain $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$ for the connected space Ω, which is impossible.

Then we can conclude that $\Omega_2 = \emptyset$. □

3. Fixed point results

Let $(X, \mathbb{A})$ be a sequentially complete Hausdorff uniform space with uniformity generated by a saturated family of pseudometrics $\mathbb{A} = \{d_\alpha(x,y) : \alpha \in A\}$, where A is an index set ([4]).

Denote by $\mathbb{N}$ the set of all positive integer numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{R}_+ = [0,\infty)$ and $\Gamma = A \times \mathbb{N}_0$.

Definition 3.1. [4] *The family of functions* $\Phi = \{\Phi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+, \alpha \in A\}$ *is said to be* Φ*-contractive if satisfies the properties (*$\Phi 1$*):*

Φ*1.1. For every* $\alpha \in A$ *the function* Φ_α *is monotone increasing and continuous from the right.*

Φ*1.2. For every* $\alpha \in A$ *the function* Φ_α *satisfies the inequality* $0 < \Phi_\alpha(t) < t$ *for each* $t > 0$.

Definition 3.2. *The family of functions* $\Phi = \{\Phi_{(\alpha,n)} : \mathbb{R}_+ \to \mathbb{R}_+, (\alpha,n) \in \Gamma\}$ *is said to be* Φ*-almost contractive if satisfies the properties (*$\Phi 2$*):*

Φ*2.1. For every* $(\alpha,n) \in \Gamma$ *the function* $\Phi_{(\alpha,n)}$ *is monotone increasing and continuous from the right.*

Φ*2.2. For every* $\alpha \in A$ *there exists a number* $n(\alpha) \in \mathbb{N}$*, such that for each* $n \geq n(\alpha)$ *the function* $\Phi_{(\alpha,n)}$ *satisfies the inequality* $0 < \Phi_{(\alpha,n)}(t) < t$ *for every* $t > 0$.

Remark 3.3. *It is easy to see that if for every* $\alpha \in A$ *we have that* $n(\alpha) = 0$*, then the* Φ*-almost contractive family is* Φ*-contractive.*

The next theorem is a slight generalization of Theorem 2.2.6 from [4]. First we introduce the class of the (Φ)-almost contractive operators. Denote by T^0 the identity operator.

Definition 3.4. *The mapping* $T : X \to X$ *is said to be* (Φ)*-almost contractive on* X *if for every fixed* $\alpha \in A$ *and* $n \in \mathbb{N}_0$ *there exists* $\Phi_{(\alpha,n)}(t) \in \Phi$ *such that the inequality* $d_\alpha(T^n x, T^n y) \leq \Phi_{(\alpha,n)}(d_{j(\alpha,n)}(x,y))$ *holds for every* $x, y \in X$*, where* $j : \Gamma \to A$ *is a mapping.*

Theorem 3.5. *Let the following conditions hold:*

1. The operator $T : X \to X$ *is* (Φ)*-almost contractive.*

2. There exists a point $x_0 \in X$ *such that for every* $\alpha \in A$ *and* $n \in \mathbb{N}_0$ *there exists* $q = q(\alpha) : A \to (0,\infty)$ *for which the inequality* $d_{j(\alpha,n)}(x_0, Tx_0) \leq q(\alpha) < \infty$ *holds.*

3. For every $\alpha \in A$ *we have that* $\sum\limits_{n=0}^{\infty} \Phi_{(\alpha,n)}(q(\alpha)) < \infty$.

Then the operator $T : X \to X$ *has at least one fixed point in* X.

Proof. Let $\alpha \in A$ be an arbitrary fixed index. Then condition 2 implies that for $n \in N_0$ the inequalities

$$d_\alpha(T^{n+m}x_0, T^n x_0) \leq \sum_{i=1}^{m} d_\alpha(T^{n+i-1}(Tx_0), T^{n+i-1}x_0)$$

$$\leq \sum_{i=1}^{m} \Phi_{(\alpha,n+i-1)}(d_{j(\alpha,n+i-1)}(x_0, Tx_0)) \leq \sum_{i=1}^{m} \Phi_{(\alpha,n+i-1)}(q(\alpha))$$

hold for each $m \in \mathbb{N}$. Then the sequence $\{T^n(x_0)\}_{n=0}^{\infty}$ is a Cauchy one and consequently there exists x^*: $x^* = \lim_{n\to\infty} T^n(x_0)$ in $(X, \mathbb{A})$. The limit x^* is the required fixed point of the operator T.

Indeed, for any fixed $\alpha \in A$ there exists $n(\alpha) \in \mathbb{N}_0$ such that for every $n \geq n(\alpha)$ we have

$$\begin{aligned} d_\alpha(x^*, Tx^*) &\leq d_\alpha(x^*, x_{n+1}) + d_\alpha(x_{n+1}, Tx^*) \\ &\leq d_\alpha(x^*, x_{n+1}) + \Phi_{(\alpha,1)}(d_{j(\alpha,1)}(x_n, x^*)) \end{aligned}$$

where $\Phi_{(\alpha,1)}$ is right-continuous function and the last expression tends to 0.

The proof is complete. □

4. Applications

Let $\Omega^* \subset \Omega$ be an arbitrary M-star set. In our discussion below we will assume that for the operator $Q : \Omega \times \Omega \times B \to B$ are fulfilled some of the following conditions (where $||f||_{M_x} := max\{||f(y)||_B : y \in M_x\}$):

S1. For every M-star set $\Omega^* \subset \Omega$ the operator Q is continuous in the set $\Omega^* \times \Omega^* \times B$.

S2. For every $f \in C_M$ and each $x \in \Omega$, there exist numbers $\delta(f, x) > 0$ and $L(f, x, \delta) > 0$ such that for every function $g \in C_M$ for which $||f - g||_{M_x} < \delta$, the inequality $||Q(x, y, f(y)) - Q(x, y, g(y))||_B \leq L(f, x, \delta)||f(y) - g(y)||_B$ holds for every $y \in M_x$.

S3. For every $x \in \Omega$ and arbitrary $r > 0$, there exists a constant $L(x, r) > 0$ such that for every two functions $f_1, f_2 \in \bar{U}(x, r) = \{f \in C_M | \ ||f||_{M_x} \leq r\}$ the inequality $||Q(x, y, f_1(y)) - Q(x, y, f_2(y))||_B \leq L(x, r)||f_1(y) - f_2(y)||_B$ holds for every $y \in M_x$.

S4. For every $x \in \Omega$ there exists a constant $L(x) > 0$ such that for every two functions $f_1, f_2 \in C_M$ the inequality $||Q(x, y, f_1(y)) - Q(x, y, f_2(y))||_B \leq L(x)||f_1(y) - f_2(y)||_B$ holds for every $y \in M_x$.

Consider the operator K defined by

$$Kf(x) = \int_{M_x} Q(x, y, f(y))\mathrm{d}\mu_y \tag{4.1}$$

where $f \in C_M(\Omega), x \in \Omega$. If for the operator Q the condition S1 holds, then the integral in (4.1) does exist (see Remark 2.6).

Lemma 4.1. *Let the conditions (A), S1 and S2 hold.*
Then the operator K defined by (4.1) maps $C_M(\Omega)$ into itself.

Proof. Let $x \in \Omega$ be an arbitrary fixed point, $x_0 \in M_x$ and let $\{x_\alpha\}_{\alpha\in A} \subset M_x$ be an arbitrary generalized sequence such that $\lim_{\alpha\in A} x_\alpha = x_0$. If $f \in C_M(\Omega)$ is an arbitrary fixed element then

$$\begin{aligned} ||Kf(x_\alpha) - Kf(x_0)||_B &\leq \int_{M_{x_\alpha}\cap M_{x_0}} ||Q(x_\alpha, y, f(y)) - Q(x_0, y, f(y))||_B \mathrm{d}\mu_y \\ &+ \int_{M_{x_\alpha}\setminus M_{x_0}} ||Q(x_\alpha, y, f(y))||_B \mathrm{d}\mu_y + \int_{M_{x_0}\setminus M_{x_\alpha}} ||Q(x_0, y, f(y))||_B \mathrm{d}\mu_y \end{aligned} \tag{4.2}$$

Let $\epsilon > 0$ be an arbitrary number. Since the set $B(f) = M_x \times M_x \times f(M_x)$ is compact then there exists a constant $Q_0(x, f) > 0, Q_0(x, f) = \sup\limits_{(s,y,v)\in B(f)} ||Q(s, y, v))||_B$. Conditions A1 and S1 imply that there exists $\alpha_0 = \alpha_0(\epsilon) \in A$, such that for each $\alpha \succ \alpha_0$ the inequalities

$$\sup_{\alpha\succ\alpha_0, y\in M_{x_0}} ||Q(x_\alpha, y, f(y)) - Q(x_0, y, f(y))||_B < \frac{\epsilon}{2\mu(M_{x_0})}$$

and

$$\mu(M_{x_\alpha}\Delta M_{x_0}) \leq \frac{\epsilon}{2Q_0(x, f)}$$

hold. Therefore from (4.2) it follows, that for each $\alpha \succ \alpha_0$ we have

$$||(Kf)(x_\alpha) - Q(Kf)(x_0)||_B < \epsilon.$$

□

Corollary 4.2. *Let the conditions (A), S1 and S2 hold.*
Then the operator K defined by (4.1) maps C_M into itself.

Definition 4.3. *We say that the equation* (2.1) *has a local solution in some M-star set $\Omega^* \subset \Omega$ for some $p \in C(\Omega^*, B)$ if there exists a point $x_p \in \Omega^*$ and a function $f \in C(M_{x_p}, B)$ for which $\mu(M_{x_p}) > 0$ and f satisfies the equation (2.1) for each $x \in M_{x_p}$. If for every $x \in \Omega^*$: $p, f \in C(M_x, B)$ and f satisfies the equation* (2.1) *then we say that f is a solution of* (2.1) *in $\Omega^* \subset \Omega$.*

For each $p \in C_M(\Omega)$ define the operator $T : Tf(x) = p(x) + Kf(x)$, $T^{n+1} = T(T^n), n \in \mathbb{N}$ and with T^0 we denote the identity operator. From Lemma 4.1 it follows that T maps $C_M(\Omega)$ into $C_M(\Omega)$ and therefore $T : C_M \to C_M$.

Theorem 4.4. *Let the following conditions be fulfilled:*
1. *The space Ω is connected.*
2. *The conditions (A), S1 and S3 hold.*
3. *For each $x \in \Omega$ the set M_x is connected.*

Then $\forall p \in C(M_\Omega, B)$ the equation (2.1) *has at most one solution $f \in C(M_\Omega, B)$.*

Proof. Suppose $f_1, f_2 \in C(M_\Omega, B)$ are two solutions of the equation (2.1). Let $x_0 \in M_\Omega$ be an arbitrary point such that $f_1(x_0) \neq f_2(x_0)$. Then there exists a point $s \in \Omega$ such that $x_0 \in M_s$ and $M_{x_0} \subset M_s$ (from **A3**).

Let $r > \sup\limits_{s\in M_s} ||f_1(x)||_B + \sup\limits_{s\in M_s} ||f_2(x)||_B > 0$ and denote by $L(s, r) > 0$ the constant existing according to condition S3. From equation (2.1) and condition S3 it follows that for each $x \in M_s$ the inequality

$$||f_1(x) - f_2(x)||_B \leq L(s, r) \int_{M_x} ||f_1(y) - f_2(y)||_B \mathrm{d}\mu_y$$

holds. Using Theorem 2 and Theorem 3 from [7] we obtain $||f_1(x) - f_2(x)||_B = 0$ for each $x \in M_s$, which contradicts our supposition. □

Let $A = \Omega$ be the index set and introduce in the linear space C_M uniform topology generated by the following saturated family of pseudometrics

$$G_M = \{d_x(f_1, f_2) = ||f_1 - f_2||_{M_x} : \ f_1, f_2 \in C_M, x \in \Omega\}.$$

It is easy to see that the family of pseudometrics is separated, i.e. if for some $f_1, f_2 \in C_M$ we have that $d_x(f_1, f_2) = 0$ holds for every $x \in \Omega$, then by necessary $f_1(s) = f_2(s)$ for $s \in M_\Omega$. Thus C_M is Hausdorff (T_2-separated) uniform space and obviously C_M is sequentially complete space.

Theorem 4.5. *Let the following conditions be fulfilled:*

1. *The space Ω is connected.*
2. *The conditions (A), S1 and S4 hold.*
3. *For each $x \in \Omega$ the set M_x is connected.*
4. *For each $n \in \mathbb{N}$ and $\forall x = x_0 \in \Omega$ the following inequality*

$$\int_{M_x} \Big(\int_{M_{x_1}} \Big(\dots \Big(\int_{M_{x_{n-1}}} \mu(M_{x_n}) \mathrm{d}\mu_{x_n} \Big) \dots \Big) \mathrm{d}\mu_{x_2} \Big) \mathrm{d}\mu_{x_1} \le \frac{C_x \mu^{n+1}(M_x)}{(n+1)!}$$

holds.

Then for each $p \in C_M(\Omega)$ the equation (2.1) *has in M_Ω at least one solution $f \in C_M$.*

Proof. Let $x \in \Omega$ be an arbitrary fixed element. For arbitrary $f_1, f_2 \in C_M$ and $n \in \mathbb{N}$ from S4 and condition 4 we have

$$\begin{aligned}
&\sup_{y \in M_x} ||T^n(f_1)(y) - T^n(f_2)(y)||_B \\
&= \sup_{y \in M_x} ||K(T^{n-1}(f_1))(y) - K(T^{n-1}(f_2))(y)||_B \\
&\le \sup_{y \in M_x} \int_{M_y} \|Q(y, x_1, T^{n-1}(f_1)(x_1)) - Q(y, x_1, T^{n-1}(f_2)(x_1))\|_B d\mu_{x_1} \\
&\le \sup_{y \in M_x} L(y) \int_{M_y} ||T^{n-1}(f_1)(x_1) - T^{n-1}(f_2)(x_1)||_B d\mu_{x_1} \\
&\le L(x) \int_{M_x} ||T^{n-1}(f_1)(x_1) - T^{n-1}(f_2)(x_1)||_B d\mu_{x_1} \le \dots \\
&\le L^n(x) \int_{M_x} \Big(\int_{M_{x_1}} \Big(\dots \Big(\int_{M_{x_{n-1}}} ||f_1 - f_2||_{M_x} \mathrm{d}\mu_{x_n} \Big) \dots \Big) d\mu_{x_2} \Big) d\mu_{x_1} \\
&\le \frac{C_x \mu^n(M_x) L^n(x)}{n!} ||f_1 - f_2||_{M_x} = c_n^x ||f_1 - f_2||_{M_x}.
\end{aligned} \tag{4.3}$$

Let $n(x) \in \mathbb{N}$ be the first number for which we have

$$c_{n(x)}^x = \frac{C_x \mu^{n(x)}(M_x) L^{n(x)}(x)}{n(x)!} < 1 \quad \text{and} \quad C_x \mu(M_x) L(x) < n(x).$$

Define the mapping $j : \Omega \times \mathbb{N}_0 \to \Omega$ as follows: $j(x, n) = x$ for $n \in \mathbb{N}_0$ and $x \in \Omega$.

Define the family $\Phi = \{\Phi_{(x,n)}(t) = c_n^x t : t \in \mathbb{R}_+, n \in \mathbb{N}_0, x \in \Omega\}$. Obviously the family Φ is Φ-almost contractive.

Assume that the function $f_0(x) \equiv 0$ for each $x \in M_\Omega$.

Since $P(x) = \sup_{s\in M_x} ||p(s)||_B < \infty$ and $Q_0(x) = \sup_{(s,y)\in M_x\times M_x} ||Q(s,y,0)||_B < \infty$, then for $n \in \mathbb{N}_0$ we have

$$\begin{aligned} d_{j(x,n)}(f_0, Tf_0) &= d_x(f_0, Tf_0) \\ &= ||Tf_0||_{M_x} \\ &< P(x) + \mu(M_x)Q_0(x) + 1 \\ &= q(x) < \infty \end{aligned} \tag{4.4}$$

From (4.4) it follows that the condition 2 of Theorem 3.5 holds too. For each $n \geq n(x)$ from (4.3) it follows that the following estimation

$$\frac{\Phi_{(x,n+1)}(q(x))}{\Phi_{(x,n)}(q(x))} = \frac{c^x_{n+1}q(x)}{c^x_n q(x)} \leq \frac{\mu(M_x)L(x)}{n}$$

holds and therefore the condition 3 of Theorem 3.5 also is fulfilled. Thus for each $p \in C_M$ the equation $f = Tf$ has at least one fixed point $f \in C_M$.

From Theorem 4.4 it follows that the fixed point f is unique and it is a continuous function in M_Ω, $f \in C(M_\Omega, B)$. □

Remark 4.6. *Obviously if Ω is a finite dimensional space then the condition 4 of Theorem 4.5 is implicitly fulfilled.*

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