Fixed Point Theory, 18(2017), No. 1, 37-46
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

MEASURES OF WEAK NONCOMPACTNESS AND FIXED POINT THEORY IN BANACH ALGEBRAS SATISFYING CONDITION ($\mathcal{P}$)

AFIF BEN AMAR* AND DONAL O'REGAN**

*Faculty of Sciences, Department of Mathematics
Sfax University, Sfax, Tunisia
E-mail: afif_ben_amar@yahoo.fr

**School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway, Ireland

Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group
Department of Mathematics, King Abdulaziz University
Jeddah, Saudi Arabia
E-mail: donal.oregan@nuigalway.ie

Abstract. The aim of this paper is to prove some new fixed point theorems in a nonempty closed convex subset of a Banach algebra satisfying a sequential condition ($\mathcal{P}$) in a weak topology setting.
Key Words and Phrases: Measure of weak noncompactness, quasi-regular, weakly condensing, weakly sequentially continuous, relatively weakly compact, fixed point theorems.
2010 Mathematics Subject Classification: 47H08, 47H09, 47H10.

1. Introduction

Existence results for certain equations in a Banach algebra setting reduce to establishing fixed point theory for nonlinear operator equations of the form

$$x = AxBx + Cx.$$

Fixed point theory in Banach algebras in a weak topology setting was discussed in [7, 8, 9, 18, 19]. In [7, 8] the authors introduced a new class of Banach algebras satisfying a certain sequential condition ($\mathcal{P}$) (see Definition 2.6 in this paper) and they presented some new fixed point theorems in a nonempty closed convex (not necessarily bounded) subset of a Banach algebra satisfying condition ($\mathcal{P}$).

The authors in [6] used the methods in the papers cited above and tried to extend these results to weakly condensing operators. However there are problems in the analysis. Theorems 3.1 and 3.2 in [6], which are the main results, are not correct. Theorem 3.5 in this paper gives a correct version of Theorem 3.1 in [6].

- In the proof of Theorem 3.1 in [6], the operator $\frac{I-C}{A}$ is not well defined. As a result we introduce the notion of quasi-regular operators to guarantee that the operator $\frac{I-C}{A}$ is well defined.

- In the proof of Theorem 3.1 in [6], the authors construct a set Λ which is not necessarily bounded and they then use its measure of weak noncompactness. To correct this we add the condition that $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is bounded.
- In the proof of Theorem 3.1 in [6], to prove the weak sequential continuity of the operator $\left(\frac{I-C}{A}\right)^{-1} B$, the authors assume that if a sequence $(x_n)_n$ converges weakly to x and a sequence $(y_n)_n$ converges weakly to y then the sequence $(x_n y_n)_n$ converges weakly to xy. Unfortunately this is not correct in a general Banach algebra.

As a consequence the result of Theorem 3.2 in [6] is not correct since it is a particular case of Theorem 3.1 ($C \equiv 0$). Also Theorem 3.3, Corollary 3.1 and Corollary 3.2 in [6] are not correct since they are based on the result of Theorem 3.1. In particular we mention that a Banach algebra satisfying condition $(\mathcal{P})$ is a WC-Banach algebra but the converse is not necessarily true.

The present paper is organized as follows. After some preliminaries, in Section 3 we provide some new fixed point theorems in a nonempty closed convex subset of a Banach algebra satisfying a sequential condition $(\mathcal{P})$, for the sum and the product of nonlinear weakly sequentially operators. The main condition in our results is formulated in terms of axiomatic measures of weak noncompactness (see Definition 2.1). Our theorems extend some results stated in [6, 7, 8].

2. Preliminaries

Definition 2.1. Let X be a Banach space and C a lattice with a least element, which is denoted by 0. By a measure of weak non-compactness on X, we mean a function Φ defined on the set of all bounded subsets of X with value in C satisfying:

(1) $\Phi(\overline{\text{conv}}(\Omega)) = \Phi(\Omega)$, for all bounded subsets $\Omega \subseteq X$, where $\overline{\text{conv}}$ denotes the closed convex hull of Ω,

(2) for any bounded subsets Ω_1, Ω_2 of X we have

$$\Omega_1 \subseteq \Omega_2 \Longrightarrow \Phi(\Omega_1) \leq \Phi(\Omega_2),$$

(3) $\Phi(\Omega \cup \{a\}) = \Phi(\Omega)$ for all $a \in E$, Ω bounded set of X,

(4) $\Phi(\Omega) = 0$ if and only if Ω is weakly relatively compact in X.

The above notion is a generalization of the well known De Blasi measure of weak noncompactness β (see [11]) defined on each bounded set Ω of X by

$$\beta(\Omega) = \inf\{\varepsilon > 0 : \text{ there exists a weakly compact set } D \text{ such that } \Omega \subseteq D + B_\varepsilon(\theta)\}.$$

Note for all bounded subsets $\Omega, \Omega_1, \Omega_2$ of X,

(5) $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$,

(6) $\beta(\lambda\Omega) = \lambda\beta(\Omega)$ for all $\lambda > 0$,

(7) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$.

Note that β is the counterpart for the weak topology of the classical Hausdorff measure of noncompactness. For more examples and properties of measures of weak noncompactness we refer the reader to [1, 4, 5, 20, 21].

Definition 2.2. Let D be a nonempty subset of Banach space X, Φ is a MWNC on X and F maps D into X. We say that

- F is Φ-condensing if F is bounded and $\Phi(F(V)) < \Phi(V)$ for all bounded subsets V of D with $\Phi(V) > 0$,
- F is weakly compact if $F(V)$ is relatively weakly compact for every bounded subset $V \subset D$.

Definition 2.3. Let X be a Banach space. An operator $F : X \longrightarrow X$ is said to be weakly sequentially continuous on X if for every sequence $(x_n)_n$ with $x_n \rightharpoonup x$, we have $Fx_n \rightharpoonup Fx$, here $\rightharpoonup$ denotes weak convergence.

The following fixed point result stated in [3] is an analogue of Sadovskii's fixed point result [2], will be used throughout the next section.

Theorem 2.4. *Let Ω be a nonempty, convex subset of a Banach space X and let Φ be a MWNC on X. Then the following assertions hold, for every Φ-condensing and weakly sequentially continuous map $T : \Omega \to \Omega$ with bounded range:*

(i) *T has a weakly-approximate fixed point sequence, i.e. a sequence $(x_n) \subset \Omega$ so that the sequence $(x_n - Tx_n)$ converges weakly to θ in X.*
(ii) *if Ω is closed, then the set $F(T)$ of fixed points of T is nonempty and weakly compact.*

Definition 2.5. An algebra X is a vector space endowed with an internal composition law noted by (.) i.e.,

$$\begin{cases} (.) : X \times X & \longrightarrow X \\ \quad (x, y) & \longrightarrow x.y \end{cases}$$

which is associative and bilinear.
A normed algebra is an algebra endowed with a norm satisfying the following property

$$\text{for all } x, y \in X;\ \|x.y\| \leq \|x\|\|y\|.$$

A complete normed algebra is called a Banach algebra.

In general, the product of two weakly sequentially continuous mappings on a Banach algebra X is not necessarily weakly sequentially continuous (the operation multiplication is not necessarily weakly sequentially continuous).

Definition 2.6. We will say that the Banach algebra X satisfies condition $(\mathcal{P})$ if

$$(\mathcal{P}) \begin{cases} \text{For any sequences } \{x_n\} \text{ and } \{y_n\} \text{ in } X \text{ such that } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y, \\ \text{then } x_n y_n \rightharpoonup xy;\ \text{ here } \rightharpoonup \text{ denotes weak convergence} \end{cases}$$

Note that, every finite dimensional Banach algebra satisfies condition $(\mathcal{P})$. If X satisfies condition $(\mathcal{P})$ then $\mathcal{C}(K, X)$ is also a Banach algebra satisfying condition $(\mathcal{P})$, where K is a compact Hausdorff space. The proof is based on Dobrokov's theorem:

Theorem 2.7. [12, Dobrakov, p. 36] *Let K be a compact Hausdorff space and X be a Banach space. Let $(f_n)_n$ be a bounded sequence in $\mathcal{C}(K, X)$, and $f \in \mathcal{C}(K, X)$.*

Then $(f_n)_n$ *is weakly convergent to* f *if and only if* $(f_n(t))_n$ *is weakly convergent to* $f(t)$ *for each* $t \in K$.

Definition 2.8. [22] Let X be a Banach space. An operator $F : X \longrightarrow X$ is said to be strongly continuous on X if for every sequence $(x_n)_n$ with $x_n \rightharpoonup x$, we have $Fx_n \longrightarrow Fx$, here $\longrightarrow$ denotes convergence in X.

Definition 2.9. Let X be a Banach space. X is said to have the Dunford-Pettis property (for short property DP) if for each Banach space Y every weakly compact linear operator $F : X \longrightarrow Y$ takes weakly compact sets in X into norm compact sets of Y.

The Dunford-Pettis property as defined above was explicitly defined by A. Grothendieck [17] who undertook an extensive study of this and related properties (see also [13]). It is well known that any L_1 space has the property DP [14].

Proposition 2.10. *Let* X *be a Dunford-Pettis space and* T *a weakly compact linear operator on* X. *Then* T *is strongly continuous.*

It was proved in [6] that any Banach algebra with the Dunford-Pettis property satisfies condition $(\mathcal{P})$.

Lemma 2.11. [8, Lemma 3.2] *Let* X *be a Banach algebra with the condition* $(\mathcal{P})$. *Then for any bounded subset* V *of* X *and weakly compact subset* K *of* X, *we have*

$$\beta(V.K) \leq \|K\|\beta(V),$$

where $\|K\| = \sup\{\|x\|, x \in K\}$.

Definition 2.12. Let X be a Banach space. A mapping $\mathcal{G} : \mathrm{X} \longrightarrow \mathrm{X}$ is called $\mathcal{D}$-Lipschitzian if there exists a continuous and nondecreasing function $\phi_{\mathcal{G}} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\|\mathcal{G}x - \mathcal{G}y\| \leq \phi_{\mathcal{G}}(\|x - y\|).$$

for all $x, y \in \mathrm{X}$, with $\phi_{\mathcal{G}}(0) = 0$. Sometimes we call the function $\phi_{\mathcal{G}}$ a $\mathcal{D}$-function of $\mathcal{G}$ on X. If $\phi_{\mathcal{G}}(r) = kr$ for some $k > 0$, then $\mathcal{G}$ is called a Lipschitzian function on X with the Lipschitz constant k. Furthermore if $k < 1$, then $\mathcal{G}$ is called a contraction on X with the contraction k.

Remark 2.13. Every Lipschitzian mapping is $\mathcal{D}$-Lipschitzian, but the converse may not be true. For example [6], take $\mathcal{G}(x) = \sqrt{|\, x\,|}$, $x \in \mathbb{R}$ and consider $\phi_{\mathcal{G}}(r) = \sqrt{r}$, $r \geq 0$. Then $\mathcal{G}$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\phi_{\mathcal{G}}$, but $\mathcal{G}$ is not Lipschitzian.

Remark 2.14. If $\phi_{\mathcal{G}}$ is not necessarily nondecreasing and satisfies $\phi_{\mathcal{G}}(r) < r$, for $r > 0$, the mapping $\mathcal{G}$ is called a nonlinear contraction with contraction function $\phi_{\mathcal{G}}$.

3. Fixed point theory

Definition 3.1. An element x of a Banach algebra X is said to be quasi-regular if for any $y \in X$ we have $xy = 0_X$ imply $y = 0_X$

Definition 3.2. Let X be a Banach algebra and $A, C : X \longrightarrow X$ two operators such that A is quasi-regular on X, i.e., A maps X into the set of all quasi-regular elements of X. We say that the operator $\frac{I-C}{A}$ is defined on $x \in X$ and we write $\frac{I-C}{A}x = y \in X$ if $x = (Ax)y + Cx$.

Remark 3.3. If A maps X into the set of all invertible elements of X, then A is quasi-regular.

Proposition 3.4. *Let X be a Banach algebra and S be a nonempty subset of X. Let $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ be three operators such that*

(i) *A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
(ii) *A is quasi-regular,*
(iii) *B is a bounded function with bound M.*

Then $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$ if $M\phi_A(r) + \phi_C(r) < r$, for all $r > 0$.

Proof. Let y be fixed in S and define the mapping

$$\begin{cases} \varphi_y : X & \longrightarrow X \\ \qquad x & \longrightarrow \varphi_y(x) = AxBy + Cx. \end{cases}$$

Let $x_1, x_2 \in X$, and from (i) we have

$$\begin{aligned} \|\varphi_y(x_1) - \varphi_y(x_2)\| &\leq \|Ax_1By - Ax_2By\| + \|Cx_1 - Cx_2\| \\ &\leq \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\ &\leq M\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|). \end{aligned}$$

Now, an application of a fixed point theorem of Boyd and Wong [10] yields that there is a unique element $x_y \in X$ such that

$$\varphi_y(x_y) = x_y = Ax_yBy + Cx_y$$

Hence, x_y verifies the equation $(I - C)x_y = Ax_yBy$ and so $\frac{I-C}{A}x_y = By$. Therefore, the mapping $\left(\frac{I-C}{A}\right)^{-1}$ is well defined on $B(S)$ and $\left(\frac{I-C}{A}\right)^{-1}By = x_y$ and the desired result is deduced.

Theorem 3.5. *Let X be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ is a MWNC on X. Let S be a nonempty closed convex subset of X and let $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ be three operators such that*

(i) *A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
(ii) *A is quasi-regular on X,*
(iii) *A, B and C are weakly sequentially continuous on S,*
(iv) *$B(S)$ is bounded with bound M,*
(v) *$\left(\frac{I-C}{A}\right)^{-1}B$ is Φ-condensing on S with $\left(\frac{I-C}{A}\right)^{-1}B(S)$ is bounded,*
(vi) *$x = AxBy + Cx \Rightarrow x \in S$, for all $y \in S$.*

Then the equation $x = AxBx + Cx$ has at least one solution in S if $M\phi_A(r) + \phi_C(r) < r$, for all $r > 0$.

Proof. From Proposition 3.4, it follows that $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$. From assumption (vi), we obtain

$$\left(\frac{I-C}{A}\right)^{-1} B(S) \subset S.$$

From Theorem 2.4, it suffices to prove that the operator $\left(\frac{I-C}{A}\right)^{-1}B$ is weakly sequentially continuous. To see this, let (u_n) be a weakly convergent sequence of S (converging to a point u of S). Now define the sequence (v_n) of the subset S by

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n.$$

We prove that the set $D = \{v_n\}$ is relatively weakly compact. It is easily seen that

$$D \subset C(D) + A(D)\{Bu_n\} \subset C(D) + A(D)\overline{\{Bu_n\}^w}.$$

Using Lemma 2.11 and Lemma 2.1 in [6] with the weak compactness of $\overline{\{Bu_n\}^w}$, we infer that

$$\beta(D) \leq \beta(C(D) + \beta(A(D)\overline{\{Bu_n\}^w}) \leq \phi_C(\Phi(D)) + M\phi_A(\beta(D)).$$

This shows that $\beta(D) = 0$. Thus D is relatively weakly compact. Therefore, there is a renamed subsequence such that

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n \rightharpoonup v.$$

However the subsequence $\{v_n\}$ satisfies

$$v_n - Cv_n = Av_nBu_n.$$

Therefore, from assumption (iii) and in view of condition $(\mathcal{P})$, we deduce that v satisfies

$$v - Cv = AvBu,$$

or, equivalently

$$v = \left(\frac{I-C}{A}\right)^{-1} Bu.$$

Next we claim that the whole sequence $\{u_n\}$ satisfies

$$\left(\frac{I-C}{A}\right)^{-1} Bu_n = v_n \rightharpoonup v.$$

Indeed, suppose that this is not the case. There is a V^w, a weakly neighborhood of v, with for all $n \in \mathbb{N}$, there exists an $N \geq n$ such that $v_N \notin V^w$. Hence, there is a renamed subsequence $\{v_n\}$ satisfying the property

$$\text{for all } n \in \mathbb{N},\ v_n \notin V^w. \tag{1}$$

However

$$\text{for all } n \in \mathbb{N},\ v_n \in \left(\frac{I-C}{A}\right)^{-1} B(S).$$

Again, there is a renamed subsequence such that

$$v_n \rightharpoonup v'.$$

Thus we have

$$v' = \left(\frac{I-C}{A}\right)^{-1} Bu,$$

and, consequently

$$v = v',$$

which is a contradiction with property (1). This yields that $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous.

Corollary 3.6. *Let X be a Banach algebra satisfying condition $(\mathcal{P})$ and S be a nonempty closed convex subset of X and let $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ be three operators such that*

- (i) *A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
- (ii) *A is quasi-regular on X,*
- (iii) *A, B and C are weakly sequentially continuous on S,*
- (iv) *$B(S)$ is bounded with bound M,*
- (v) *$\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact,*
- (vi) *$x = AxBy + Cx \Rightarrow x \in S$, for all $y \in S$.*

Then the equation $x = AxBx + Cx$ has at least one solution in S if $M\phi_A(r) + \phi_C(r) < r$, for all $r > 0$.

Proof. This is an immediate consequence of Theorem 3.5 since $\left(\frac{I-C}{A}\right)^{-1} B$ is Φ-condensing on S, where Φ is an arbitrary MWNC on X.

Remark 3.7. Corollary 3.6 improves Theorem 3.5 in [8].

Corollary 3.8. *Let X be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ is a MWNC on X. Let S be a nonempty closed convex subset of X and let $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ be three operators such that*

- (i) *C is $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_C,*
- (ii) *A is nonexpansive and quasi-regular on X,*
- (iii) *A, B and C are weakly sequentially continuous on S,*
- (iv) *$B(S)$ is bounded with bound M,*
- (v) *$\left(\frac{I-C}{A}\right)^{-1} B$ is Φ-condensing on S with $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is bounded,*
- (vi) *$x = AxBy + Cx \Rightarrow x \in S$, for all $y \in S$.*

Then the equation $x = AxBx + Cx$ has at least one solution in S if $Mr + \phi_C(r) < r$, for all $r > 0$.

Taking $C \equiv x_0 \in X$ in Theorem 3.5, we obtain this following result.

Theorem 3.9. *Let X be a Banach algebra satisfying condition $(\mathcal{P})$, $x_0 \in X$ and Φ is a MWNC on X. Let S be a nonempty closed convex subset of X and let $A : X \longrightarrow X$ and $B : S \longrightarrow X$ be two operators such that*

- (i) *A is $\mathcal{D}$-Lipschitzian with the $\mathcal{D}$-functions ϕ_A,*
- (ii) *A is quasi-regular on X,*
- (iii) *A and B are weakly sequentially continuous on S,*
- (iv) *$B(S)$ is bounded with bound M,*
- (v) *$\left(\frac{I-x_0}{A}\right)^{-1} B$ is Φ-condensing on S with $\left(\frac{I-x_0}{A}\right)^{-1} B(S)$ is bounded,*
- (vi) *$x = AxBy + x_0 \Rightarrow x \in S$, for all $y \in S$.*

Then the equation $x = AxBx + x_0$ has at least one solution in S if $M\phi_A(r) < r$, for all $r > 0$.

Remark 3.10. Theorem 3.9 improves the conditions in Theorem 3.2 in [6] (and note the result of Theorem 3.2 in [6] is not correct).

Taking $A \equiv x_0$ a quasi-regular element of X in Theorem 3.5, we obtain this following result.

Theorem 3.11. *Let X be a Banach algebra satisfying condition $(\mathcal{P})$, $x_0 \in X$ a quasi-regular element and Φ is a MWNC on X. Let S be a nonempty closed convex subset of X and let $C : X \longrightarrow X$ and $B : S \longrightarrow X$ be two operators such that*

- *(i) C is $\mathcal{D}$-Lipschitzian with the $\mathcal{D}$-functions ϕ_C,*
- *(ii) C and B are weakly sequentially continuous on S,*
- *(iii) $B(S)$ is bounded with bound M,*
- *(iv) $\left(\frac{I-C}{x_0}\right)^{-1} B$ is Φ-condensing on S with $\left(\frac{I-C}{x_0}\right)^{-1} B(S)$ is bounded.*
- *(vi) $x = x_0By + Cx \Rightarrow x \in S$, for all $y \in S$.*

Then the equation $x = x_0Bx + Cx$ has at least one solution in S if $\phi_C(r) < r$, for all $r > 0$.

In particular if we take $x_0 = 1_X$, where 1_X is the unit element of the Banach algebra X, we obtain a Krasnoselskii type fixed point theorem.

Corollary 3.12. *Let X be a Banach algebra and Φ is a MWNC on X. Let S be a nonempty closed convex subset of X and let $C : X \longrightarrow X$ and $B : S \longrightarrow X$ be two operators such that*

- *(i) C is $\mathcal{D}$-Lipschitzian with the $\mathcal{D}$-functions ϕ_C,*
- *(ii) C and B are weakly sequentially continuous on S,*
- *(iii) $B(S)$ is bounded with bound M,*
- *(iv) $(I - C)^{-1} B$ is Φ-condensing on S with $(I - C)^{-1} B(S)$ is bounded,*
- *(vi) $x = By + Cx \Rightarrow x \in S$, for all $y \in S$.*

Then the equation $x = Bx + Cx$ has at least one solution in S if $\phi_C(r) < r$, for all $r > 0$.

Remark 3.13. It turns out that Corollary 3.12 remains valid in any Banach space. As a result we do not require the sequential condition $(\mathcal{P})$.

Finally we prove the following useful result.

Theorem 3.14. *Let S be a nonempty closed of a Banach algebra X satisfying condition $(\mathcal{P})$ and let $A, C : X \longrightarrow X$ and $B : S \longrightarrow X$ be three operators such that*

- *(i) A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
- *(ii) A is quasi-regular on X,*
- *(iii) A, B and C are weakly sequentially continuous on S,*
- *(iv) $B(S)$ is bounded with bound M,*
- *(v) B is weakly compact on S and $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is bounded,*
- *(vi) $x = AxBy + Cx \Rightarrow x \in S$, for all $y \in S$.*

Then the equation $x = AxBx + Cx$ has at least one solution in S if $M\phi_A(r) + \phi_C(r) < r$, for all $r > 0$.

Proof. According to Theorem 3.5, it suffices to prove that the operator $\left(\frac{I-C}{A}\right)^{-1} B$ is Φ-condensing where Φ is an arbitrary MWNC on X. To see this, let D be a bounded subset of S and $H = \left(\frac{I-C}{A}\right)^{-1} B(D)$. It is easily seen that

$$H \subset C(H) + A(H)B(D) \subset C(H) + A(H)\overline{B(D)^w}.$$

We claim that $\|\overline{B(D)^w}\| \leq M$. Indeed, let $x \in \overline{B(D)^w}$. By the Eberlein-Smulian theorem, there exists a sequence $(x_n) \subset B(D)$ such that $x_n \rightharpoonup x$. Since $\|x\| \leq \liminf \|x_n\|$ and for all n, $\|x_n\| \leq M$, we obtain that $\|x\| \leq M$. Using Lemma 2.11 and Lemma 2.1 in [6] with the weak compactness of $\overline{B(D)^w}$, we infer that

$$\beta(H) \leq \beta(C(H) + \beta(A(H)\overline{B(D)^w}) \leq \phi_C(\beta(H)) + M\phi_A(\beta(H)).$$

This shows that $\beta(H) = 0$. Thus H is relatively weakly compact, and $\left(\frac{I-C}{A}\right)^{-1} B$ is Φ-condensing where Φ is an arbitrary MWNC on X. The result follows from Theorem 3.5.

Remark 3.15. Theorem 3.14 improves Corollary 3.1 in [7].

Remark 3.16. Theorem 3.3 in [6] is not correct and Theorem 3.14 gives a correct version of this theorem.

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Received: February 6, 2014; Accepted: March 27, 2014.