Fixed Point Theory, 18(2017), No. 1, 27-36
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

KRASNOSEL'SKII-TYPE FIXED-SET RESULTS UNDER WEAK TOPOLOGY CIRCUMSTANCES AND APPLICATIONS

AFIF BEN AMAR*, MOHAMED BOUMAIZA** AND SANA HADJ AMOR**

*Faculté des Sciences de Sfax
Rue Soukra 3038 Sfax, Tunisia
E-mail:afif.benamar@ipeis.rnu.tn

**Institut Supérieure des Sciences et de Technologies de Hammam Sousse
Rue Lamin El Abbassi 4011 Hammam Sousse, Tunisia
E-mail:mohamed.boumaiza@essths.rnu.tn, sana.hadjamor@yahoo.fr

Abstract. In this paper, using the technique of measure of weak noncompacteness, we prove some fixed set results of Krasnosel'skii type for the sum of two multivalued operators in the setting of weak topology, without the assumption of the convexity of their common domain. Applications to the theory of self-similarity are also given.
Key Words and Phrases: Krasnosel'skii theorem, fixed set, weak topology, self-similarity theory.
2010 Mathematics Subject Classification:47H10, 47H09.

1. Introduction

Krasnosel'skii stated in [9] one of most important theorem in nonlinear analysis, which is the following:

Theorem 1.1. *Let M be a non-empty closed convex subset of a Banach space E and S and T be two mappings from M into E such that*

(1) *S is compact and continuous,*
(2) *T is a strict contraction mapping,*
(3) *$S(M) + T(M) \subset M$.*

Then there exists an $x \in M$ such that $Sx + Tx = x$.

It has been extensively used and has also been generalized in many directions (see [10, 2, 17, 4] and the references therein). In [18, 3, 13, 14, 15], the authors extended a number of existing generalizations or modifications of Krasnosel'skii fixed point theorem for the weak topology.

In [12], Ok studied the case when the convexity of the set M is relaxed and he proved a fixed (sometimes also called, invariant) set i.e. find a non-empty subset K of M such that

$$S(K) + T(K) = K.$$

In [16], Thagafi and Shahzad proved several new Krasnosel'skii type fixed set theorems for the sum $S+T$, where S is a multivalued mapping and T is a single-valued mapping. Their results provided a positive answer to the question of Ok [12] and showed that the fixed set is compact.

In this paper, we extend, generalize and improve several fixed set results including that given in [12] and [16] under weak topology circumstances. Applications to the theory of self-similarity are also given.

2. Preliminaries

We present some notations and preliminary facts which we will need in what follows.

Let Y be a Hausdorff linear topological space and let M be a non empty subset of a Banach space X.

We note 2^Y, the set of all subsets of Y. The multi-valued mapping $T : M \to 2^Y$ is said to have weakly sequentially closed graph if for every sequence $\{x_n\} \subset M$ with $x_n \rightharpoonup x$ in M and for every sequence $\{y_n\}$ with $y_n \in T(x_n), \forall n \in \mathbb{N}, y_n \rightharpoonup y$ in Y implies $y \in T(x)$, where $\rightharpoonup$ denotes weak convergence.

T is called weakly completely continuous if T has a weakly sequentially closed graph and, if A is a bounded subset of M, then $T(A)$ is a relatively weakly compact subset of Y.

If T is a single valued mapping, then T is said weakly sequentially continuous whenever for every sequence $\{x_n\} \subset M$ with $x_n \rightharpoonup x \in M$, we have $T(x_n) \rightharpoonup T(x)$.

Definition 2.1. Let M be a nonempty subset of a Banach space X, and let $T : M \to X$. One says that T is a nonlinear contraction if there exists an upper semicontinuous map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) < t$ for every $t > 0$ such that $d(Tx, Ty) \leq \varphi(d(x, y))$ for all $x, y \in M$.

Definition 2.2. Let X a Banach space and C be a lattice with a least element, which is denoted by 0. By a measure of weak non-compactness (MNWC) on X, we mean a function Φ defined on the set of all bounded subsets of X with values in C satisfying:

1. $\Phi(\overline{conv}(\Omega)) = \Phi(\Omega)$, for all bounded subsets $\Omega \subseteq X$, where $\overline{conv}$ denotes the closed convex hull of Ω.
2. For any bounded subsets Ω_1, Ω_2 of X we have
$$\Omega_1 \subseteq \Omega_2 \Longrightarrow \Phi(\Omega_1) \leq \Phi(\Omega_2).$$
3. $\Phi(\Omega \cup \{a\}) = \Phi(\Omega)$ for all $a \in X, \Omega$ bounded set of X.
4. $\Phi(\Omega) = 0$ if and only if Ω is relatively weakly compact in X.

The MNWC Φ is said positive homogeneous provided $\Phi(\lambda\Omega) = \lambda\Phi(\Omega)$ for all $\lambda > 0$ and Ω is a bounded set in X.

The above notion is a generalization of the important well known De Blasi measure of weak non-compactness β (see [6]) defined on each bounded set Ω of X by

$$\beta(\Omega) = \inf\{\varepsilon > 0; \text{there exists a weakly compact set } D \text{ such that } \Omega \subseteq D + B_\varepsilon(0)\}.$$

It is well known that β enjoys these properties for all bounded subsets $\Omega, \Omega_1, \Omega_2$ of X

1. $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$
2. $\beta(\lambda\Omega) = \lambda\beta(\Omega)$ for all $\lambda > 0$.

(3) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$.

Definition 2.3. Let Ω be a nonempty subset of Banach space X and Φ a MNWC on X. If F maps Ω into X. We say that F is $\Phi-$condensing if $\Phi(F(D)) < \Phi(D)$ for all bounded sets $D \subseteq \Omega$ with $\Phi(D) \neq 0$.

3. Fixed-set results

In this section we prove, in the setting of weak topology, some new Karasnosel'skii type theorems. Our purpose is to obtain the existence of a fixed set (instead of a fixed point) for the sum of two operators, by removing the convexity hypothesis.

Theorem 3.1. *Let M be a non-empty weakly closed subset of a Banach space E and Φ a semiadditive MNWC on E. Assume $S : M \to 2^E$ and $T : M \to 2^E$ satisfying the following conditions:*

(1) *S is weakly completely continuous.*
(2) *T is Φ-condensing and has weakly sequentially closed graph.*
(3) *$S(M) + T(M)$ is a bounded set of M.*

Then
i) there exists a minimal K weakly compact subset of M such that $K = S(K) + T(K)$;
ii) there exists a maximal $A \in 2^M$ such that $A = S(A) + T(A)$.

Proof. Let $y \in S(M) + T(M)$ and

$$\mathcal{C} = \{C \subset M, \text{ weakly closed such that } y \in C \text{ and } S(C) + T(C) \subset C\}$$

By 3), $\mathcal{C}$ is non-empty. Let $C_0 = \bigcap_{C \in \mathcal{C}} C$. It is clear that $C_0 \in \mathcal{C}$. We define

$$L = \overline{S(C_0) + T(C_0) \cup \{y\}}^w.$$

The set L satisfies

$$S(L) + T(L) \subset S(C_0) + T(C_0) \subset L.$$

Then, we deduce that $C_0 = L = \overline{S(C_0) + T(C_0) \cup \{y\}}^w$. On the other hand L is bounded and if $\Phi(L) > 0$, we have

$$\Phi(L) \leq \Phi(S(L)) + T(L)) < \Phi(L).$$

Then L is weakly compact. Let now

$$\mathcal{F} = \{F \subset M \text{ weakly compact, such that } F \subset L \text{ and } S(F) + T(F) \subset F\}.$$

Note that $\mathcal{F}$ is nonempty since $L \in \mathcal{F}$. Any chain in the posed $(\mathcal{F}, \supseteq)$ has the finite intersection property, so as L is weakly compact the intersection of all members of any chain in $(\mathcal{F}, \supseteq)$ is non-empty. Then, any chain in $(\mathcal{F}, \supseteq)$ has a lower bound in $\mathcal{F}$. Therefore, by Zorn's lemma $(\mathcal{F}, \supseteq)$ has a minimal element, say K. By definition, we have that K is weakly compact, $K \subset L$ and $S(K) + T(K) \subset K$. Let

$$N = \overline{S(K) + T(K)}^{\omega}.$$

N is also weakly compact and $N \subset K$. It follows that

$$S(N) + T(N) \subset S(K) + T(K) \subset \overline{S(K) + T(K)}^{\omega} \subset N$$

and hence $N \in \mathcal{F}$. Thus $N = K = \overline{S(K)+T(K)}^{\omega}$. We prove now that $S(K)+T(K)$ is weakly closed. Let $x \in \overline{S(K)+T(K)}^{w}$, by the Eberlein-Smulian theorem (see [7] Theorem 8.12.4 p.549), there exists a sequence $\{x_n\} \subset S(K)+T(K)$ such that $x_n \rightharpoonup x$. Then, there exist a sequence $\{\alpha_n\} \subset S(K)$ and a sequence $\{\beta_n\} \subset T(K)$ such that

$$x_n = \alpha_n + \beta_n$$

with $\alpha_n \in S(y_n)$ and $\beta_n \in T(z_n)$, for some $y_n, z_n \in K$. Since $\{y_n\}$ is bounded and S is weakly completely continuous, by the Eberlein-Smulian theorem, there exists a subseqence $\alpha_{n_k} \rightharpoonup \alpha$. On the other hand, since K is weakly compact, by the Eberlein-Smulian theorem, there exists a subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ which converge weakly to $y \in K$. Since S has weakly sequentially closed graph, we get $\alpha \in S(y)$. It is clear that $\beta_{n_{k_j}} \rightharpoonup x - \alpha$. By the Eberlein-Smulian theorem, there exists a subsequence of $\{z_{n_{k_j}}\}$ which converge weakly to $z \in K$. Since T has weakly sequentially closed graph, we get $\beta \in T(z)$ and $x = \alpha + \beta \in S(y) + T(z) \subset S(K) + T(K)$. Hence

$$K = \overline{S(K)+T(K)}^{w} = S(K) + T(K).$$

Which proves $i)$. For $ii)$, let

$$\mathcal{C} = \{C \subset M; C \subset S(C) + T(C)\}$$

and $A = \bigcup_{C \in \mathcal{C}} C$. Clearly A is nonempty since $K \in \mathcal{C}$. We have $A \subset S(A) + T(A)$. Take $y \in S(A) + T(A)$. It follows that

$$A \cup \{y\} \subset S(A) + T(A) \subset S(A \cup \{y\}) + T(A \cup \{y\})$$

and hence $A \cup \{y\} \in \mathcal{C}$ and $y \in A$. Thus $S(A) + T(A) = A$.

Remark 3.2. By using the techniques of measures of weak compactness, we establish in Theorem 3.1 a weak topology version of Theorem 3.1 in [16] for the case of multivalued functions.

Corollary 3.3. *Let M be a non-empty weakly closed subset of a Banach space E and Φ a semiadditive MNWC on E. Assume $S : M \to 2^E$ and $T : E \to E$ satisfying the following conditions:*

(1) *S is weakly completely continuous.*
(2) *T is a nonlinear contraction and weakly sequentially continuous.*
(3) *$S(M) + T(M)$ is a bounded set of M.*

Then,
$i)$ there exists a minimal K weakly compact subset of M such that $K = S(K)+T(K)$;
$ii)$ there exists a maximal $A \in 2^M$ such that $A = S(A) + T(A)$.

Proof. T is a nonlinear contraction, then it is $\beta-$condensing (see [1]). We apply now Theorem 3.1 and we use the fact that every weakly sequentially continuous single valued mapping has weakly sequentially closed graph.

Theorem 3.4. *Let M be a non-empty weakly closed subset of a Banach space E and Φ MNWC on E. Assume $S : M \to 2^E$ and $T : M \to E$ satisfying the following conditions:*

(1) *S has weakly sequentially closed graph.*

(2) *T is weakly sequentially continuous.*
(3) *$\Phi(S(A)+T(A)) < \Phi(A)$, for all bounded subset A in M with $\Phi(A) > 0$.*
(4) *$S(M)+T(M)$ is a bounded set of M.*

Then,
i) there exists a minimal K weakly compact subset of M such that $K = S(K)+T(K)$;
ii) there exists a maximal $A \in 2^M$ such that $A = S(A)+T(A)$.

Proof. As in the proof of Theorem 3.1, we prove that there exists a weakly compact K such that $S(K)+T(K) \subset K$ and $K = \overline{S(K)+T(K)}^w$. It suffices now to show that $S(K)+T(K)$ is weakly closed. Let $x \in \overline{S(K)+T(K)}^w$. Since $\overline{S(K)+T(K)}^w$ is weakly compact then by the Eberlein-Smulian theorem, there exists a sequence $\{x_n\} \subset S(K)+T(K)$ such that $x_n \rightharpoonup x$. Then, there exist a sequence $\{\alpha_n\} \subset S(K)$ and a sequence $\{\beta_n\} \subset T(K)$ such that

$$x_n = \alpha_n + \beta_n$$

with $\alpha_n \in S(y_n)$ and $\beta_n \in T(z_n)$, for some $y_n, z_n \in K$. Since K is weakly compact, by the Eberlein-Smulian theorem, there exists a subseqence $z_{n_k} \rightharpoonup z \in K$. Since T is weakly sequently continuous, then $x_{n_k} - T(z_{n_k}) \rightharpoonup x - T(z)$. Similarly, by the Eberlein-Smulian theorem, there exists a subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ which converges weakly to $y \in K$. Since S has weakly sequentially closed graph, we get $x - T(z) \in S(y)$ which implies that $x \in S(K)+T(K)$. Hence

$$K = \overline{S(K)+T(K)}^w = S(K)+T(K).$$

Which proves i). Finally, ii) follows by the same proof as in Theorem 3.1.

Corollary 3.5. *Let M be a non-empty weakly closed subset of a Banach space E and Φ a measure of weak noncompactness (MNWC) on E. Assume $S : M \to 2^E$ satisfying the following conditions:*

(1) *S is Φ-condensing and has weakly sequentially closed graph.*
(2) *$S(M)$ is a bounded set of M.*

Then
i) there exists K a weakly compact subset of M such that $S(K) = K$;
ii) there exists a maximal $A \in 2^M$ such that $A = S(A)$. Moreover $A = \bigcap_{k\geq 1} S^k(M)$.

Proof. First, we apply Theorem 3.1 or Theorem 3.4 with $T = 0$. So to finish the proof, it is sufficient to prove that $A = \bigcap_{k\geq 1} S^k(M)$. Define

$$\mathcal{F} = \{F \in M \text{ such that } S(F) = F\}.$$

By i), it is clear that $\mathcal{F}$ is not empty, since $K \in \mathcal{F}$. We put $A = \cup_{F\in\mathcal{F}} F$. We have that A is maximal and $A = S(A)$. On the other hand, let $B = \bigcap_{n\geq 1} S^n(M)$. For $F \in \mathcal{F}$, we have

$$F = S(F) \subset S(M).$$

Then, $F = S(F) \subset S^2(M)$ and by iteration, we obtain $F \subset S^n(M)$, for all $n \geq 1$. So

$$A = \cup_{F\in\mathcal{F}} F \subset S^n(M), \text{ for all } n \geq 1.$$

Consequently,

$$A = \cup_{F\in\mathcal{F}} F \subset \bigcap_{n\geq 1} S^n(M) = B.$$

Since $S(M) \subset M$, we have that $S(B) = B$. By the facts that A is maximal, $B \in \mathcal{F}$ and $A \subset B$, it follows that $A = B$.

Remark 3.6. By Corollary 3.5 and using the techniques of measures of weak compacteness, we establish a weak topology version of Proposition 4.2 in [11] and Lemma 2.3 in [12].

Corollary 3.7. *Let M be a non-empty weakly closed subset of a Banach space E. Assume $S : M \to 2^M$ such that*

(1) *$S(M)$ is bounded,*
(2) *$S(N)$ is relatively weakly compact, for all bounded set $N \subset M$.*

Then,
i) there exists K a weakly compact subset of M such that $\overline{S(K)}^{\omega} = K$.
ii) there exists a maximal $A \in 2^M$ such that $A = S(A)$. Moreover $A = \bigcap_{k\geq 1} S^k(M)$.

Proof. This is an immediate consequence of Corollary 3.5, since S is $\Phi-$condensing for any measure of weak noncompactness (MNWC) on E.

Theorem 3.8. *Let M be a non-empty weakly closed subset of a Banach space E and Φ a semiadditive MNWC on E. Assume $S : M \to 2^E$ and $T : E \to 2^E$ satisfying the following conditions:*

(1) *S is weakly completely continuous.*
(2) *T is Φ-condensing and has weakly sequentially closed graph.*
(3) *$S(M) \subset (I - T)(E)$ and $x \in T(x) + S(y),\ y \in M \Longrightarrow x \in M$.*
(4) *$(I - T)^{-1}S(M)$ is bounded.*

Then,
i) there exists a minimal K weakly compact subset of M such that $(I-T)(K) = S(K)$ and $K \subset S(K) + T(K)$
ii) there exists a maximal A subset of M such that $S(A) + T(A) = A$.

Proof. First of all, we check that $(I - T)^{-1}S(M) \subset M$. Let $x \in (I - T)^{-1}S(M)$, then there exists $y \in M$ such that $x \in (I - T)^{-1}(S(y))$. It follows that $x \in T(x) + S(y)$ and by assumption 3) we get $x \in M$. Hence, $(I - T)^{-1}S(M) \subset M$. We define

$$N := (I - T)^{-1} \circ S : M \to 2^M$$

Let $x_0 \in M$. Define

$$\mathcal{A} = \{A \subset M \text{ weakly closed such that } x_0 \in A \text{ and } N(A) \subset A\}.$$

Note that $\mathcal{A}$ is non-empty since $\overline{N(M) \cup \{x_0\}}^{\omega} \in \mathcal{A}$. Take $A_0 = \bigcap_{A\in\mathcal{A}} A$. As A_0 is weakly closed, $x_0 \in A_0$, and $N(A_0) \subset A_0$, we have $A_0 \in \mathcal{A}$. Let

$$L = \overline{N(A_0) \cup \{x_0\}}^{\omega}.$$

So L is a weakly closed subset of $A_0, x_0 \in L$, and

$$N(L) \subset N(A_0) \subset \overline{N(A_0) \cup \{x_0\}}^{\omega} = L.$$

This shows that $L \in \mathcal{A}$ and $L = A_0$. Notice that $\overline{((I-T)^{-1}S(M)) \cup \{x_0\}}^{\omega}$ is a bounded subset of M containing L and $\Phi(A_0) = \Phi(\overline{N(A_0) \cup \{x_0\}}^{\omega}) = \Phi(N(A_0))$. But

$$N(A_0) = (I-T)^{-1}S(A_0) \subset T(I-T)^{-1}S(A_0) + S(A_0) \subset T(A_0) + S(A_0).$$

So

$$\Phi(N(A_0)) \leq \Phi(T(A_0) + S(A_0)) \leq \Phi(T(A_0)) + \Phi(S(A_0))$$

Since $\overline{S(A_0)}^{\omega}$ is weakly compact and T is Φ-condensing, then

$$\Phi(N(A_0)) \leq \Phi(T(A_0)) < \Phi(A_0).$$

Hence, it follows that $\Phi(A_0) = 0$ and A_0 is weakly compact. Therefore, $N : A_0 \to 2^{A_0}$ is weakly compact. By Corollary 3.7 there exists a minimal K weakly compact of M such that $\overline{N(K)}^{\omega} = K$. Let $x \in \overline{N(K)}^{\omega}$. By the Eberlein Smulian theorem, there exists a sequence $\{x_n\} \subset N(K)$ such that $x_n \rightharpoonup x$. We have $x_n \in N(y_n) = (I-T)^{-1}S(y_n)$, for all $n \geq 1$ with $y_n \in K$. Then $(I-T)x_n \cap S(y_n) \neq \emptyset$, for all $n \geq 1$ and we have

$$x_n \in Tx_n + \alpha_n,$$

where $\alpha_n \in S(y_n)$. Since $\{y_n\}$ is bounded, by assumption 1) $S(\{y_n\})$ is relatively weakly compact, then, by the Eberlein Smulian theorem, there is a subsequence $\{\alpha_{n_k}\}$ such that $\alpha_{n_k} \rightharpoonup \alpha$, Since S has weakly sequentially closed graph and using a subsequence $y_{n_{k_j}} \rightharpoonup y \in K$, we get $\alpha \in S(y)$. On the other hand, $x_{n_{k_j}} - \alpha_{n_{k_j}} \rightharpoonup x - \alpha$. Since T has weakly sequentially closed graph, we get $x - \alpha \in T(x)$, then $x \in (I-T)^{-1}S(y)$. We deduce that

$$K = N(K),$$

which gives that $(I-T)(K) = S(K)$, and hence $K \subset S(K) + T(K)$, which proves $i)$. For $ii)$, the result follows by the same proof as in Theorem 3.1.

Theorem 3.9. *Let M be a non-empty weakly closed subset of a banach space E and Φ a semiadditive MNWC on E. Assume $S : M \to 2^E$ and $T : M \to E$ satisfying the following conditions:*

(1) *S is weakly completely continuous.*
(2) *T is Φ-condensing and weakly sequentially continuous.*
(3) *$S(M) \subset (I-T)(M)$ and $(I-T)^{-1}S(M)$ is bounded.*
(4) *$(I-T)^{-1}$ is a single-valued map on $S(M)$.*

Then,
$i)$ there exists a minimal K weakly compact subset of M such that $(I-T)(K) = S(K)$ and $K \subset S(K) + T(K)$
$ii)$ there exists a maximal A subset of M such that $S(A) + T(A) = A$.

Proof. Let $y \in M$. Then, by (3), there exists $A \subset M$ such that $Sy \subset (I-T)A$, and, as $(I-T)^{-1}$ is a single valued map on $S(M)$,

$$((I-T)^{-1} \circ S)y = (I-T)^{-1}(Sy) \subset A \subset M.$$

So $N := (I-T)^{-1} \circ S : M \to 2^M$. Let $x_0 \in M$. As in the proof of Theorem 3.8, there exists a minimal K weakly compact such that $\overline{N(K)}^{\omega} = K$. Let $x \in \overline{N(K)}^{\omega}$. By the Eberlein Smulian theorem, there exists a sequence $\{x_n\} \subset N(K)$ such that

$x_n \rightharpoonup x$. We have $x_n \in N(y_n) = (I-T)^{-1}S(y_n)$, for all $n \geq 1$ with $y_n \in K$. Then $(I-T)x_n \in S(y_n)$, for all $n \geq 1$ and we have

$$x_n = Tx_n + \alpha_n,$$

where $\alpha_n \in S(y_n)$. Since $\{y_n\} \subset K$ and K is weakly compact, there is a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \rightharpoonup y \in K$. Since now T is weakly sequentially continuous, $\alpha_{n_k} = x_{n_k} - T(x_{n_k})$ converges weakly to $x - T(x)$. Since S has weakly sequentially closed graph, we get $x - T(x) \in S(y) \subset S(K)$. hence $x \in (I-T)^{-1}S(K)$. Consequently,$K = N(K)$. Wich gives that $(I-T)(K) = S(K)$, and hence $K \subset S(K) + T(K)$. The remained proof follows along the lines of Theorem 3.1.

Theorem 3.10. *Let M be a non-empty weakly closed subset of a Banach space E. Assume $S : M \to 2^E$ and $T : E \to E$ satisfying the following conditions:*

1. *S has a weakly sequentially closed graph and $S(M)$ is relatively weakly compact.*
2. *T is a nonlinear contraction and weakly sequentially continuous.*
3. *$S(M) + T(M) \subset M$.*

Then,
i) there exists a minimal K weakly compact subset of M such that $(I-T)(K) = S(K)$ and $K \subset S(K) + T(K)$
ii) there exists a maximal A subset of M such that $S(A) + T(A) = A$.

Proof. Let $z \in \overline{S(M)}^w$. By (2) and (4) the map $x \mapsto z + Tx$ is a nonlinear contraction from M into M. So, there exists a unique $x_0 \in M$ such that $x_0 = z + Tx_0$. Then $z = x_0 - Tx_0 \in (I-T)M$, and so $S(M) \subset (I-T)M$. Since the map $x \mapsto z + Tx$ has a unique fixed-point, its fixed-point set $(I-T)^{-1}z$ is a singleton. So $(I-T)^{-1} : \overline{S(M)}^w \to M$ is a single-valued map.

In the following, we proof that $(I-T)^{-1}$ is weakly sequentially continuous on $\overline{S(M)}^\omega$. Let $z_n, z \in \overline{S(M)}^\omega$, such that $z_n \rightharpoonup z$. Define

$$x_n = (I-T)^{-1}(z_n) \text{ and } x = (I-T)^{-1}z.$$

We show that $x_n \rightharpoonup x$. If we suppose that $\{x_n\}$ is not weakly convergent to x, then there exists a neighborhood V of x and a subsequence $\{x_{n_j}\}$ such that for all $j \geq 1$, $x_{n_j} \notin V$. On the other hand $z_{n_j} = (I-T)(x_{n_j}) \rightharpoonup z$. Since T is $\beta-$condensing (see [1]), we have as explained in [5], that if $\beta(\{x_n\}) > 0$, then

$$\beta(\{x_n\}) \leq \beta(\{(I-T)(x_n)\}) + \beta(\{Tx_n\}) \leq \beta(\{Tx_n\}) \leq \beta(\{x_n\}).$$

So $\beta(\{x_n\}) = 0$. By the Eberlein Smulian theorem, there exists a subsequence $\{x_{n_{j_k}}\}$ which converges weakly to $x_0 \in M$. Then

$$(I-T)(x_{n_{j_k}}) \rightharpoonup z$$

and, so, $T(x_{n_{j_k}}) \rightharpoonup x_0 - z$. Consequently $x_0 - z = T(x_0)$ and $x_0 = (I-T)^{-1}z = x$. This contradict the choice of V, hence $x_n = (I-T)^{-1}(z_n) \rightharpoonup x = (I-T)^{-1}z$, which means that $(I-T)^{-1}$ is weakly sequentially continuous on $\overline{S(M)}^\omega$. On the other hand $\overline{S(M)}^\omega$ is weakly compact, then the multimap

$$(I-T)^{-1} \circ S : M \longrightarrow 2^M$$

is weakly compact. By Corollary 3.3, there exists a weakly compact set $K \subset M$ such that

$$\overline{(I-T)^{-1}S(K)}^{w} = K.$$

It suffices now to show that $(I-T)^{-1}S(K)$ is weakly closed. Let $z \in \overline{(I-T)^{-1}S(K)}^{w}$, there exists a sequence $z_n \in (I-T)^{-1}S(K)$ such that $z_n \rightharpoonup z$ and a sequence $x_n \in K$ such that $z_n \in (I-T)^{-1}S(x_n)$ and then

$$z_n - T(z_n) \in S(x_n).$$

Since K is weakly compact, we assume that $x_n \rightharpoonup x \in K$. Since T is weakly sequentially continuous, $z_n - T(z_n) \rightharpoonup z - T(z)$. We use the fact that S has a weakly sequentially closed graph, we get $z - T(z) \in S(x)$ which implies that $z \in (I-T)^{-1}S(x) \subset (I-T)^{-1}S(K)$. Which proves i). For ii) the result follows as in Theorem 3.1.

4. An application: the existence of self-similar sets

Let M be a non-empty weakly closed subset of a Banach space E and $\mathcal{F}$ be a family of self-maps of M. For any $x \in M$ and $S \subset M$, we let

$$\mathcal{F}(x) = \{f(x),\ f \in \mathcal{F}\},\ \mathcal{F}(S) = \bigsqcup\{f(S),\ f \in \mathcal{F}\}.$$

A non-empty set S is said to be self similar if $\mathcal{F}(S) = S$. If $\mathcal{F} = \{F_1, ..., F_n\}$ is a finitely family of self-maps, then $(M, \{F_1, ..., F_n\})$ is called an iterated function system (IFS). We say that an (IFS) is continuous (resp. contraction, Φ-condensing, etc.) if each F_i is so. A well-known theorem of fractal geometry says that there exists a unique compact self-similar with respect to any contraction (IFS) (see [8]). In [11], OK has extend this result to a continuous and Φ- condensing (IFS). For a Φ-condensing (IFS) in weak topology circumstances we show the following results.

Theorem 4.1. *Let M be a non-empty weakly closed subset of a Banach space E and Φ a measure of weak non compactness on E. Let $(M, \{F_1, ..., F_n\})$ be a $\Phi-$condensing (IFS) such that*

(1) *$F_1 \cup \cdots \cup F_n$ has weakly sequentially closed graph,*
(2) *$F_1(M) \cup ... \cup F_n(M)$ is bounded.*

Then there exists a weakly compact self-similar set with respect to $(M, \{F_1, ..., F_n\})$.

Proof. According to Corollary 3.5, it suffices to show that

$$F : M \longrightarrow 2^M,\ x \longmapsto F_1(x) \cup ... \cup F_n(x)$$

is $\Phi-$condensing. In fact, for all bounded set $A \subset M$, we have

$$\Phi(F(A)) = \Phi(F_1(A) \cup ... \cup F_n(A)) = max(\Phi(F_1(A)), ..., \Phi(F_n(A))) < \Phi(A).$$

The following result is an immediate consequence of Theorem 3.10.

Theorem 4.2. *Let M be a non empty closed subset of a Banach space E and Φ a semi-additive measure of weak non compactness on E. Let $(M, \{F_1, ..., F_n, F_{n+1}\})$ be a (IFS) such that*

(1) *$F_1 \cup \cdots \cup F_n$ has weakly sequentially closed graph and $F_1(M) \cup ... \cup F_n(M)$ is relatively weakly compact,*
(2) *F_{n+1} is a nonlinear contraction and weakly sequentially continuous,*
(3) *$F_i(M) + F_{n+1}(M) \subset M, \ \forall i = 1, \cdots, n.$*

Then there exists a weakly compact set K of M such that

$$(I - F_{n+1})(K) = F_1(K) \cup ... \cup F_n(K).$$

Remark 4.3. This Theorem provides information about the stability of self-similar sets with respect to some perturbations.

References

[1] A., Ben Amar, *Krasnoselskii type fixed point theorems for multivalued mapping with weakly sequentially closed graph*, Ann. Univ. Ferrara, **58**(2012), 1-10.

[2] A. Ben Amar, A. Jeribi, M. Mnif, *On a generalization of the Schauder and Krasnosel'skii fixed point theorems on Dunford-Pettis space and applications*, Math. Methods Appl. Sci., **28**(2005), 1737-1756.

[3] A. Ben Amar, A. Jeribi, M. Mnif, *Some fixed point theorems and application to biological model*, Numer. Funct. Anal. Optim., **29**(2008), no. 1, 1-23.

[4] C.S. Barrosso, E.V., Teixeira, *A topological and geometric approach to fixed points results for sum of operators and applications*, Nonlinear Anal., **60**(2005), 625-650.

[5] A. Ben Amar, T. Xiang, *Critical type of Krasnoselskii fixed point theorem in weak topology circumstances*, Submitted.

[6] F.S. De Blasi, *On a property of the unit sphere in a Banach space*, Bull. Math. Soc. Sci. Math. Roumanie, **21**(1977), no. 69, 259-262.

[7] R.E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Reinhart and Winston, 1965.

[8] J. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J., **30**(1981), 713-747.

[9] M.A. Krasnosel'skii, *Some problems of nonlinear analysis*, Math. Soc. Transl. Ser. 2, **10**(1958), no. 2, 345-409.

[10] Y.C. Liu, Z.X. Li, *Krasnosel'skii type fixed point theorems and applications*, Proc. Amer. Math. Soc., **136**(2008), 1213-1220.

[11] E.A. Ok, *Fixed set theory for closed correspondences with applications to self similarity and games*, Nonlinear Anal., **56**(2004), no. 3, 309-330.

[12] E.A. Ok, *Fixed set theorems of Krasnosel'skii type*, Proc. Amer. Math. Soc., **137**(2009), no. 2, 511-518.

[13] D. O'Regan, *Fixed point theory for weakly contractive maps with applications to operator inclusions in Banach spaces relative to the weak topology*, Z. Anal. Anwendungen, **17**(1998), 282-296.

[14] D. O'Regan, *Fixed point theorems for weakly sequentially closed maps*, Arch. Math., **36**(2000), 61-70.

[15] M.A. Taoudi, *Krasnoselskii type fixed point theorems under weak topology features*, Nonlinear Anal., **72**(2010), 478-482.

[16] M.A. Al-Thagafi, N. Shahzad, *Krasnoselskii-type fixed point results*, Fixed Point Theory and Applications, ID 394139 (2009), 9 pages.

[17] T. Xiang, R. Yuan, *Critical type of Krasnosel'skii fixed point theorem*, Proc. Amer. Math. Soc., **139**(2011), 1033-1044.

[18] T. Xiang, R. Yuan, *Krasnoselskii-type fixed point theorems under weak topology settings and applications*, Electronic J. Diff. Equations, **35**(2010), 1-15.

Received: February 26, 2014; Accepted: February 28, 2015.