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SOME STABILITY CONCEPTS FOR ABSTRACT FRACTIONAL DIFFERENTIAL EQUATIONS WITH NOT INSTANTANEOUS IMPULSES

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Abstract. In this paper, we investigate some uniqueness and Ulam's type stability concepts for functional abstract fractional differential equations with not instantaneous impulses in Banach spaces.
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1. Introduction

The fractional calculus deals with extensions of derivatives and integrals to noninteger orders. It represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [5], Kilbas *et al.* [15], Miller and Ross [16], the papers of Abbas *et al.* [1, 2, 3, 4, 6, 7, 8], Benchohra *et al.* [9], Diethelm [10], Kilbas and Marzan [14], Podlubny [19], Vityuk and Golushkov [22], and the references therein.

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In [13] Hernández and O'Regan initially offered to study a new class of abstract semilinear impulsive differential equations with not instantaneous impulses in a PC-normed Banach space. Meanwhile, in [18, 20, 23, 24] the authors continue to study other new classes of differential equations with not instantaneous impulses. Motivated by the above works, we investigate the uniqueness and Ulam-Hyers-Rassias stability of the following abstract fractional differential equations with not instantaneous impulses of the form

$$\begin{cases} {}^cD^r_{s_k}u(t) = Au(t) + f(t, u(t)); \text{ if } t \in I_k,\ k = 0, \dots, m, \\ u(t) = g_k(t, u(t)); \text{ if } t \in J_k,\ k = 1, \dots, m, \\ u(0) = u_0 \in E, \end{cases} \tag{1.1}$$

where $I_k := (s_k, t_{k+1}],\ J_k := (t_k, s_k],\ {}^cD^r_{s_k}$ is the fractional Caputo derivative of order $r \in (0, 1],\ 0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < s_{m-1} \le t_m \le s_m \le t_{m+1} = a,\ f : I_k \times E \to E;\ k = 0, \dots, m,\ g_k : J_k \times E \to E;\ k = 1, \dots, m$ are given continuous functions, E is a (real or complex) separable Banach space and A is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{T(t);\ t > 0\}$ in E. We present sufficient conditions to obtain some uniqueness of mild solutions and the Ulam-Hyers-Rassias stability for the abstract impulsive problem (1.1). An example is also provided to illustrate the main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $J = [0, a];\ a > 0$, denote $L^1(J)$ the space of Bochner-integrable functions $u : J \to E$ with the norm

$$\|u\|_{L^1} = \int_0^a \|u(t)\|_E dt,$$

where $\|.\|_E$ denotes a suitable complete norm on E.
As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into E, and $\mathcal{C} := C(J)$ is the Banach space of all continuous functions from J into E with the norm $\|.\|_\infty$ defined by $\|u\|_\infty = \sup_{t \in J} \|u(t)\|_E$. Consider the Banach space

$$\begin{aligned} PC = &\big\{u : J \to E : u \in C((t_k, t_{k+1}]);\ k = 0, 1, \dots, m, \text{ and there} \\ &\text{exist } u(t_k^-) \text{ and } u(t_k^+);\ k = 1, \dots, m, \text{ with } u(t_k^-) = u(t_k)\big\}, \end{aligned}$$

with the norm $\|u\|_{PC} = \sup_{t \in J} \|u(t)\|_E$.

Let $r > 0$. For $u \in L^1(J)$, the expression

$$(I_0^r u)(t) = \frac{1}{\Gamma(r)} \int_0^t (t - \tau)^{r-1} u(\tau) d\tau,$$

is called the left-sided mixed Riemann-Liouville integral of order r, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\varsigma) = \int_0^\infty t^{\varsigma - 1} e^{-t} dt;\ \varsigma > 0$.

In particular,

$$(I_0^0 u)(t) = u(t),\ (I_0^1 u)(t) = \int_0^t u(\tau)d\tau;\ \text{for almost all } t \in J.$$

For instance, $I_0^r u$ exists for all $r \in (0, \infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_0^r u) \in C(J)$.

Example 2.1. Let $\lambda \in (-1, 0) \cup (0, \infty),\ r \in (0, \infty)$ and $h(t) = t^\lambda;\ \ t \in J$. We have $h \in L^1(J)$, and we get

$$(I_0^r h)(t) = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+r)} t^{\lambda+r};\ \text{for almost all } t \in J.$$

Definition 2.2. [21] Let $r \in (0, 1]$ and $u \in L^1(J)$. The Caputo fractional-order derivative of order r of u is defined by the expression

$${}^c D_0^r u(t) = (I_\theta^{1-r} \frac{d}{dt} u)(t) = \frac{1}{\Gamma(1-r)} \int_0^t (t-\tau)^{-r} \frac{d}{d\tau} u(\tau) d\tau.$$

Example 2.3. Let $\lambda \in (-1, 0) \cup (0, \infty)$ and $r \in (0, 1]$, then

$${}^c D_0^r t^\lambda = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-r)} t^{\lambda-r};\ \text{for almost all } t \in J.$$

Let $a_1 \in [0, a],\ J_1 = (a_1, a],\ r > 0$. For $u \in L^1(J_1)$, the expression

$$(I_{a^+}^r u)(t) = \frac{1}{\Gamma(r)} \int_{a_1^+}^t (t-\tau)^{r-1} u(\tau) d\tau,$$

is called the left-sided mixed Riemann-Liouville integral of order r of u.

Definition 2.4. [21] For $u \in L^1(J_1)$ where $\frac{d}{dt}u$ is Bochner integrable on J_1, the Caputo fractional order derivative of order r of u is defined by the expression

$$({}^c D_{a^+}^r u)(t) = (I_{a^+}^{1-r} \frac{d}{dt} u)(t).$$

Now, we consider the Ulam stability for the problem (1.1). Let $\epsilon > 0,\ \Psi \geq 0$ and $\Phi : J \to [0, \infty)$ be a continuous function. We consider the following inequalities

$$\begin{cases} \|{}^c D_{s_k}^r u(t) - Au(t) - f(t, u(t))\|_E \leq \epsilon; & \text{if } t \in I_k,\ k = 0, \dots, m, \\ \|u(t) - g_k(t, u(t))\|_E \leq \epsilon; & \text{if } t \in J_k,\ k = 1, \dots, m. \end{cases} \tag{2.1}$$

$$\begin{cases} \|{}^c D_{s_k}^r u(t) - Au(t) - f(t, u(t))\|_E \leq \Phi(t); & \text{if } t \in I_k,\ k = 0, \dots, m, \\ \|u(t) - g_k(t, u(t))\|_E \leq \Psi; & \text{if } t \in J_k,\ k = 1, \dots, m. \end{cases} \tag{2.2}$$

$$\begin{cases} \|{}^c D_{s_k}^r u(t) - Au(t) - f(t, u(t))\|_E \leq \epsilon\Phi(t); & \text{if } t \in I_k,\ k = 0, \dots, m, \\ \|u(t) - g_k(t, u(t))\|_E \leq \epsilon\Psi; & \text{if } t \in J_k,\ k = 1, \dots, m. \end{cases} \tag{2.3}$$

Definition 2.5. [20, 24, 6] Problem (1.1) is Ulam-Hyers stable if there exists a real number $c_{f,g_k} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (2.1) there exists a solution $v \in PC$ of problem (1.1) with

$$\|u(t) - v(t)\|_E \leq \epsilon c_{f,g_k};\ t \in J.$$

Definition 2.6. [20, 24, 6] Problem (1.1) is generalized Ulam-Hyers stable if there exists $c_{f,g_k} \in C([0,\infty),[0,\infty))$ with $c_{f,g_k}(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (2.1) there exists a solution $v \in PC$ of problem (1.1) with

$$\|u(t) - v(t)\|_E \leq c_{f,g_k}(\epsilon);\ t \in J.$$

Definition 2.7. [20, 24, 6] Problem (1.1) is Ulam-Hyers-Rassias stable with respect to (Φ,Ψ) if there exists a real number $c_{f,g_k,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (2.3) there exists a solution $v \in PC$ of problem (1.1) with

$$\|u(t) - v(t)\|_E \leq \epsilon c_{f,g_k,\Phi}(\Psi + \Phi(t));\ t \in J.$$

Definition 2.8. [20, 24, 6] Problem (1.1) is generalized Ulam-Hyers-Rassias stable with respect to (Φ,Ψ) if there exists a real number $c_{f,g_k,\Phi} > 0$ such that for each solution $u \in PC$ of the inequality (2.2) there exists a solution $v \in PC$ of problem (1.1) with $\|u(t) - v(t)\|_E \leq c_{f,g_k,\Phi}(\Psi + \Phi(t));\ t \in J.$

Remark 2.9. It is clear that: (i) Definition 2.5 $\Rightarrow$ Definition 2.6, (ii) Definition 2.7 $\Rightarrow$ Definition 2.8, (iii) Definition 2.7 for $\Phi(.) = \Psi = 1$ $\Rightarrow$ Definition 2.5.

Remark 2.10. A function $u \in PC$ is a solution of the inequality (2.2) if and only if there exist a function $G \in PC$ and a sequence $G_k;\ k = 1,\dots,m$ in E (which depend on u) such that

(*i*) $\|G(t)\|_E \leq \Phi(t)$ and $\|G_k\|_E \leq \Psi;\ k = 1,\dots,m,$
(*ii*) ${}^cD^r_{s_k}u(t) = Au(t) + f(t,u(t)) + G(t);$ if $t \in I_k,\ k = 0,\dots,m,$
(*iii*) $u(t) = g_k(t,u(t)) + G_k;$ if $t \in J_k,\ k = 1,\dots,m,$

One can have similar remarks for the inequalities (2.1) and (2.3). So, the Ulam stabilities of the impulsive fractional differential equations are some special types of data dependence of the solutions of impulsive fractional differential equations.

In the sequel we will make use of the following Gronwall's lemma.

Lemma 2.11. *(Gronwall lemma)* [12] *Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r < 1$ such that*

$$\upsilon(t) \leq \omega(t) + c\int_0^t (t-\tau)^{-r}\upsilon(\tau)d\tau,$$

then there exists a constant $\delta = \delta(r)$ such that for every $t \in J$, we have

$$\upsilon(t) \leq \omega(t) + \delta c\int_0^t (t-\tau)^{-r}\omega(\tau)d\tau.$$

3. Uniqueness and Ulam stabilities results

In this section, we discuss the uniqueness of mild solutions and we present conditions for the Ulam stability for the problem (1.1). Following [11, 25], we will introduce now the definition of mild solution to (1.1).

Definition 3.1. A function $u : J \to E$ is said to be a mild solution of (1.1) if u satisfies

$$\begin{cases} u(t) = S_r(t)u_0 + \int_0^t (t-\tau)^{r-1}T_r(t-\tau)f(\tau,u(\tau))d\tau; \ \text{ if } \ t \in [0,t_1], \\ \\ u(t) = S_r(t-s_k)g_k(s_k,u(s_k)) \\ + \int_{s_k}^t (t-\tau)^{r-1}T_r(t-\tau)f(\tau,u(\tau))d\tau; \ \text{ if } \ t \in I_k, \ k = 1,\dots,m, \\ \\ u(t) = g_k(t,u(t)); \ \text{if } t \in J_k, \ k = 1,\dots,m, \\ \\ u(0) = u_0, \end{cases}$$

where

$$S_r(t) = \int_0^\infty \xi_r(\theta)T(t^r\theta)d\theta, \ T_r(t) = r\int_0^\infty \theta\xi_r(\theta)T(t^r\theta)d\theta,$$

$$\xi_r(\theta) = \frac{1}{r}\theta^{-1-\frac{1}{r}}\overline{w}_r(\theta^{-\frac{1}{r}}) \geq 0,$$

and

$$\overline{w}_r(\theta) = \frac{1}{\pi}\sum_{n=0}^\infty (-1)^{n-1}\theta^{-nr-1}\frac{\Gamma(nr+1)}{n!}\sin(nr\pi); \ \theta \in (0,\infty).$$

ξ_q is a probability density function on $(0,\infty)$, that is $\int_0^\infty \xi_r(\theta)d\theta = 1$.

Remark 3.2. It is not difficult to verify that for $v \in [0,1]$,

$$\int_0^\infty \theta^v\xi_r(\theta)d\theta = \int_0^\infty \theta^{-rv}\overline{w}_r(\theta)d\theta = \frac{\Gamma(1+v)}{\Gamma(1+rv)}.$$

Lemma 3.3. [25] *For any $t \geq 0$, the operators $S_r(t)$ and $T_r(t)$ have the following properties:*

(a) *For any fixed $t \geq 0$, S_r and T_r are linear and bounded operators, ie., for any $u \in E$,*

$$\|S_r(t)u\|_E \leq M\|u\|_E, \quad \|T_r(t)u\|_E \leq \frac{M}{\Gamma(r)}\|u\|_E.$$

(b) *$\{S_r(t); \ t \geq 0\}$ and $\{T_r(t); \ t \geq 0\}$ are strongly continuous.*

(c) *For every $t \geq 0$, $S_r(t)$ and $T_r(t)$ are also compact operators.*

Lemma 3.4. *If $u \in PC$ is a solution of the inequality (2.2) then u is a solution of the following integral inequalities*

$$\begin{cases} \|u(t) - S_r(t)u_0 - \int_0^t (t-\tau)^{r-1} T_r(t-\tau) f(\tau, u(\tau)) d\tau\|_E \\ \leq M I_0^r \Phi(t); \quad \textit{if} \;\; t \in [0, t_1], \\ \\ \|u(t) - S_r(t-s_k) g_k(s_k, u(s_k)) - \int_{s_k}^t (t-\tau)^{r-1} T_r(t-\tau) f(\tau, u(\tau)) d\tau\|_E \\ \leq M I_{s_k}^r \Phi(t); \quad \textit{if} \;\; t \in I_k, \;\; k = 1, \dots, m, \\ \\ \|u(t) - g_k(t, u(t))\|_E \leq \Psi; \;\; \textit{if} \; t \in J_k, \;\; k = 1, \dots, m. \end{cases} \tag{3.1}$$

Proof. By Remark 2.10 we have that

$$\begin{cases} {}^c D_{\theta_k}^r u(t) = Au(t) + f(t, u(t)) + G(t); \;\; \text{if } t \in I_k, \;\; k = 0, \dots, m, \\ u(t) = g_k(t, u(t)) + G_k; \;\; \text{if } t \in J_k, \;\; k = 1, \dots, m. \end{cases}$$

Then

$$\begin{cases} u(t) = S_r(t)u_0 + \int_0^t (t-\tau)^{r-1} T_r(t-\tau)(f(\tau, u(\tau)) + G(\tau)) d\tau; \;\; \text{if} \;\; t \in [0, t_1], \\ \\ u(t) = S_r(t-s_k) g_k(s_k, u(s_k)) \\ + \int_{s_k}^t (t-\tau)^{r-1} T_r(t-\tau)(f(\tau, u(\tau)) + G(\tau)) d\tau; \;\; \text{if} \;\; t \in I_k, \;\; k = 1, \dots, m, \\ \\ u(t) = g_k(t, u(t)) + G_k; \; \text{if } t \in J_k, \;\; k = 1, \dots, m. \end{cases}$$

Thus, it follows that

$$\begin{cases} \|u(t) - S_r(t)u_0 - \int_0^t (t-\tau)^{r-1} T_r(t-\tau) f(\tau, u(\tau)) d\tau\|_E \\ = \| \int_0^t (t-\tau)^{r-1} T_r(t-\tau) G(\tau) d\tau\|_E \\ \leq \frac{M}{\Gamma(r)} \int_0^t (t-\tau)^{r-1} \|G(\tau)\|_E d\tau \\ \leq \frac{M}{\Gamma(r)} \int_0^t (t-\tau)^{r-1} \Phi(\tau) d\tau; \;\; \text{if} \;\; t \in [0, t_1], \\ \\ \|u(t) - S_r(t-s_k) g_k(s_k, u(s_k)) - \int_{s_k}^t (t-\tau)^{r-1} T_r(t-\tau) f(\tau, u(\tau)) d\tau\|_E \\ = \| \int_{s_k}^t (t-\tau)^{r-1} T_r(t-\tau) G(\tau) d\tau\|_E \\ \leq \frac{M}{\Gamma(r)} \int_{s_k}^t (t-\tau)^{r-1} \|G(\tau)\|_E d\tau \\ \leq \frac{M}{\Gamma(r)} \int_{s_k}^t (t-\tau)^{r-1} \Phi(\tau) d\tau; \;\; \text{if} \;\; t \in I_k, \;\; k = 1, \dots, m, \\ \\ \|u(t) - g_k(t, u(t))\|_E = \|G_k\|_E \leq \Psi; \; \text{if } t \in J_k, \;\; k = 1, \dots, m. \end{cases}$$

Hence, we obtain (3.1).

Remark 3.5. We have similar results for the solutions of (2.1) and (2.3).

Theorem 3.6. *Assume that the following hypotheses hold:*

(H_1) *The semigroup $T(t)$ is compact for $t > 0$,*

(H_2) *For each $t \in J$, the function $f(t,.) : E \to E$ is continuous and for each $v \in E$, the function $f(.,v) : J \to E$ is strongly measurable,*

(H_3) *There exists a constant $l_f > 0$ such that*

$$\|f(t,u) - f(t,\overline{u})\|_E \le l_f \|u - \overline{u}\|_E, \text{ for each } t \in J, \text{and each } u, \overline{u} \in E,$$

(H_4) *There exist constants $l_{g_k} > 0;\ k = 1, \dots, m$, such that*

$$\|g_k(t,u) - g_k(t,\overline{u})\|_E \le l_{g_k} \|u - \overline{u}\|_E,$$

for each $t \in J_k$, and each $u, \overline{u} \in E,\ k = 1, \dots, m$.

If

$$\ell := M l_g + \frac{M l_f a^r}{\Gamma(r)} < 1, \tag{3.2}$$

where $l_g = \max\limits_{k=1,\dots,m} l_{g_k}$, then the problem (1.1) has a unique mild solution on J.

Furthermore, if the following hypothesis

(H_5) *There exists $\lambda_\Phi > 0$ such that for each $t \in J$, we have*

$$I^r_{s_k} \Phi(t) \le \lambda_\Phi \Phi(t);\ k = 0, \dots, m,$$

holds, then the problem (1.1) is generalized Ulam-Hyers-Rassias stable.

Proof. Consider the operator $N : PC \to PC$ defined by

$$\begin{cases} (Nu)(t) = S_r(t)u_0 + \int_0^t (t-\tau)^{r-1} T_r(t-\tau) f(\tau, u(\tau)) d\tau; & \text{if } \ t \in [0, t_1], \\ (Nu)(t) = S_r(t - s_k) g_k(s_k, u(s_k)) & \\ + \int_{s_k}^t (t-\tau)^{r-1} T_r(t-\tau) f(\tau, u(\tau)) d\tau; & \text{if } \ t \in I_k,\ k = 1, \dots, m, \\ (Nu)(t) = g_k(t, u(t)); \text{ if } t \in J_k,\ k = 1, \dots, m. & \end{cases}$$

Clearly, the fixed points of the operator N are solution of the problem (1.1). We shall use the Banach contraction principle to prove that N has a fixed point. *N is a contraction.* Let $u, v \in PC$, then, for each $t \in J$, we have

$$\begin{cases} \|(Nu)(t) - (Nv)(t)\|_E \le \| \int_0^t (t-\tau)^{r-1} T_r(t-\tau) \\ \times [f(\tau, u(\tau)) - f(\tau, v(\tau))] d\tau \|_E; \text{ if } \ t \in [0, t_1], \\ \|(Nu)(t) - (Nv)(t)\|_E \le \|S_r(t - s_k)(g_k(s_k, u(s_k)) - g_k(s_k, v(s_k))\|_E \\ + \| \int_{s_k}^t (t-\tau)^{r_1 - 1} T_r(t - s_k)[f(\tau, u(\tau)) - f(\tau, v(\tau))] d\tau \|_E; \ \text{ if } \ t \in I_k,\ k = 1, \dots, m, \\ \|(Nu)(t) - (Nv)(t)\|_E = \|g_k(t, u(t)) - g_k(t, v(t))\|_E; \text{ if } t \in J_k,\ k = 1, \dots, m. \end{cases}$$

Thus, we get

$$
\begin{cases}
\|(Nu)(t)-(Nv)(t)\|_E \le \int_0^t (t-\tau)^{r-1}T_r(t-\tau)l_f\|T_r(t-\tau)(u(\tau)-v(\tau))\|_E d\tau;\\
\le \frac{Ml_f a^r}{\Gamma(r)}\|u-v\|_{PC};\ \text{if}\ \ t\in[0,t_1],\\
\|(Nu)t-(Nv)t\|_E \le l_g\|S_r(t-s_k)(u(t)-v(t))\|_E\\
+\int_{s_k}^t (t-\tau)^{r-1}l_f\|T_r(t-\tau)(u(\tau)-v(\tau))\|_E d\tau\\
\le \left(Ml_g+\frac{Ml_f a^r}{\Gamma(r)}\right)\|u-v\|_{PC};\ \text{if}\ \ t\in I_k,\ k=1,\dots,m,\\
\|(Nu)(t)-(Nv)(t)\|_E \le l_g\|u-v\|_{PC}; \text{if}\ t\in J_k,\ k=1,\dots,m.
\end{cases}
$$

Hence $\|N(u)-N(v)\|_{PC} \le \ell\|u-v\|_{PC}$ and by (3.2), we conclude that N is a contraction. As a consequence of Banach fixed point theorem, we deduce that N has a unique fixed point v which is the unique mild solution defined on J of the problem (1.1). Then we have

$$
\begin{cases}
v(t)=S_r(t)u_0+\int_0^t (t-\tau)^{r-1}T_r(t-\tau)f(\tau,v(\tau))d\tau;\ \ \text{if}\ \ t\in[0,t_1],\\
v(t)=S_r(t-s_k)g_k(s_k,v(s_k))\\
+\int_{s_k}^t (t-\tau)^{r-1}T_r(t-\tau)f(\tau,v(\tau))d\tau;\ \ \text{if}\ \ t\in I_k,\ k=1,\dots,m,\\
v(t)=g_k(t,v(t));\ \ \text{if}\ \ t\in J_k,\ k=1,\dots,m.
\end{cases}
$$

Let $u\in PC$ be a solution of the inequality (2.2). By Lemma 3.4 and (H_5), for each $t\in J$, we get

$$
\begin{cases}
\|u(t)-S_r(t)u_0-\int_0^t (t-\tau)^{r-1}T_r(t-\tau)f(\tau,u(\tau))d\tau\|_E\\
\le M\lambda_\Phi\Phi(t);\ \ \text{if}\ \ t\in[0,t_1],\\
\|u(t)-S_r(t-s_k)g_k(s_k,u(s_k))-\int_{s_k}^t (t-\tau)^{r-1}T_r(t-\tau)f(\tau,u(\tau))d\tau\|_E\\
\le M\lambda_\Phi\Phi(t);\ \ \text{if}\ \ t\in I_k,\ k=1,\dots,m,\\
\|u(t)-g_k(t,u(t))\|_E \le \Psi;\ \text{if}\ t\in J_k,\ k=1,\dots,m.
\end{cases}
$$

Thus

$$\begin{cases} \|u(t)-v(t)\|_E \leq M\lambda_\Phi \Phi(t) + \| \int_0^t (t-\tau)^{r-1} T_r(t-\tau) \\ \times [f(\tau,u(\tau)) - f(\tau,v(\tau))] d\tau \|_E; \text{ if } \ t\in[0,t_1], \\ \\ \|u(t)-v(t)\|_E \leq \lambda_\Phi \Phi(t) + M\|g_k(s_k,u(s_k)) - g_k(s_k,v(s_k)\|_E \\ + \int_{s_k}^t (t-\tau)^{r_1-1} \|T_r(t-\tau)(f(\tau,u(\tau)) - f(\tau,v(\tau)))\|_E d\tau; \\ \ \text{if } \ t\in I_k, \ k=1,\dots,m, \\ \\ \|u(t)-v(t)\|_E \leq \Psi + \|g_k(t,u(t)) - g_k(t,v(t))\|_E; \text{ if } t\in J_k, \ k=1,\dots,m. \end{cases}$$

Hence

$$\begin{cases} \|u(t)-v(t)\|_E \leq \lambda_\Phi \Phi(t) \\ +\frac{Ml_f}{\Gamma(r)} \int_0^t (t-\tau)^{r-1} \|u(\tau)-v(\tau)\|_E d\tau; \ \text{ if } \ t\in[0,t_1]\times[0,b], \\ \\ \|u(t)-v(t)\|_E \leq M\lambda_\Phi \Phi(t) + Ml_g\|u(t)-v(t)\|_E \\ +\frac{Ml_f}{\Gamma(r)} \int_{s_k}^t (t-\tau)^{r_1-1} \|u(\tau)-v(\tau)\|_E d\tau; \ \text{ if } \ t\in I_k, \ k=1,\dots,m, \\ \\ \|u(t)-v(t)\|_E \leq \Psi + l_g\|u(t)-v(t)\|_E; \text{ if } t\in J_k, \ k=1,\dots,m. \end{cases}$$

For each $t\in[0,t_1]$, we have

$$\|u(t)-v(t)\|_E \leq M\lambda_\Phi \Phi(t) + \frac{Ml_f}{\Gamma(r)} \int_0^t (t-\tau)^{r-1} \|u(\tau)-v(\tau)\|_E d\tau.$$

From Lemma 2.11, there exists a constant $\delta_1 := \delta_1(r)$ such that

$$\begin{aligned} \|u(t)-v(t)\|_E &\leq M\lambda_\Phi \left(\Phi(t) + Ml_f\delta_1 I_0^r \Phi(t)\right) \\ &\leq M\lambda_\Phi \left(1 + Ml_f\delta_1\lambda_\Phi\right)\Phi(t) \\ &:= c_{1,f,g_k,\Phi}\Phi(t). \end{aligned}$$

Thus, for each $t\in[0,t_1]\times[0,b]$, we get

$$\|u(t)-v(t)\|_E \leq c_{1,f,g_k,\Phi}(\Psi+\Phi(t)).$$

Now, for each $t\in I_k, \ k=1,\dots,m$, we have

$$\begin{aligned} &\|u(t)-v(t)\|_E \leq M\lambda_\Phi\Phi(t) + Ml_g\|u(t)-v(t)\|_E \\ &+ \frac{Ml_f}{\Gamma(r)} \int_{s_k}^t (t-\tau)^{r-1} \|u(\tau)-v(\tau)\|_E d\tau. \end{aligned}$$

Then, we obtain

$$\begin{aligned} &\|u(t)-v(t)\|_E \leq \frac{M\lambda_\Phi}{1-Ml_g}\Phi(t) \\ &+ \frac{Ml_f}{(1-Ml_g)\Gamma(r)} \int_{s_k}^t (t-\tau)^{r-1} \|u(\tau)-v(\tau)\|_E d\tau. \end{aligned}$$

Again, from Lemma 2.11, there exists a constant $\delta_2 := \delta_2(r)$ such that

$$\begin{aligned}\|u(t)-v(t)\|_E &\leq \frac{M\lambda_\Phi}{1-Ml_g}\left(\Phi(t)+\frac{Ml_f\delta_2}{1-Ml_g}I^r_{s_k}\Phi(t)\right)\\ &\leq \frac{M\lambda_\Phi}{1-Ml_g}\left(1+\frac{Ml_f\delta_2\lambda_\Phi}{1-Ml_g}\right)\Phi(t)\\ &:= c_{2,f,g_k,\Phi}\Phi(t).\end{aligned}$$

Hence, for each $t \in I_k,\ k = 1,\dots,m$, we get

$$\|u(t)-v(t)\|_E \leq c_{2,f,g_k,\Phi}(\Psi+\Phi(t)).$$

Now, for each $t \in J_k,\ k = 1,\dots,m$, we have

$$\|u(t)-v(t)\|_E \leq \Psi + l_g\|u(t)-v(t)\|_E.$$

This gives,

$$\|u(t)-v(t)\|_E \leq \frac{\Psi}{1-l_g} := c_{3,f,g_k,\Phi}\Psi.$$

Thus, for each $t \in J_k,\ k = 1,\dots,m$, we get

$$\|u(t)-v(t)\|_E \leq c_{3,f,g_k,\Phi}(\Psi+\Phi(t)).$$

Set $c_{f,g_k,\Phi} := \max_{i\in\{1,2,3\}} c_{i,f,g_k,\Phi}$. Hence, for each $t \in J$, we obtain

$$\|u(t)-v(t)\|_E \leq c_{f,g_k,\Phi}(\Psi+\Phi(t)).$$

Consequently, problem (1.1) is generalized Ulam-Hyers-Rassias stable.

4. An Example

As an application of our results, we present the fractional differential equations with not instantaneous impulses of the form

$$\begin{cases} D^r_{0,t}z(t,x) = \frac{\partial^2 z}{\partial x^2}(t,x) + Q(t,z(t,x)); & t \in [0,1]\cup(2,3], \quad x \in [0,\pi],\\ z(t,x) = g(t,z(t,x)); & t \in (1,2], \quad x \in [0,\pi],\\ z(t,0) = z(t,\pi) = 0; & t \in [0,1]\cup(2,3],\\ z(0,x) = \phi(x); & x \in [0,\pi], \end{cases} \tag{4.1}$$

where $D^r_{0,t} := \frac{\partial^r}{\partial t^r}$ is the Caputo fractional partial derivative of order $r \in (0,1]$ with respect to t. It is defined by the expression

$$^cD^t_{0,t}z(t,x) = \frac{1}{\Gamma(1-r)}\int_0^t (t-\tau)^{-r}\frac{\partial}{\partial \tau}z(\tau,x)d\tau,$$

$Q : ([0,1]\cup(2,3])\times\mathbb{R}\to\mathbb{R}$ and $g : (1,2]\times\mathbb{R}\to\mathbb{R}$ are given by

$$Q(t,z(t,x)) = \frac{1}{(1+110e^t)(1+|z(t,x)|)};\ t \in ([0,1]\cup(2,3])\times[0,1],\ x \in [0,\pi],$$

$$g(t,z(t,x)) = \frac{1}{1+110e^{t+x}}\ln(1+t^2+|z(t,x)|);\ t \in (1,2]\times[0,1],\ x \in [0,\pi],$$

and $\phi : [0,\pi]\to\mathbb{R}$ is a continuous function.

Let $E = L^2([0,\pi],\mathbb{R})$ and define $A : D(A) \subset E \to E$ by $Aw = w''$ with domain

$$D(A) = \{w \in E : w,\ w' \text{ are absolutely continuous},\ w'' \in E,\ w(0) = w(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup on E (see [17]). Then

$$Aw = -\sum_{n=1}^{\infty} n^2 < w, e_n > e_n; w \in D(A),$$

where

$$e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx);\ x \in [0,\pi],\ n = 1,2,3,...$$

Clearly A generates a compact semigroup $T(t);\ t > 0$ given by

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} < w, e_n > e_n;\ w \in E.$$

Hence the assumptions (H_1) and (H_2) are satisfied.

For $x \in [0,\pi]$, set $u(t)(x) = z(t,x); \quad t \in [0,3], \quad u_0(x) = z(0,x) = \phi(x)$,

$$Au(t)(x) = \frac{\partial^2 z}{\partial x^2}(t,x); \quad t \in [0,1] \cup (2,3],$$

$$f(t,u(t))(x) = Q(t,z(t,x)); \quad t \in [0,1] \cup (2,3],$$

and

$$g(t,u(t))(x) = g(t,z(t,x)); \quad t \in (1,2].$$

Thus, under the above definitions of $\phi,\ A,\ f$ and g, the system (4.1) can be represented by the functional abstract problem (1.1).

For each $z,\overline{z},\in E,\ t \in [0,1] \cup (2,3]$ and $x \in [0,\pi]$, we have

$$|f(t,z(t))(x) - f(t,\overline{z}(t))(x)| \leq \frac{1}{111}|z(t,x) - \overline{z}(t,x)|,$$

then, we obtain

$$\|f(t,z) - f(t,\overline{z}\|_E \leq \frac{1}{111}\|z - \overline{z}\|_E.$$

Also, for each $z,\overline{z},\in E,\ t \in (1,2]$ and $x \in [0,\pi]$, we can easily get

$$\|g(t,z) - g(t,\overline{z}\|_E \leq \frac{1}{111}\|z - \overline{z}\|_E.$$

Thus the conditions (H_3) and (H_4) are satisfied with $l_f = l_g = \frac{1}{111}$. We shall show that condition (3.2) holds with $a = 3$ and $M = 1$. Indeed, for each $r \in (0,1]$ we get

$$\ell = Ml_g + \frac{Ml_f a^r}{\Gamma(r)} = \frac{1}{111} + \frac{3^r}{111\Gamma(r)} < \frac{7}{111} < 1.$$

Hence, we ensure the existence of unique mild solution defined on $[0,3]$ for the problem (4.1). Finally, the hypothesis (H_5) is satisfied with $\Phi(t)=t$ and $\lambda_\Phi=\frac{3^r}{\Gamma(2+r)}$. Indeed, for each $(t,x)\in[0,3]\times[0,1]$ we get

$$(I_0^r\Phi)(t)=\frac{\Gamma(2)t^{1+r}}{\Gamma(2+r)}\leq\frac{3^rt}{\Gamma(2+r)}=\lambda_\Phi\Phi(t).$$

Consequently, Theorem 3.6 implies that the problem (4.1) is generalized Ulam-Hyers-Rassias stable.

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