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SOME RESULTS ON COMMON FIXED POINTS WITH APPLICATIONS

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Abstract. In this paper we prove some common fixed point theorems for a quadruple of self mappings on a complete metric space satisfying a weak compatibility condition and a rational inequality. Subsequently, we utilize our main result to obtain common solutions of certain functional equations arising in dynamic programming.

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1. Introduction

The study of finding a common fixed point of a pair of commuting mappings seems to be of vital interest in view of a historically significant and negatively settled problem that a pair of commuting continuous self mappings on the unit interval [0,1] need not have a common fixed point [4, 8]. Since then, there have been several attempts to find weaker forms of commutativity that may ensure the existence of a common fixed point for a pair of self maps on a metric space. In this context, the notion of compatible mappings, introduced by

Jungck [9] (see also Sessa [23]) has been of significant interest and has proven useful for generalizing results in metric fixed point theory for single-valued and multivalued mappings [7, 9-22,24-25]. Recall that self mappings S and T of a metric space (X,d) are *compatible* if $\lim_n d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n Sx_n = \lim_n Tx_n = z$ for some $z \in X$. It is well known that two commuting mappings are compatible but the reverse implication need not be true in general [10, Example, page 285]. Further to this, the notion of *weakly compatible* mappings was recently introduced and studied by Jungck and Rhoades [14] which has been found more general than many of it's counterparts, including the compatible mappings . Self mappings S and T of a metric space (X,d) are called *weakly compatible* if $Sx = Tx$ implies that $STx = TSx$ for all $x \in X$

Notice that two mappings may fail to be weakly compatible only if they possess a coincidence point at which they do not commute. This means that weak compatibility is the minimal condition for mappings to have a common fixed point as a common fixed point is also a point of coincidence. Also, it is interesting to note that compatible mappings are again weakly compatible but not conversely (see [14, Example 5.1]). The same observation applies to several other forms of compatibility, such as compatibility of type A, B or C etc. [13, 18,19] (see for example, [7, Example 3]).

On the other hand $\varphi-$*contractions* were used earlier by Bhakta and Mitra [3] to obtain some existence theorems for functional equations that arise in certain type of continuous multistage decision process (related to dynamic programming). Subsequent results in this direction appear, among others, in [1,6,15, 17, 25].

Motivated by the above results, we prove some common fixed point theorems for a quadruple of self mappings of a complete metric space satisfying weak compatibility condition and a rational inequality. Subsequently, we utilize our main theorem (Theorem 2.1) to obtain common solutions of certain functional equations arising in dynamic programming. The results obtained here in extend and improve some results in [1,3,6,10, 13,15,17,26] and others.

2. Common fixed point theorems

Now onward, we denote by Φ the collection of all functions $\varphi : [0,\infty) \to [0,\infty)$ which are upper semi-continuous from the right, non-decreasing and satisfy

$$\lim_{s\to t+} \sup \varphi(s) < t, \varphi(t) < t \text{ for all } \quad t > 0$$

Throughout, unless stated otherwise, X will denote a metric space (X, d). The set of natural numbers will be denoted by $\mathbb{N}$.

First we have the following lemmas.

Lemma 2.1 If $\varphi_i \in \Phi$, $i \in I$, where I is some finite indexing set, then there exists a $\varphi \in \Phi$ such that $\max\{\varphi_i(t) : i \in I\} \leqslant \varphi(t)$ for all $t > 0$.

The proof of the above lemma can easily be constructed.

Lemma 2.2 [5, Lemma 2] Let $\varphi \in \Phi$ and let $\{\tau_n\}$ be a sequence of non-negative real numbers. If $\tau_{n+1} \leqslant \varphi(\tau_n)$ for $n \in \mathbb{N}$, then the sequence $\{\tau_n\}$ converges to 0.

Now, let A, B, S and T be self-mappings of a metric space (X, d) such that

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{2.1}$$

$$\begin{aligned} d(Ax, By) \leq & \frac{p \max\{d(Ax, Sx)d(By, Ty), d(Ax, Ty)d(By, Sx)\}}{1 + pd(Sx, Ty)} + \\ & \frac{1}{1 + pd(Sx, Ty)} \max\{\varphi_1(d(Sx, Ty)), \varphi_2(d(Ax, Sx)), \\ & \varphi_3(d(By, Ty)), \varphi_4(\frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\} \end{aligned} \tag{2.2}$$

for all $x, y \in X$, $\varphi_i \in \Phi (i = 1, 2, 3, 4)$ and $p \geqslant 0$.

Next, we construct a sequence $\{x_n\}$ in X as follows. Pick $x_o \in X$. By (2.1), since $A(X) \subset T(X)$ we can choose a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Again, since $B(X) \subset S(X)$ for $x_1 \in X$, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$. Continuing in this way, we can choose a sequence $\{x_n\}$ in X such that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1} \; (n \in \mathbb{N}) \tag{2.3}$$

For the sake of brevity, let $d_n = d(y_n,y_{n-1}),\ n \in \mathbb{N}$. Then we have the following:

Lemma 2.3. $\lim_{n\to\infty} d_n = 0$.

Proof. By setting $x = x_{2n}$ and $y = x_{2n-1}$ in (2.2) and using (2.3) along with the brevity notation $d_n = d(y_n,y_{n-1}),\ n \in \mathbb{N}$ and a fairly standard calculation, we obtain

$$\begin{aligned}[1 + pd_{2n}]d_{2n+1} \le p \max\{&d_{2n+1}d_{2n}, d(y_{2n+1},y_{2n-1})d(y_{2n}, y_{2n})\}\\ &+ \max\{\varphi_1(d_{2n}), \varphi_2(d_{2n+1}), \varphi_3(d_{2n}),\\ &\varphi_4(\frac{1}{2}[d(y_{2n+1}, y_{2n-1}) + d(y_{2n}, y_{2n})])\}.\end{aligned}$$

Since $d(y_{2n+1},\ y_{2n-1}) \le d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})$, the above inequality reduces to

$$[1 + pd_{2n}]d_{2n+1} \le$$
$$pd_{2n+1}d_{2n} + \max\{\varphi_1(d_{2n}), \varphi_2(d_{2n+1}), \varphi_3(d_{2n}), \varphi_4(\frac{1}{2}[d_{2n+1} + d_{2n}])\}$$

and we obtain

$$d_{2n+1} \le \max\{\varphi_1(d_{2n}), \varphi_2(d_{2n+1}), \varphi_3(d_{2n}), \varphi_4(\frac{1}{2}[d_{2n+1} + d_{2n}])\} \tag{2.4}$$

Similarly, by setting $x = x_{2n-2}$ and $y = x_{2n-1}$ in (2.2) and using similar arguments as above, we obtain

$$d_{2n} \le \max\{\varphi_1(d_{2n-1}), \varphi_2(d_{2n-1}), \varphi_3(d_{2n}), \varphi_4(\frac{1}{2}[d_{2n} + d_{2n-1}])\} \tag{2.5}$$

If $d_{2n} < d_{2n+1}$ for some $n \in \mathbb{N}$, then $\frac{1}{2}[d_{2n+1} + d_{2n}] < d_{2n+1}$. By Lemma 2.1, there exists a $\varphi \in \Phi$ so that from (2.4) we have

$$\begin{aligned}d_{2n+1} &\le \max\{\varphi_1(d_{2n+1}), \varphi_2(d_{2n+1}), \varphi_3(d_{2n+1}), \varphi_4(d_{2n+1})\}\\ &\le \varphi(d_{2n+1}) < d_{2n+1},\end{aligned}$$

a contradiction. Consequently, we have $d_{2n+1} \le d_{2n}$ for all $n \in \mathbb{N}$. This fact together with (2.4) and Lemma 2.1 imply that

$$d_{2n+1} \leq \varphi(d_{2n}) \text{ for all } n \in \mathbb{N} \text{ and some } \varphi \in \Phi. \tag{2.6}$$

A similar argument applied to (2.5) will result in

$$d_{2n} \leq \varphi(d_{2n-1}) \text{ for all } n \in \mathbb{N}, \tag{2.7}$$

where $\varphi \in \Phi$ is assumed to be the same as in the previous case. Therefore $d_{n+1} \leq \varphi(d_n)$ for all $n \in \mathbb{N}$, and by Lemma 2.2 we have $\lim_{n\to\infty} d_n = 0.\square$

Lemma 2.4 The sequence $\{y_n\}$ defined in (2.3) is a Cauchy sequence in X.

Proof. In view of Lemma 2.3, it suffices to show that a subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in X. Suppose $\{y_{2n}\}$ is not Cauchy. Then there exists an $\epsilon_0 > 0$ such that for each even integer $2k$ there exist even integers $2m(k), 2n(k) \in \mathbb{N}$ with $2m(k) > 2n(k) \geq 2k$ such that

$$d(y_{2n(k)}, y_{2m(k)}) \geq \varepsilon_0 \text{ and } d(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon_0 \tag{2.8}$$

that is, $2m(k)$ is the least positive even integer so that $2m(k) > 2n(k)$ and

$$d(y_{2n(k)}, y_{2m(k)}) \geq \varepsilon_0$$

Hence for each even integer $2k$, we have

$$\begin{aligned}\varepsilon_0 &\leq d(y_{2n(k)}, y_{2m(k)})\\ &\leqslant d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)})\\ &< \varepsilon_0 + d_{2m(k)-1} + d_{2m(k)}.\end{aligned}$$

Hence by Lemma 2.3 and (2.8) it follows that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \varepsilon_0 \tag{2.9}$$

By making use of the triangle inequalities

$$\begin{aligned}d(y_{2n(k)}, y_{2m(k)-1}) &\leq d(y_{2n(k)}, y_{2m(k)}) + d(y_{2m(k)}, y_{2m(k)-1}),\\ d(y_{2n(k)}, y_{2m(k)}) &\leq d(y_{2n(k)}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)})\end{aligned}$$

we obtain

$$\left|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})\right| \leqslant d(y_{2m(k)}, y_{2m(k)-1}) = d_{2m(k)} \quad (2.10)$$

Similarly we have

$$\begin{aligned}&\left|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})\right| \\ &\leqslant d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2n(k)+1}, y_{2n(k)}) \\ &= d_{2m(k)} + d_{2n(k)+1} \end{aligned} \quad (2.11)$$

By Lemma 2.2 and inequalities (2.10) and (2.11) we obtain

$$\lim_{k\to\infty} d(y_{n(k)}, y_{2m(k)-1}) = \varepsilon_0 = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) \quad (2.12)$$

Now using (2.2) with $x = x_{2n(k)}$ and $y = x_{2m(k)-1}$ along with (2.3) and a rearrangement we obtain

$$\begin{aligned}&[1 + p\, d(y_{2n(k)}, y_{2m(k)-1})]d(y_{2n(k)+1}, y_{2m(k)}) \\ &\leqslant p \max\{d(y_{2n(k)+1}, y_{2n(k)})d(y_{2m(k)}, y_{2m(k)-1}), \\ &d(y_{2n(k)+1}, y_{2m(k)-1})d(y_{2m(k)}, y_{2n(k)})\} \\ &+ \max\{\varphi_1(d(y_{2n(k)}, y_{2m(k)-1})), \varphi_2(d(y_{2n(k)+1}, y_{2n(k)})), \\ &\varphi_3(d(y_{2m(k)}, y_{2m(k)-1})), \varphi_4(\frac{1}{2}[d(y_{2n(k)+1}, y_{2m(k)-1}) \\ &+ d(y_{2m(k)}, y_{2n(k)})])\}.\end{aligned}$$

Letting $k \to \infty$ and using Lemma 2.2, (2.9) and (2.12) and the fact that $\varphi_i \in \Phi (i = 1, 2, 3, 4)$ we have

$$\begin{aligned}\varepsilon_0 + p\varepsilon_0^2 &\leqslant p\varepsilon_0^2 + \max\{\varphi_1(\varepsilon_0), \varphi_2(0), \varphi_3(0), \varphi_4(\varepsilon_0)\} \\ &\leqslant p\varepsilon_0^2 + \max\{\varphi_1(\varepsilon_0), \varphi_2(\varepsilon_0), \varphi_3(\varepsilon_0), \varphi_4(\varepsilon_0)\}.\end{aligned}$$

Hence by Lemma 2.1 with $\varphi \in \Phi$ we have

$$\varepsilon_0 \leqslant \max\{\varphi_1(\varepsilon_0), \varphi_2(\varepsilon_0), \varphi_3(\varepsilon_0), \varphi_4(\varepsilon_0)\} \leq \varphi(\varepsilon_0) < \varepsilon_0,$$

a contradiction. Hence $\{y_{2n}\}$ is a Cauchy sequence in X. This proves that

$\{y_n\}$ is Cauchy in X.□

The following result is our main theorem of this section.

Theorem 2.1. Let A, B, S and T be self mappings of a complete metric space X satisfying (2.1) and (2.2). If the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible and that $T(X)$ or $S(X)$ is closed, then A, B, S and T have a unique common fixed point in X.

Proof. Since X is complete, it follows from Lemma 2.4 that the sequence $\{y_n\}$ converges to a point z in X. Consequently, the subsequences $\{Ax_{2n}\}, \{Bx_{2n-1}\}, \{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the same limit z.

Now suppose that $T(X)$ is closed. Then since $\{Tx_{2n+1}\} \subset T(X)$, there exists a point $u \in X$ such that $z = Tu$. Then by using (2.2) with $x = x_{2n}$ and $y = u$ we get

$$\begin{aligned}
&d(Ax_{2n}, Bu)\\
&\le \frac{p \max\{d(Ax_{2n}, Sx_{2n})d(Bu, Tu), d(Ax_{2n}, Tu)d(Bu, Sx_{2n})\}}{1 + pd(Sx_{2n}, Tu)} +\\
&\frac{1}{1 + pd(Sx_{2n}, Tu)}(\max\{\varphi_1(d(Sx_{2n}, Tu)), \varphi_2(d(Ax_{2n}, Sx_{2n})),\\
&\varphi_3(d(Bu, Tu)), \varphi_4(\frac{1}{2}[d(Ax_{2n}, Tu) + d(Bu, Sx_{2n})]).
\end{aligned}$$

Letting $n \to \infty$ and using the properties of the functions $\varphi_i \in \Phi(i = 1, 2, 3, 4)$ and the fact that $z = Tu$ we get

$$\begin{aligned}
&d(z, Bu)\\
&\le \max\{\varphi_1(0), \varphi_2(0), \varphi_3(d(Bu, Tu), \varphi_4(\frac{1}{2}d(Bu, z))\}\\
&\le \max\{\varphi_1(d(z, Bu)), \varphi_2(d(z, Bu)), \varphi_3(d(Bu, z), \varphi_4(\frac{1}{2}d(Bu, z))\}\\
&\le \varphi(d(z, Bu)), .
\end{aligned}$$

where $\varphi \in \Phi$ is ensured by Lemma 2.1. This implies that $z = Bu$, otherwise

$$d(z, Bu) \le \varphi(d(z, Bu)) < d(z, Bu),$$

a contradiction. Therefore $Tu = z = Bu$. Hence by the weak compatibility of the pair $\{B, T\}$ it immediately follows that $BTu = TBu$, that is, $Bz = Tz$.

Next, we shall show that z is a common fixed point of B and T. By setting $x = x_{2n}$ and $y = z$ in (2.2) we have

$$\begin{aligned}
&d(Ax_{2n}, Bz)\\
&\leq \frac{p\max\{d(Ax_{2n}, Sx_{2n})d(Bz, Tz), d(Ax_{2n}, Tz)d(Bz, Sx_{2n})\}}{1 + pd(Sx_{2n}, Tz)} +\\
&\frac{1}{1 + pd(Sx_{2n}, Tz)} \max\{\varphi_1(d(Sx_{2n}, Tz)), \varphi_2(d(Ax_{2n}, Sx_{2n})),\\
&\varphi_3(d(Bz, Tz)), \varphi_4(\frac{1}{2}[d(Ax_{2n}, Tz) + d(Bz, Sx_{2n})])\}.
\end{aligned}$$

Letting $n \to \infty$ and noting that $\lim\limits_{n\to\infty} Ax_{2n} = z = \lim\limits_{n\to\infty} S\, x_{2n}$ and $Bz = Tz$,we get

$$\begin{aligned}
d(z, Bz) &\leq \frac{pd(z, Tz)d(z, Bz)}{1 + pd(z, Tz)}\\
&+ \frac{\max\{\varphi_1(d(z, Tz), \varphi_2(0), \varphi_3(0), \varphi_4(\frac{1}{2}[d(z, Tz) + d(Bz, z)])\}}{1 + pd(z, Tz)}\\
&\leq \frac{pd(z, Bz)d(z, Bz)}{1 + pd(z, Bz)}\\
&+ \frac{\max\{\varphi_1(d(z, Bz)), \varphi_2(d(z, Bz)), \varphi_3(d(z, Bz)), \varphi_4(d(z, Bz)))\}}{1 + pd(z, Bz)}
\end{aligned}$$

Since by Lemma 2.1, there exists a $\varphi \in \Phi$ such that

$$\max\{\varphi_1(d(z, Bz)), \varphi_2(d(z, Bz)), \varphi_3(d(z, Tz)), \varphi_4(d(z, Tz)))\} \leq \varphi(d(z, Bz)),$$

the above inequality reduces to

$$d(z, Bz) \leq \frac{pd(z, Bz)d(z, Bz)}{1 + pd(z, Bz)} + \frac{\varphi(d(z, Bz))}{1 + pd(z, Bz)}$$

If $d(z, Bz) \neq 0$,then by the properties of φ, we have $\varphi(d(z, Bz)) < d(z, Bz)$ and hence from the above inequality we have

$$d(z, Bz) < \frac{pd(z, Bz)d(z, Bz)}{1 + pd(z, Bz)} + \frac{d(z, Bz)}{1 + pd(z, Bz)} = d(z, Bz)$$

a contradiction. Therefore $d(z, Bz) = 0 = d(z, Tz)$,i.e. $z = Bz = Tz$ and thus z is a common fixed point of B and T.

Further, $z = Bz$ implies that $z \in B(X) \subset S(X)$, by (2.1). Therefore there exists a point $v \in X$ such that $z = Sv$. We now show that $Av = Sv$. Indeed, by setting $x = v$ and $y = x_{2n-1}$ in (2.2), making $n \to \infty$ and using the properties of the functions $\varphi_i \in \Phi(i = 1, 2, 3, 4)$ and Lemma 2.1 and following similar arguments as in the case of mappings B and T, we can find a $\varphi \in \Phi$ with

$$d(Av, z) \leq \varphi(d(Av, z) < d(Av, z)$$

which by contradiction implies that $Av = z$. Hence $Av = z = Sv$. Then by the weak compatibility of the pair $\{A, S\}$ we immediately have $SAv = Sz = ASv = Az$. Hence $Az = Sz$.

Now, by setting $x = z$ and $y = x_{2n-1}$in (2.2) and following the earlier arguments, it can easily be verified that z is a common fixed point of A and S as well. Hence z is a common fixed point of A, B, S and T.

The uniqueness of z as a common fixed point of A, B, S and T can easily be verified. In fact, if $w \neq z$ is another common fixed point of the given mappings, then by setting $x = z$ and $y = w$ in (2.2) we get

$$\begin{aligned} d(z, w) = d(Az, Bw) &\leq \frac{p[d(z, w)]^2}{1 + pd(Sz, Tw)} \\ &+ \frac{\max\{\varphi_1(d(z, w)), \varphi_2(0), \varphi_3(0), \varphi_4(\frac{1}{2}[d(z, w)) + d(w, z)])\}}{1 + pd(Sz, Tw)} \\ &\leq \frac{p[d(z, w)]^2}{1 + pd(z, w)} \\ &+ \frac{\max\{\varphi_1(d(z, w)), \varphi_2(d(z, w)), \varphi_3(d(z, w)), \varphi_4(d(z, w))\}}{1 + pd(z, w)}. \end{aligned}$$

Since by Lemma 2.1, there exists a $\varphi \in \Phi$ such that

$$\max\{\varphi_1(d(z, w)), \varphi_2(d(z, w)), \varphi_3(d(z, w)), \varphi_4(d(z, w))\} \leq \varphi(d(z, w)) < d(z, w),$$

the above inequality reduces to

$$d(z, w) < \frac{p[d(z, w)]^2}{1 + pd(z, w)} + \frac{d(z, w)}{1 + pd(z, w)} = \frac{d(z, w)[1 + pd(z, w)]}{1 + pd(z, w)} = d(z, w),$$

a contradiction. Therefore $w = z$, proving the uniqueness of z as a common fixed point of A, B, S and T.□

Remark 2.1. The above theorem remains valid if one assumes $S(X)$ to be closed instead of $T(X)$. The same thing applies if $A(X)$ or $B(X)$ is assumed to be closed.

Now we have the following corollary in respect of *compatible mappings of type* (P) [16]. Recall that self mappings S and T of a metric space X are compatible of type (P) if

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. A glance into [16] in conjunction with the examples in [7] will quickly reveal that compatible mappings of type (P) are weakly compatible but not conversely.

Corollary 2.1. Let A, B, S and T be self mappings of a complete metric space X satisfying conditions (2.1) and (2.2). If the pairs $\{A, S\}$ and $\{B, T\}$ are compatible of type (P) and that T(X) or S(X) is closed, then A, B, S and T have a unique common fixed point in X.

Remark 2.2. The advantage of the assumption that $T(X)$ or $S(X)$ is closed in Theorem 2.1 (and Corollary 2.1) is that the mappings A, B, S or T need not be continuous. However, if we assume one of the mappings to be continuous, then the requirement that $T(X)$ or $S(X)$ is closed can be dispensed with for the above results still to hold.

Corollary 2.2 Let A and B be self-mappings of a complete metric X space satisfying

$$\begin{aligned} d(Ax, By) \le \frac{p \max\{d(x,Ax)d(y,By), d(x,By)d(y,Ax)\}}{1+pd(x,y)} + \\ \frac{1}{1+pd(x,y)} \max\{\varphi_1(d(x,y)), \varphi_2(d(x,Ax)), \varphi_3(d(y,By)), \\ \varphi_4(\frac{1}{2}[d(x,By)+d(y,Ax)])\} \end{aligned}$$

for all $x, y \in X$ and $p \geqslant 0$, where $\varphi_i \in \Phi$ $(i = 1, 2, 3, 4)$. Then A and B have a unique common fixed point in X.

Proof. By setting $S = T = I$, the identity mapping, it is not difficult to see that the conditions of Theorem 2.1 are satisfied.□

Theorem 2.2. Let S, T and A_n $(n \in \mathbb{N})$ be self-mapping of a complete metric space X. Suppose further that for any $n \in \mathbb{N}$, the pairs $\{A_{2n-1}, S\}$ and $\{A_{2n}, T\}$ are weakly compatible and that

$$A_{2n-1}(X) \subset T(X),\ A_{2n}(X) \subset S(X)$$

If $S(X)$ or $T(X)$ is closed and that for any $i \in \mathbb{N}$, the following condition is satisfied for all $x, y \in X$ and $p \geqslant 0$

$$\begin{aligned}
&d(A_i x, A_{i+1} y)\\
&\leq \frac{p \max(d(Sx, A_i x) d(Tx, A_{i+1} y), d(Sx, A_{i+1} y) d(Ty, A_i x)\}}{1 + pd(Sx, Ty)} +\\
&\frac{1}{1 + pd(Sx, Ty)} \max\{\varphi_1(d(Sx, Ty)), \varphi_2(d(Sx, A_i x)),\\
&\varphi_3(d(Ty, A_{i+1} y)), \varphi_4(\frac{1}{2}[d(Sx, A_{i+1} y) + d(Ty, A_i x)])\}
\end{aligned}$$

where $\varphi_i \in \Phi$ $(i = 1, 2, 3, 4)$, then S, T and $A_n\,(n \in N)$ have a unique common fixed point in X.

Remark 2.3. If we drop the condition that $S(X)$ or $T(X)$ is closed in Theorem 2.2 and replace the weak compatibility by compatibility of type (P) and assume one of the mappings S or T to be continuous, the theorem will still remain valid. Under that form we get an extension of the results of Pathak et al. [17, Theorem 3.3] and Jungck [10, Theorem 3.1] which in turn include several known results, for example, the main results of Chang [5] and Singh and Singh [26].

3. Applications

Throughout this section, we assume that X and Y are Banach spaces, $S \subset X$ is the state space and $D \subset Y$ is the decision space. Let $\mathbb{R} = (-\infty, \infty)$ and denote by $B(S)$ the set of all bounded real valued functions on S.

Following Bellman and Lee [2], the basic form of the functional equation of dynamic programming is given by

$$f(x) = opt_y H(x, y, f(T(x, y))),$$

where x and y represent the state and decision vectors respectively, T represents the transformation of the process and $f(x)$ represents the optimal return with initial state x (where opt denotes max or min).

In this section, we study the existence and uniqueness of a common solution of the following functional equations arising in dynamic programming.

$$f_i(x) = \sup_{y \in D} H_i(x, y, f_i(T(x, y))), \; x \in S, \tag{3.1}$$

$$g_i(x) = \sup_{y \in D} F_i(x, y, g_i(T(x, y))), x \in S, \tag{3.2}$$

where $T: \; S \times D \to S$ and $H_i\,, F_i: \; S \times D \times \mathbb{R} \to \mathbb{R}, \; i = 1, 2.$

Suppose the mappings A_i and T_i $(i = 1, 2)$ are defined by

$$\begin{cases} A_i h(x) = \sup_{y \in D} H_i(x, y, h(T(x, y))), \text{ for all } x \in S, h \in B(S), i = 1, 2. \\ T_i k(x) = \sup_{y \in D} F_i(x, y, k(T(x, y))), \text{ for all } x \in S, k \in B(S), i = 1, 2 \end{cases} \tag{3.3}$$

Theorem 3.1 Suppose that the following conditions are satisfied:

(i) H_i and F_i are bounded for $i = 1, 2$

(ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))|$

$$\begin{aligned} \leqslant M^{-1}(p \max\{&|T_1h(t) - A_1h(t)| \,.\, |T_2k(t) - A_2k(t)|\,, \\ &|T_1h(t) - A_2k(t)| \,.\, |T_2k(t) - A_1h(t)|\} \\ &+\max\,\{\varphi_1(|T_1h(t) - T_2k(t)|), \varphi_2(|T_1h(t) - A_1h(t)|), \\ &\varphi_3(|T_2k(t) - A_2k(t)|), \, \varphi_4(\tfrac{1}{2}[| \; T_1h(t) - A_2k(t) \; | \\ &+ | \; T_2k(t) - A_1h(t) \; |])\}) \end{aligned}$$

for all $(x, y) \in S \times D, \; h, \; k \in B(S), \; t \in S, \; p \geq 0$,where

$$M = [1 + p \sup_{t \in S} |T_1h(t) - T_2k(t)|],$$

and $\varphi_i \in \Phi (i = 1, 2, 3, 4)$ and the mappings A_i and T_i $(i = 1, 2)$ are as defined in (3.3).

(iii) For any sequence $\{k_n\} \subset B(S)$ and $k \in B(S)$ with

$$\lim_{n\to\infty} \sup_{x\in S} |k_n(x) - k(x)| = 0,$$

there exist $h_i \in B(S)$ such that $k = T_i h_i$ for $i = 1$ or $i = 2$.

(iv) For any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$ such that

$$A_1 h(x) = T_2 k_1(x),\ A_2 h(x) = T_1 k_2(x), x \in S.$$

(v) For any $h \in B(S)$ with $A_i h = T_i h$ $(i = 1, 2)$ we have $T_i A_i h = A_i T_i h$.

Then the system of functional equations (3.1) and (3.2) have a unique common solution in $B(S)$.

Proof. It is well known that $B(S)$ endowed with the metric

$$d(h, k) = \underset{x\in S}{sup} |h(x) - k(x)| \text{ for any } h,\ k \in B(S)$$

is a complete metric space. Moreover, by condition (i), A_i and T_i are self mappings of $B(S)$ and by condition (iv) it is clear that

$$A_1(B(S)) \subset T_2(B(S)) \text{ and } A_2(B(S)) \subset T_1(B(S)).$$

Also, by condition (v), the pairs $\{A_i, T_i\}$ are weakly compatible for $i = 1, 2$. Moreover, by (3.3) and (i) we have that for any $\eta > 0$ there exist y_1, $y_2 \in D$ such that

$$A_i h_i(x) < H_i(x_i, y_i, h_i(x)) + \eta, \tag{3.4}$$

where $x_i = T(x, y_i), i = 1, 2$ Also,

$$A_1 h_1(x) \geqslant H_1(x, y_2, h_1(x_2)), \tag{3.5}$$

$$A_2 h_2(x) \geqslant H_2(x, y_1, h_2(x_1)). \tag{3.6}$$

Then from (3.4), (3.5), (3.6) and (ii), we have

$$\begin{aligned} & A_1 h_1(x) - A_2 h_2(x) \\ & \leqslant H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \eta \\ & \leqslant |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1 h_2(x_1))| + \eta \\ \leqslant M^{-1}(p \max\{ & |T_1 h_1(x_1) - A_1 h_1(x_1)| \,.\, |T_2 h_2(x_1) - A_2 h_2(x_1)| \,, \end{aligned}$$

$$|T_1h_1(x_1) - A_2h_2(x_1)| \, . \, |T_2h_2(x_1) - A_1h_1(x_1)|\} +$$
$$max\{\varphi_1(|T_1h_1(x_1) - T_2h_2(x_1)|), \varphi_2(|T_1h_1(x_1) - A_1h_1(x_1)|),$$
$$\varphi_3(|T_2h_2(x_1) - A_2h_2(x_1)|), \varphi_4(\frac{1}{2}[|T_1h_1(x_1) - A_2h_2(x_1)|$$
$$+ |T_2h_2(x_1) - A_1h_1(x_1)|])\}) + \eta$$
$$\leqslant M^{-1}(p \max\{d(T_1h_1, A_1h_1)d(T_2h_2, A_2h_2), d(T_1h_1, A_2h_2)d(T_2h_2, A_1h_1)\}$$
$$+ \max\{\varphi_1\left(d\left(T_1h_1, T_2h_2\right)\right), \varphi_2(d(T_1h_1, A_1h_1)), \varphi_3(d(T_2h_2, A_2h_2)),$$
$$\varphi_4(\frac{1}{2}[d(T_1h_{1,}A_2h_2) + \varphi(T_2h_2, A_1h_1)]\}) + \eta. \tag{3.7}$$

From (3.4), (3.5) and (ii), we have

$$A_1h_1(x) - A_2h_2(x)$$
$$\geqslant -M^{-1}(p \max\{d(T_1h_1, A_1h_1)d(T_2h_2, A_2h_2), d(T_1h_1, A_2h_2)d(T_2h_2, A_1h_1)\}$$
$$+ \max\{\varphi_1 d(T_1h_1, T_2h_2)), \varphi_2(d(T_1h_1, A_1h_1)), \varphi_3(d(T_2h_2, A_2h_2)),$$
$$\varphi_4(\frac{1}{2}[d(T_1h_1, A_2h_2) \ + d(T_2h_2, A_1h_1)])\}) - \eta. \tag{3.8}$$

Using (3.7) and (3.8), we obtain

$$|A_1h_1(x) - A_2h_2(x)|$$
$$\leqslant M^{-1}(p \max\{d(T_1h_1, A_1h_1)d(T_2h_2, A_2h_2), d(T_1h_1, A_2h_2)d(T_2h_2, A_1h_1)\}$$
$$+ \max\{\varphi_1(d(T_1h_1, T_2h_2))\varphi_2(d(T_1h_1, A_1h_1)), \varphi_3(d(T_2h_2, A_2h_2)),$$
$$\varphi_4(\frac{1}{2}[d(T_1h_1, A_2h_2) + d(T_2h_2, A_1h_1)])\}) + \eta. \tag{3.9}$$

Since (3.9) is true for any $x \in S$ and $\eta > 0$ is arbitrary, by taking supremum over all $x \in S$ we have,

$$d(A_1h_1, A_2h_2)$$
$$\leqslant \frac{p}{1 + pd(T_1h_1, T_2h_2)} \max\{d(T_1h_1, A_1h_1)d(T_2h_2, A_2h_2),$$
$$d(T_1h_1, A_2h_2)d(T_2h_2, A_1h_1)\}$$
$$+ \frac{1}{1 + pd(T_1h_1, T_2h_2)} \max\{\varphi_1(d(T_1h_1, T_2h_2)), \varphi_2(d(T_1h_1, A_1h_1),$$
$$\varphi_3(d(T_2h_2, A_2h_2)), \ \varphi_4(\tfrac{1}{2}[d(T_1h_1, A_2h_2) + d(T_2h_2, A_1h_1)])\}.$$

Therefore condition (2.2) is satisfied by mappings A_1, A_2, T_1 and T_2 and hence by Theorem 2.1, they have a unique common fixed point $h^* \in B(S)$, i.e., $h^*(x)$ is a unique common solution of the functional equations (3.1) and (3.2).□

As an immediate consequence of Theorem 3.1 and Corollary 2.2 we have the following.

Theorem 3.2 Suppose that the following conditions are satisfied.

(i) H_i is bounded for $i = 1, 2$,

(ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))|$

$$\leqslant L^{-1}(p \max\{|h(t) - A_1h(t)|\ |k(t) - A_2k(t)|, |h(t) - A_2k(t)|\ |k(t) - A_1h(t)|\}$$
$$+ \max\{\varphi_1(|\ h(t) - k(t)\ |), \varphi_2(|\ h(t) - A_1h(t)\ |, \varphi_3(|k(t) - A_2k(t)|),$$
$$\varphi_4(\frac{1}{2}[|h(t) - A_2k(t)| + |k(t) - A_1h(t)|])\}),$$

for all $(x, y) \in S \times D,\ h,\ k \in B(S),\ t \in S,\ p \geqslant 0$,where

$$L = \{1 + p\sup_{t \in S} |h(t) - k(t)|\}$$

and $\varphi_i \in \Phi$ $(i = 1, 2, 3, 4)$ and A_i $(i = 1, 2)$ are is defined in (3.3)

Then the functional equations (3.1) and (3.2) have a unique common solution in $B(S)$.

Remark.3.1 In view of Remark 2.3, we may drop condition (v) of weak compatibility and replace it by the compatibility of type (P) under the following form

$(v)^*$ For any sequence $\{k_n\} \subset B(S)$, if there exists an $h \in B(S)$ such that

$$\lim_{n\to\infty} \sup_{x \in S} |A_ik_n(x) - h(x)| = \lim_{n\to\infty} \sup_{x \in S} |T_ik_n(x) - h(x)| = 0 \text{ for } i = 1, 2,$$

then

$$\lim_{n\to\infty} \sup_{x \in S} |T_iT_ik_n(x) - A_iA_i(x)| = 0 \text{ for } i = 1, 2.$$

Moreover, if we dispense with condition (iii) in Theorem 3.1 which requires that $T_1(B(S))$ or $T_2(B(S))$ is closed, then we will have to impose a continuity condition on the mappings T_i $(i = 1, 2)$ that may be stated as follows:

$(iii)^*$ For any sequence $\{k_n\} \subset B(S)$ and any $k \in B(S)$,

$$\lim_{n\to\infty} \sup_{x\in S} |k_n(x) - k(x)| = 0 \Longrightarrow \lim_{n\to\infty} \sup_{x\in S} |T_i k_n(x) - T_i k(x)| = 0$$

for $i = 1$ or $i = 2$.

Under this form Theorem 3.1 still remains valid and extends the results of Pathak and Fisher [15, Theorem 3] and Pathak et al. [17, Theorem 5.1]. Moreover, Theorem 3.2 extends the results of Baskaran and Subrahmanyam [1, Theorem 2.1], Bhakta and Mitra [3, Theorem 2.1], Pathak and Fisher [15, Theorem 4] and Pathak et al. [17, Theorem 5.2].

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