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# PARAMETER DEPENDENCE OF THE SOLUTION OF A DELAY INTEGRO-DIFFERENTIAL EQUATION ARISING IN INFECTIOUS DISEASES

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**Abstract.** In this paper we consider a model for the spread of certain infectious disease governed by a delay integro-differential equation where appear a parameter as an index for the severity of disease. We obtain an existence and uniqueness theorem of the positive solution for the considered equation and the smooth dependence by parameter of this solution and his derivative.

**Key Words and Phrases**: Delay integro-differential equation, parameter dependence, Perov's fixed point theorem, successive approximations method.

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## 1. Introduction

In this paper we consider a model for the spread of certain infections disease with a contact rate that varies seasonally. This model is govern by the following integro-differential equation, with the solution depending by the parameter $\lambda \in [a, b]$ which can be an index for severity of the disease:

$$x(t, \lambda) = \int_{t-\tau}^{t} f\left(s, x(s, \lambda), x'_s(s, \lambda), \lambda\right) ds, \qquad t \in \mathbb{R}, \tag{1}$$

where:
(i) $0 \leq t \leq T$;
(ii) $x(t, \lambda)$ is the proportion of infectious in the population at time $t$ with the

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index of severity $\lambda$;
(iii) $\tau$ is the length of time in which an individual remains infectious;
(iv) $x'(t,\lambda)$ is the speed of infection spreading at moment $t$;
(v) $f(t, x(t,\lambda), x'(t,\lambda), \lambda)$ is the proportion of new infectives on unit time.

Here, we study the existence, uniqueness and smooth dependence by the parameter $\lambda$ of a positive solution for an initial value problem associated to the equation (1). We will use a method developed by Rus in [13] which applies a generalization of a result of Hirsch and Pugh from [5].

A similar integral equation which models the problem of the infection spreading and population dynamics in an environment with periodic life conditions:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds, \qquad t \in \mathbb{R}, \tag{2}$$

has been considered in [3], [4], [10], [15], [12] where sufficient conditions are given for the existence of nontrivial periodic nonnegative and continuous solutions for this equation in case of a periodic contact rate: $f(t+\omega, x) = f(t,x)$, $\forall t \in \mathbb{R}$.The tools were: Banach fixed point principle in [12], topological fixed point theorems in [3], [4], [10], [15], fixed point index theory in [4] and monotone technique in [4], [10]. Also, a system of integral equations in the form (2) has been studied in [2] and [13] using: the monotone technique in [2] and the Perov's fixed point theorem for data dependence by the parameter in [13].

In other papers is studied the initial value problem,

$$x(t) = \begin{cases} \int_{t-\tau}^{t} f(s, x(s))\, ds, & t \in [0,T] \\ \varphi(t), & t \in [-\tau, 0]. \end{cases} \tag{3}$$

The existence of a positive solution of (3) has been studied in [9] using the Leray-Schauder continuation principle and in [11], using the monotone iterations technique..Using a Lipschitz's condition, in [6], it obtains the existence and uniqueness of the positive, bounded solution for (3) and is given a numerical method to approximate this solution. This numerical method is based on the sequence of successive approximations and on the trapezoidal quadrature rule.

Also, for the initial value problem

$$x(t) = \begin{cases} \int\limits_{t-\tau}^{t} f(s, x(s), x'(s))\, ds, & t \in [0, T] \\ \varphi(t), & t \in [-\tau, 0] \end{cases} \tag{4}$$

in [1] is studied the existence and uniqueness of a positive solution using the Perov's fixed point theorem, is given a Lipschitz property of the derivative of this solution and a numerical method which approximate it.

In the following, if $X$ is a nonempty set then by a generalized metric $d$ on $X$ we understand a function $d : X \times X \to \mathbb{R}^n$ which fulfils the following:
(gm1) $0_{\mathbb{R}^n} \leq d(x, y), \forall x, y \in X$ and $d(x, y) = 0_{\mathbb{R}^n} \Leftrightarrow x = y$;
(gm2) $d(x, y) = d(y, x), \forall x, y \in X$;
(gm3) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$,
where for $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ from $\mathbb{R}^n$ we have $x \leq y \Leftrightarrow x_i \leq y_i$, for any $i = \overline{1, n}$. The pair $(X, d)$ will be called generalized metric space ( see [7], [8] and [14] ).

On a generalized metric space we have the following generalizations of the Banach's fixed point theorem

**Theorem 1.** *(Perov, [7]) Let $(X, d)$ a complete generalized metric space. If $T : X \to X$ is a map for which exists a matrix $Q \in \mathcal{M}_n(\mathbb{R})$ such that:*
*(i) $d(T(x), T(y)) \leq Qd(x, y)$, $\forall x, y \in X$;*
*(ii) the eigenvalues of $Q$ lies in the open unit disc from $\mathbb{R}^2$,*
*then:*
*(1) $T$ has a unique fixed point $x^*$;*
*(2) the sequence of successive approximations $x_m = T^m(x_0)$ converges to $x^*$ for any $x_0 \in X$;*
*(3) the estimation*

$$d(x_m, x^*) \leq Q^m (I_n - Q)^{-1} d(x_0, x_1), \forall m \in \mathbb{N}^* \text{ holds.}$$

**Theorem 2.** *(Rus, [13, Theorem 4])Let $(X, d)$ be a generalized metric space with $d(x, y) \in \mathbb{R}^p$ and $(Y, \rho)$ a generalized complete metric space with $\rho(x, y) \in \mathbb{R}^q$. Let $A : X \times Y \to X \times Y$, $B : X \to X$ and $C : X \times Y \to Y$ continuous maps. We suppose that:*
*(i) $A(x, y) = (B(x), C(x, y)), \forall x \in X, \forall y \in Y$.*
*(ii) the map $B$ has a unique fixed point $x^*$ and, for any $x_0 \in X$, the sequence*

$(B^n(x_0))_n$ *converges to* $x^*$.
*(iii) there exists a matrix* $Q \in M_{qq}(\mathbb{R}_+)$, *with* $Q^n \overset{n\to\infty}{\longrightarrow} 0$, *such that*

$$\rho(C(x,y_1),C(x,y_2)) \leq Q\rho(y_1,y_2), \forall x \in X, \forall y_1,y_2 \in Y.$$

*Then* $A$ *has a unique fixed point* $(x^*,y^*)$ *(where* $y^*$ *is the unique fixed point of* $C(x^*,\cdot)$*) and for any* $(x_0,y_0) \in X \times Y$ *the sequence* $(A^n(x_0,y_0))_n$ *converges to* $(x^*,y^*)$.

## 2. Main result

Here we study the existence and uniqueness of the solution and the dependence by parameter of this solution of the following initial value problem,

$$x(t,\lambda) = \begin{cases} \int_{t-\tau}^{t} f(s,x(s,\lambda),x'(s,\lambda),\lambda)\,ds, & t \in [0,T] \\ \varphi(t,\lambda), & t \in [-\tau,0] \end{cases} \quad (5)$$

where $\varphi \in C([-\tau,0]\times[a,b])$, in the following conditions:

$(C1)$ $f \in C([-\tau,T]\times\mathbb{R}_+\times\mathbb{R}\times[a,b])$, $\varphi \in C^1([-\tau,0]\times[a,b])$ and

$$f(s,u,v,\cdot) \in C^1[a,b], \forall (s,u,v) \in [-\tau,T]\times\mathbb{R}_+\times\mathbb{R}.$$

$(C2)$ (boundeness conditions) there exists $m, M \geq 0$ such that

$$m \leq f(t,u,v,\lambda) \leq M, \forall (t,u,v,\lambda) \in [-\tau,T]\times\mathbb{R}_+\times\mathbb{R}\times[a,b]$$

and

$$\varphi(t,\lambda) \geq 0, \forall (t,\lambda) \in [-\tau,0]\times[a,b].$$

$(C3)$ (first compatibility conditions) :

$\varphi(0,\lambda) = \int_{-\tau}^{0} f(s,\varphi(s,\lambda),\varphi'_s(s,\lambda),\lambda)\,ds,\ \forall\lambda \in [a,b],$

$\varphi'_t(0,\lambda) = f(0,\varphi(0,\lambda),\varphi'_t(0,\lambda),\lambda) - f(-\tau,\varphi(-\tau,\lambda),\varphi'_t(-\tau,\lambda),\lambda), \forall\lambda \in [a,b].$

$(C4)$ (second compatibility conditions) :

$$\varphi \in C^2([-\tau,0]\times[a,b]), f(s,\cdot,\cdot,\cdot) \in C^1(\mathbb{R}_+\times\mathbb{R}\times[a,b]), \forall s \in [-\tau,T]$$

and for any $\lambda \in [a,b]$ we have

$\varphi'_\lambda(0,\lambda) = \int_{-\tau}^{0} \left[\frac{\partial f}{\partial \lambda}(s,\varphi,\varphi'_s,\lambda) + \frac{\partial f}{\partial x}(s,\varphi,\varphi'_s,\lambda)\cdot\varphi'_\lambda + \frac{\partial f}{\partial y}(s,\varphi,\varphi'_s,\lambda)\cdot\varphi''_{s\lambda}\right] ds,$

$\varphi''_{t\lambda}(0,\lambda) = [\frac{\partial f}{\partial \lambda}(0,\varphi(0,\lambda),\varphi'_t(0,\lambda),\lambda) + \frac{\partial f}{\partial x}(0,\varphi(0,\lambda),\varphi'_t(0,\lambda),\lambda)\cdot\varphi'_\lambda(0,\lambda) +$

$+\frac{\partial f}{\partial y}(0,\varphi(0,\lambda),\varphi'_t(0,\lambda),\lambda)\cdot\varphi''_{t\lambda}(0,\lambda)] - [\frac{\partial f}{\partial \lambda}(-\tau,\varphi(-\tau,\lambda),\varphi'_t(-\tau,\lambda),\lambda) +$

$+\frac{\partial f}{\partial x}(-\tau,\varphi(-\tau,\lambda),\varphi'_t(-\tau,\lambda),\lambda)\cdot\varphi'_\lambda(-\tau,\lambda) +$
$\frac{\partial f}{\partial y}(-\tau,\varphi(-\tau,\lambda),\varphi'_t(-\tau,\lambda),\lambda)\cdot\varphi''_{t\lambda}(-\tau,\lambda)]$.

$(C5)$ (Lipschitz condition) : there exists $\alpha,\beta > 0$ such that, $\forall t \in [-\tau,T]$, $\forall\lambda \in [a,b]$ and $\forall (x_i,y_i) \in \mathbb{R}_+ \times \mathbb{R}, i = 1,2$ we have that

$$|f(t,x_1,y_1,\lambda) - f(t,x_2,y_2,\lambda)| \leq \alpha|x_1 - y_1| + \beta|y_1 - y_2|.$$

We consider the product generalized metric spaces $(X,d)$ and $(Y,\rho)$ where

$$X := C([-\tau,T]\times[a,b],\mathbb{R}_+)\times C([-\tau,T]\times[a,b]),$$

$$Y := C([-\tau,T]\times[a,b])\times C([-\tau,T]\times[a,b])$$

which are complete spaces, where the metrics are given by

$$\rho((x_1,y_1),(x_2,y_2)) = (\|x_1 - x_2\|,\|y_1 - y_2\|)$$

and $d := \rho|_{X\times X}$ with

$$\|g\| := \max_{t\in[-\tau,T],\lambda\in[a,b]} |g(t,\lambda)|.$$

We consider the following initial value problems,

$$\begin{cases} x(t,\lambda) = \begin{cases} \int\limits_{t-\tau}^{t} f(s,x(s,\lambda),y(s,\lambda),\lambda)\,ds, & t\in[0,T] \\ \varphi(t,\lambda), & t\in[-\tau,0] \end{cases} \\ y(t,\lambda) = \begin{cases} f(t,x(t,\lambda),y(t,\lambda),\lambda) - f(t-\tau,x(t-\tau,\lambda),y(t-\tau,\lambda),\lambda), \\ , \quad t\in[0,T] \\ \varphi'_t(t,\lambda), \quad t\in[-\tau,0] \end{cases} \end{cases} \tag{6}$$

and

$$\begin{cases} u(t,\lambda) = \begin{cases} \int\limits_{t-\tau}^{t} \left[\frac{\partial f}{\partial \lambda}(s,x,y,\lambda) + \frac{\partial f}{\partial x}(s,x,y,\lambda) \cdot u + \frac{\partial f}{\partial y}(s,x,y,\lambda) \cdot v\right] ds, \\ , \qquad t \in [0,T] \\ \varphi'_{\lambda}(t,\lambda), \qquad t \in [-\tau,0] \end{cases} \\ v(t,\lambda) = \begin{cases} [\frac{\partial f}{\partial \lambda}(t,x,y,\lambda) + \frac{\partial f}{\partial x}(t,x,y,\lambda) \cdot u(t,\lambda) + \frac{\partial f}{\partial y}(t,x,y,\lambda) \cdot v(t,\lambda)] \\ -[\frac{\partial f}{\partial \lambda}(t-\tau,x,y,\lambda) + \frac{\partial f}{\partial x}(t-\tau,x,y,\lambda) \cdot u(t-\tau,\lambda) + \\ +\frac{\partial f}{\partial y}(t-\tau,x,y,\lambda) \cdot v(t-\tau,\lambda)], \qquad t \in [0,T] \\ \varphi''_{t\lambda}(t,\lambda), \qquad t \in [-\tau,0]. \end{cases} \end{cases} \tag{7}$$

We define the operators $B : X \to X$,

$$B(x,y)(t\,\lambda) = \\ = \begin{cases} \begin{pmatrix} \int\limits_{t-\tau}^{t} f(s,x(s,\lambda),y(s,\lambda),\lambda)\,ds, \\ f(t,x(t,\lambda),y(t,\lambda),\lambda) - f(t-\tau,x(t-\tau,\lambda),y(t-\tau,\lambda),\lambda) \end{pmatrix}, \\ , \qquad t \in [0,T] \\ (\varphi(t,\lambda), \varphi'_t(t,\lambda)), \qquad t \in [-\tau,0], \end{cases}$$

$C : X \times Y \to Y$,

$$C((x,y),(u,v))(t,\lambda) = \\ = \begin{cases} (\int\limits_{t-\tau}^{t} \left[\frac{\partial f}{\partial \lambda}(s,x,y,\lambda) + \frac{\partial f}{\partial x}(s,x,y,\lambda) \cdot u(s,\lambda) + \frac{\partial f}{\partial y}(s,x,y,\lambda) \cdot v(s,\lambda)\right] ds \\ , \; [\frac{\partial f}{\partial \lambda}(t,x,y,\lambda) + \frac{\partial f}{\partial x}(t,x,y,\lambda) \cdot u(t,\lambda) + \frac{\partial f}{\partial y}(t,x,y,\lambda) \cdot v(t,\lambda)] - \\ [\frac{\partial f}{\partial \lambda}(t-\tau,x,y,\lambda) + \frac{\partial f}{\partial x}(t-\tau,x,y,\lambda) \cdot u(t-\tau,\lambda) + \\ +\frac{\partial f}{\partial y}(t-\tau,x,y,\lambda) \cdot v(t-\tau,\lambda)]), \qquad t \in [0,T] \\ (\varphi'_{\lambda}(t,\lambda), \varphi''_{t\lambda}(t,\lambda)), \qquad t \in [-\tau,0] \end{cases}$$

and $A : X \times Y \to X \times Y$, $A((x,y),(u,v)) = (B(x,y), C((x,y),(u,v)))$.

We will study now the existence and uniqueness of the fixed point of operators $B$ and $A$.

**Theorem 3.** *(i) In the conditions $(C1)-(C3)$ and $(C5)$ with $2\beta+\alpha\tau<1$ the operator $B$ has a unique fixed point $(x^*,y^*)$ which is the solution of the problem (6) in $X$ and we have that $x^*\in C^1([-\tau,T]\times[a,b])$, with $y^*=(x^*)'_t$.*
*(ii) In the conditions $(C1)-(C5)$ with $2\beta+\alpha\tau<1$ and for $(x^*,y^*)$ (the fixed point of $B$ from above) the oparator $A$ has a unique fixed point $((x^*,y^*),(u^*,v^*))$ which is solution of the problem (7). Also we have that*

$$y^*(t,\cdot)\in C^1[a,b],\ \forall t\in[-\tau,T] \quad \text{and } u^*=(x^*)'_\lambda,\ v^*=(y^*)'_\lambda.$$

*Proof.* (i) From $(C1)$ and $(C2)$ we obtain that $B(X)\subseteq X$, hence $B$ is well defined.
For any $\lambda\in[a,b]$ and $t\in[-\tau,0]$ we have

$$d(B(x_1,y_1),B(x_2,y_2))(t,\lambda)=(0,0),\quad \forall(x_i,y_i)\in X,\ i=1,2. \tag{8}$$

For any $\lambda\in[a,b]$ and $t\in[0,T]$ we have

$$\left|\int_{t-\tau}^{t} f(s,x_1(s,\lambda),y_1(s,\lambda),\lambda)\,ds-\int_{t-\tau}^{t} f(s,x_2(s,,\lambda),y_2(s,\lambda),\lambda)\,ds\right|\overset{C5}{\leq}$$

$$\leq\int_{t-\tau}^{t}(\alpha|x_1(s,\lambda)-x_2(s,\lambda)|+\beta|y_1(s,\lambda)-y_2(s,\lambda)|)\,ds\leq$$

$$\leq\alpha\tau\|x_1-x_2\|+\beta\tau\|y_1-y_2\|.$$

Hence

$$\max_{t\in[0,T]}\left|\int_{t-\tau}^{t} f(s,x_1(s,\lambda),y_1(s,\lambda),\lambda)\,ds-\int_{t-\tau}^{t} f(s,x_2(s,\lambda),y_2(s,\lambda),\lambda)\,ds\right|\leq$$
$$\leq\alpha\tau\|x_1-x_2\|+\beta\tau\|y_1-y_2\|,\quad\forall(x_i,y_i)\in X,\ i=1,2. \tag{9}$$

$$|f(t,x_1(t,\lambda),y_1(t,\lambda),\lambda)-f(t-\tau,x_1(t-\tau,\lambda),y_1(t-\tau,\lambda),\lambda)-$$
$$-(f(t,x_2(t,\lambda),y_2(t,\lambda),\lambda)-f(t-\tau,x_2(t-\tau,\lambda),y_2(t-\tau,\lambda),\lambda))|\overset{C5}{\leq}$$
$$\leq\alpha|x_1(t,\lambda)-x_2(t,\lambda)|+\beta|y_1(t,\lambda)-y_2(t,\lambda)|+$$
$$+\alpha|x_1(t-\tau,\lambda)-x_2(t-\tau,\lambda)|+\beta|y_1(t-\tau,\lambda)-y_2(t-\tau,\lambda)|\leq$$
$$\leq2\alpha\|x_1-x_2\|+2\beta\|y_1-y_2\|.$$

Hence

$$\max_{t\in[0,T]} \left| f(t,x_1(t,\lambda),y_1(t,\lambda),\lambda) - f(t-\tau,x_1(t-\tau,\lambda),y_1(t-\tau,\lambda),\lambda) - \right.$$
$$\left. -f(t,x_2(t,\lambda),y_2(t,\lambda),\lambda) + f(t-\tau,x_2(t-\tau,\lambda),y_2(t-\tau,\lambda),\lambda) \right| \leq \tag{10}$$
$$\leq 2\alpha \|x_1 - x_2\| + 2\beta \|y_1 - y_2\|, \quad \forall (x_i, y_i) \in X,\ i = 1,2.$$

From (8), (9), (10) we have that

$$d(B(x_1,y_1), B(x_2,y_2)) \leq Qd((x_1,y_1),(x_2,y_2)), \forall (x_i,y_i) \in X, i = 1,2,$$

where $Q = \begin{pmatrix} \alpha\tau & \beta\tau \\ 2\alpha & 2\beta \end{pmatrix}$. The eigenvalues of $Q$ are

$$\lambda_1 = 0 \text{ and } \lambda_2 = 2\beta + \alpha\tau < 1.$$

Applying Theorem 1 we obtain the existence and uniqueness of a fixed point $(x^*, y^*)$ of $B$.
From condition $(C_1)$ we infer that $x^* \in C^1([-\tau,T]\times[a,b])$. We prove now that

$$(x^*)'(t) = y^*(t), \forall t \in [-\tau, T].$$

For this aim, in the case $t \in [-\tau, 0]$, we have,

$$B(x^*(t,\lambda), y^*(t,\lambda)) = (\varphi(t,\lambda), \varphi_t'(t,\lambda)) = (x^*(t,\lambda), y^*(t,\lambda)),$$

so, $x^*(t,\lambda) = \varphi(t,\lambda)$ and $y^*(t,\lambda) = \varphi_t'(t,\lambda)$, which means $(x^*)'(t,\lambda) = y^*(t,\lambda)$.
In the case $t \in [0,T]$, we have, $(x^*(t,\lambda), y^*(t,\lambda)) = B(x^*(t,\lambda), y^*(t,\lambda))$,

that is,

$$\begin{cases} x^*(t,\lambda) = \int\limits_{t-\tau}^{t} f(s, x^*(s,\lambda), y^*(s,\lambda), \lambda)\, ds \\ y^*(t,\lambda) = f(t, x^*(t,\lambda), y^*(t,\lambda)) - f(t-\tau, x^*(t-\tau,\lambda), y^*(t-\tau,\lambda), \lambda). \end{cases}$$

We derive with respect by $t$ the first above relation and obtain that

$$(x^*)_t'(t,\lambda) = y^*(t,\lambda), \quad \forall t \in [0,T], \quad \forall \lambda \in [a,b].$$

(ii) For $(x^*, y^*)$ found above we will prove now that $C((x^*,y^*),\cdot) : Y \to Y$ is a $Q-$ contraction, that is, $Q \in M_{22}(\mathbb{R}_+)$, is such that $Q^n \overset{n\to\infty}{\longrightarrow} 0$.

For any $\lambda \in [a,b]$ and $t \in [-\tau, 0]$ we have

$$\rho\left(C\left(\left(x^*, y^*\right), \left(u_1, v_1\right)\right), C\left(\left(x^*, y^*\right)\left(u_2, v_2\right)\right)\right)(t) = (0,0), \forall \left(u_i, v_i\right) \in Y, i = 1,2. \tag{11}$$

For any $\lambda \in [a,b]$ and $t \in [0, T]$ we have

$$| \int_{t-\tau}^{t} [\frac{\partial f}{\partial \lambda}(s,x,y,\lambda) + \frac{\partial f}{\partial x}(s,x,y,\lambda) \cdot u_1(s,\lambda) + \frac{\partial f}{\partial y}(s,x,y,\lambda) \cdot v_1(s,\lambda)]ds$$

$$- \int_{t-\tau}^{t} [\frac{\partial f}{\partial \lambda}(s,x,y,\lambda) + \frac{\partial f}{\partial x}(s,x,y,\lambda) \cdot u_2(s,\lambda) + \frac{\partial f}{\partial y}(s,x,y,\lambda) \cdot v_2(s,\lambda)]ds \, |$$

$$\leq (\alpha\tau \left\| u_1 - u_2 \right\| + \beta\tau \left\| v_1 - v_2 \right\|). \tag{12}$$

On the other hand,

$$| \, [\frac{\partial f}{\partial \lambda}(t,x,y,\lambda) + \frac{\partial f}{\partial x}(t,x,y,\lambda) \cdot u_1(t,\lambda) + \frac{\partial f}{\partial y}(t,x,y,\lambda) \cdot v_1(t,\lambda)$$

$$-\frac{\partial f}{\partial \lambda}(t-\tau,x,y,\lambda) - \frac{\partial f}{\partial x}(t-\tau,x,y,\lambda) \cdot u_1(t-\tau,\lambda) - \frac{\partial f}{\partial y}(t-\tau,x,y,\lambda) \cdot v_1(t-\tau,\lambda)] -$$

$$-[\frac{\partial f}{\partial \lambda}(t,x,y,\lambda) + \frac{\partial f}{\partial x}(t,x,y,\lambda) \cdot u_2(t,\lambda) + \frac{\partial f}{\partial y}(t,x,y,\lambda) \cdot v_2(t,\lambda) -$$

$$\frac{\partial f}{\partial \lambda}(t-\tau,x,y,\lambda) - \frac{\partial f}{\partial x}(t-\tau,x,y,\lambda) \cdot u_2(t-\tau,\lambda) - \frac{\partial f}{\partial y}(t-\tau,x,y,\lambda) \cdot v_2(t-\tau,\lambda)] \, | \leq$$

$$\leq \left| \frac{\partial f}{\partial x}(t,x,y,\lambda) \right| \cdot |u_1(t,\lambda) - u_2(t,\lambda)| + \left| \frac{\partial f}{\partial y}(t,x,y,\lambda) \right| \cdot |v_1(t,\lambda) - v_2(t,\lambda)| +$$

$$+ \left| \frac{\partial f}{\partial x}(t-\tau,x,y,\lambda) \right| \cdot |u_1(t-\tau,\lambda) - u_2(t-\tau,\lambda)| + \left| \frac{\partial f}{\partial y}(t-\tau,x,y,\lambda) \right| \cdot$$

$$\cdot |v_1(t-\tau,\lambda) - v_2(t-\tau,\lambda)| \leq 2\alpha \left\| u_1 - u_2 \right\| + 2\beta \left\| v_1 - v_2 \right\|. \tag{13}$$

Since

$$f(s,\cdot,\cdot,\cdot) \in C^1\left(\mathbb{R}_+ \times \mathbb{R} \times [a,b]\right), \forall s \in [-\tau, T],$$

we have $\alpha = \left\| \frac{\partial f}{\partial x} \right\|$ and $\beta = \left\| \frac{\partial f}{\partial y} \right\|$.

From (12) and (13) we infer that,

$$\rho\left(C\left(\left(x^*, y^*\right), \left(u_1, v_1\right)\right), C\left(\left(x^*, y^*\right)\left(u_2, v_2\right)\right)\right) \leq Q \cdot \rho(\left(u_1, v_1\right), \left(u_2, v_2\right))$$

and so, $C((x^*, y^*), \cdot)$ is a $Q$ - contraction.

Since $\alpha\tau + 2\beta < 1$ we have that $Q^n \to 0$ as $n \to \infty$.

Now, we can apply Theorem 2 which lead to the existence and uniqueness of the fixed point $((x^*, y^*), (u^*, v^*))$ of $A$, that is $B(x^*, y^*) = (x^*, y^*)$ and

$$C((x^*, y^*), (u^*, v^*)) = (u^*, v^*).$$

Then, choosing $x_0 \in X \cap C^2\left([-\tau, T] \times [a, b], \mathbb{R}_+\right)$, $y_0 = \frac{\partial x_0}{\partial t}$, $u_0 = \frac{\partial x_0}{\partial \lambda}$, $v_0 = \frac{\partial y_0}{\partial \lambda}$, the sequence

$$(A^n((x_0, y_0), (u_0, v_0)))_n = ((x_n, y_n), (u_n, v_n))_n$$

converges to $((x^*, y^*), (u^*, v^*))$. Consequently, $y^*(t, \cdot) \in C^1[a, b]$, $\forall t \in [-\tau, T]$ and

$$x_n \rightrightarrows x^* \ , \quad y_n = \frac{\partial x_n}{\partial t} \rightrightarrows y^*$$
$$u_n = \frac{\partial x_n}{\partial \lambda} \rightrightarrows u^* \ , \quad v_n = \frac{\partial y_n}{\partial \lambda} \rightrightarrows v^*,$$

that is

$$y^* = \frac{\partial x^*}{\partial t}, \quad u^* = \frac{\partial x^*}{\partial \lambda}, \quad v^* = \frac{\partial y^*}{\partial \lambda}.$$

□

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