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SOME SUITABLE METRICS ON FUZZY METRIC SPACES

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Abstract. A *fuzzy metric* is a function of the form $X \times X \ni (p,q) \rightarrow^{\mathcal{F}} F_{pq} \in \Delta^+$, where Δ^+ is the set of all distance distribution functions, and in many cases $\mathcal{F}$ generates a metrizable uniformity. Starting from this fundamental property, we present several *metric-like functions* determined by fuzzy metrics and we emphasize their role in getting and proving fixed point theorems for different types of contractions. There are identified large *classes of t-norms* and *general formulae* of (extended) metrics, which are seen to generalize the distances of M. Fréchet, P. Lévy and Ky Fan.

Key Words and Phrases: Fuzzy metric, extended metric, probabilistic contraction, fixed point.

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1. Preliminaries on triangular norms and fuzzy metrics

The concept of a triangular norm has been introduced in [71] by slightly modifying the Menger's axioms [36]. A mapping $T : [0,1]\times[0,1] \to [0,1] = I$ is called a **triangular norm** (shortly a **t-norm**) if $(tn1)$ $T(a,1) = a,\ (\forall)a \in I$, $(tn2)$ T is symmetric, $(tn3)$ T is nondecreasing in each variable and $(tn4)$ T is associative. We will make use of three basic t-norms, namely the *minimum* operator T_M, the *algebraic product* T_P, and the *Lukasiewicz t-norm* T_L defined by: $T_M(x,y) = Min(x,y) = \min\{x,y\}$, $T_P(x,y) = Prod(x,y) =$

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$xy, T_L(x,y) = W(x,y) = \max(0, x+y-1)$. These t-norms are ranked as: $T_L \le T_P \le T_M$; in fact, T_M is the strongest t-norm.

A *continuous* t-norm T is *Archimedean* iff there exists a **continuous strictly decreasing** mapping $g : [0,1] \to [0,\infty]$ such that $g(1) = 0$, $g(0) \in \{1,\infty\}$ and we have the representation

$$T(a,b) = T_g(a,b) := g^{(-1)}(g(a)+g(b)),\ \forall\ u,v \in [0,1], \tag{1.1.1}$$

where $[0,\infty] \ni x \to g^{(-1)}(x) = g^{-1}(Min\{x, g(0)\})$ defines the pseudo-inverse of **the additive generator** g. The *powers of a t-norm* T are defined by the following formulae:

$$T^0(t_1) = t_1, T^m(t_1,...,t_{m+1}) = T(T^{m-1}(t_1,...,t_m), t_{m+1}), (\forall)\ m \ge 1.$$

Let $T_m(t) := T^m(t,t,...,t), \forall t \in [0,1]$. We say that T is of **Hadžić type** (of **h-type** or an **h-t-norm**) if the family $\{T_m\}_{m\in\mathbf{N}}$ is equicontinuous at $t = 1$. More details can be found in the monographs [72], [31] and [18].

The set of all left-continuous on $\mathbf{R}$ and nondecreasing functions $F : [-\infty,+\infty] \to [0,1]$, for which $F(0) = 0$ and $F(+\infty) = 1$, is denoted by Δ^+ while its subset of all F with $\lim_{x\to+\infty} F(x) = 1$ is denoted by D^+. For every $a \ge 0$, ε_a is the unique element of Δ^+ for which $\varepsilon_a(t) = 0$ if $t < a$ and $\varepsilon_a(t) = 1$ if $t > a$. Let X be a nonempty set and suppose we are given a mapping $\mathcal{F} : X \times X \longrightarrow \Delta^+$. Its values $\mathcal{F}(x,y)$ are usually denoted by F_{xy}. $(X,\mathcal{F})$ is called a **fuzzy semimetric space** (shortly **FSM-space**) if and only if, for all $x,y,z \in X$,

$$F_{xy} = \varepsilon_0 \textit{ if and only if } x = y, \tag{$FSM1$}$$

$$F_{xy} = F_{yx}. \tag{$FSM2$}$$

If at least the following implication holds, then we use the generic term **fuzzy metric space** (**FM-space**):

$$[F_{xy}(t) = 1, F_{yz}(s) = 1] \Rightarrow F_{xz}(t+s) = 1. \tag{$FM3_{SS}$}$$

If there exists a t-norm T such that

$$F_{xy}(t+s) \ge T(F_{xz}(t), F_{zy}(s)), \tag{$FM3_M$}$$

then we say that $(X,\mathcal{F},T)$ is a **fuzzy Menger space,** or a generalized Menger space ([32], [27]). A more general form of triangle inequality, defining **fuzzy S-metric spaces** and **fuzzy S-Menger spaces,** respectively, can be formulated by replacing the addition $(t,s) \to t+s$ with some operations $(t,s) \to S(t,s)$ on $[0,\infty]$. For example, if $S = \max$ we have **fuzzy ultrametric spaces** (or nonArchimedean fuzzy metric spaces). We can also use some operations τ on Δ^+ (*Šerstnev-functions*) and the following triangle inequality for **fuzzy Šerstnev spaces**:

$$F_{pq} \geq \tau(F_{pr}, F_{rq}), \forall p,q,r \in X. \tag{$FM3_S$}$$

Generally, the mapping $\mathcal{F}$ is called a **fuzzy (semi) metric**. If it happens that $F_{xy} \in D^+$ for every $x, y \in X$, then $\mathcal{F}$ is called a **probabilistic (semi) metric** and $(X, \mathcal{F}, T)$ is a **Menger space** ($(X, \mathcal{F}, \tau)$ is a **Šerstnev space**). Notice that our definition includes properly the fuzzy metric spaces considered in [6] or [10], for which we shall use the term *strong fuzzy metric spaces.*

For every FSM-space $(X, \mathcal{F})$ we can consider the sets

$$U_{\varepsilon,\lambda} = \{(p,q) \in X \times X,\ F_{pq}(\varepsilon) > 1-\lambda\ \},\ \varepsilon > 0, \lambda \in (0,1),$$

which generate a semiuniformity $\mathcal{U}_F$ and a topology $\mathcal{T}_{\mathcal{F}}$, called the $\mathcal{F}-$topology or the strong topology. $\mathcal{U}_{\mathcal{F}}$ is also generated by the family of the sets $V_\delta := U_{\delta,\delta}$ and the following result ([72], see also [73]) holds: **If** $\sup_{a<1} T(a,a) = 1$, **then** $\mathcal{U}_{\mathcal{F}}$ **is a uniformity, and it is metrizable**. As shown by B. Morrel & J. Nagata [44] and U. Höhle [25], the above condition is *the weakest* one ensuring the existence of the $\mathcal{F}-$uniformity $\mathcal{U}_{\mathcal{F}}$. A sequence in $(X, \mathcal{F})$ is a *Cauchy sequence* (or $\mathcal{F}$-*Cauchy*) iff $\forall \varepsilon > 0, \forall \lambda \in (0,1), \exists n_0 :\ F_{x_n x_{n+k}}(\varepsilon) > 1-\lambda, \forall n \geq n_0, \forall k \geq 0$, and $(X, \mathcal{F})$ is said to be *complete* if every Cauchy sequence is $\mathcal{F}$-convergent. Notice that, by definition,

$$p_n \xrightarrow{\mathcal{F}} p \text{ iff } \forall \varepsilon > 0, \forall \lambda \in (0,1), \exists n_0 \text{ such that } F_{p_n p}(\varepsilon) > 1-\lambda, \forall n \geq n_0.$$

In [76] and [9] one can find details on the completion of Menger spaces and strong fuzzy metric spaces, respectively. The notions not given here, as well as other contributions and developments in the domain of nonlinear probabilistic analysis can be found in [72, 14, 5, 60, 17, 3, 18, 64, 66].

1.1. **Fundamental examples I.** **Example 1.1.1.** We obtain a complete fuzzy semimetric space on $X = [0, \infty]$ if

$$F_{xy}(t) := \frac{2^{\min(x,y)}t}{2^{\min(x,y)}t + |x-y|} =: F_{yx}(t), \forall t \in (0, \infty), \forall x, y \in [0, \infty], x \neq y^1.$$

Example 1.1.2. Let (X, d) be an (extended) metric space.

a. If we set $\mathcal{F} = \mathcal{E}_d$, that is $F_{xy} = \varepsilon_{d(x,y)}$, then $(X, \mathcal{F}, T)$ is a (fuzzy) Menger space for every t-norm T. The $\mathcal{F}$ -uniformity is exactly $\mathcal{U}_d$.

b. One can define $F_{xy}(t) = \frac{t}{t+d(x,y)}, \forall t \in (0, \infty)$. Obviously we obtain a fuzzy metric space (under any triangular norm) whose $\mathcal{F}$-uniformity is $\mathcal{U}_d$.

c.The same result is obtained if $F_{xy} \in \Delta^+$ and

$$F_{xy}(t) := \begin{cases} \frac{t}{d(x,y)}, & 0 < t < d(x,y) \\ 1 & t > d(x,y) \end{cases} .$$

In all cases $(X, \mathcal{F})$ is complete iff (X, d) is complete.

Example 1.1.3. Let $F_{xx} := \varepsilon_0$ and $F_{xy}(t) := \frac{\min(x,y)}{\max(x,y)}, \forall t \in (0, \infty), \forall x, y \geq 0, x \neq y$.

a. $X_0 = (0, \infty)$ is a complete strict fuzzy metric space under the Archimedean triangular norm T_P, for a sequence (x_n) is Cauchy if and only if there exists $x > 0$ such that $|x_n - x| \to 0$ in $\mathbf{R}$.

b. $X = [0, \infty)$ is a complete fuzzy Menger space under the triangular norm T_P.

c. $t \to \left(\frac{\min(x,y)}{\max(x,y)}\right)^{\frac{1}{t}}$, for $x, y \geq 0, x \neq y$, defines a fuzzy Menger space on $[0, \infty)$ and a Menger space on $(0, \infty)$, under T_P.

All these spaces are nonArchimedean.

Example 1.1.4. Let $X = X(\Omega, \mathcal{K}, P)$ be the space of all classes of real random variables on a probability measure space $(\Omega, \mathcal{K}, P)$. If we define, for $p, q \in X$

$$F_{pq}(t) = P\{\omega \in \Omega,\ d(p(\omega), q(\omega)) < t\},$$

that is the distribution function of $d(p, q) = |p - q|$, then $(X, (p, q) \to F_{pq}, T_L)$ is a Menger space whose strong topology is the topology of the convergence in probability. If the random variables take their values in a separable extended metric space, then we obtain a fuzzy Menger space under T_L.

[1]We automatically assume that $F_{xx} = \varepsilon_0$ as well as $F(t) = 0, \forall t \leq 0$ and $F(\infty) = 1$, for every $F \in \Delta^+$

2. Extended metrics and Hicks-type contractions

2.1. **Fuzzy metric spaces of type M and fixed points for $\Phi - M$ contractions.** The proof of the following simple result is easy to reproduce:

Proposition 2.1.1. *Let $(X, \mathcal{F})$ be a FSM-space and $\boldsymbol{K}(x, y) := \sup\{t \mid t \leq 1 - F_{xy}(t)\}$. Then*

$$\boldsymbol{K}(x, y) < \delta \Leftrightarrow F_{xy}(\delta) > 1 - \delta \ \forall \delta > 0. \tag{2.1.1}$$

Therefore $\boldsymbol{K}$ generates the topology $\mathcal{T}_F$ and the semi-uniformity $\mathcal{U}_F$, so that $\boldsymbol{K}$ is a $[0, 1]-$valued semi-metric on X.

Example 2.1.2. (i) If d is an extended semi-metric on X and we set $F_{xy} := \varepsilon_{d(x,y)}$ then $(X, \varepsilon_{d(.,.)})$ is an FSM-space and $\mathbf{K}(x, y) = \min(d(x, y), 1)$. (ii) If $X = X(\Omega, \mathcal{K}, P)$ and $\mathcal{F}(x, y)$ is the distribution function of $d(x, y)$ (see Example 1.1.4), then $\mathbf{K}$ is the Ky Fan metric for the convergence in probability.

Generally, **K need not to be a metric**. Notice that in order to ensure the triangle inequality for **K**, T. L. Hicks identified the following condition[2] for a probabilistic metric space $(X, \mathcal{F})$:

$$[F_{xy}(t) > 1 - t, F_{yz}(s) > 1 - s] \Rightarrow F_{xz}(t + s) > 1 - (t + s) \tag{III^1}$$

and observed that III^1 holds for every Menger space $(X, \mathcal{F}, T)$ with $T \geq T_L$ (see [22]). Actually, one has the following

Proposition 2.1.3. *Let T be a t-norm such that the property (III^1) holds for every fuzzy Menger space $(X, \mathcal{F}, T)$. Then $T \geq T_L$.*

The **proof** follows from the next example: Let $X = \{x, y, z\}, F_{xy} = F_{yx}, F_{yz} = F_{zy}, F_{xz} = F_{zx}$, where $F_{xy}(t) = \begin{cases} 0 & t \leq 0 \\ a & t > 0 \end{cases}$, $F_{yz}(t) = \begin{cases} 0 & t \leq 0 \\ b & t > 0 \end{cases}$, $F_{zx}(t) = \begin{cases} 0 & t \leq 0 \\ T(a, b) & t > 0 \end{cases}$, and $F_{xx} = F_{yy} = F_{zz} = \varepsilon_0$. Then $(X, \mathcal{F}, T)$ is a Menger

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[2] Recall that in [23] was proposed the following triangle inequality for a probabilistic metric (structure):

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s. t. } [F_{xy}(\delta) > 1 - \delta, F_{yz}(\delta) > 1 - \delta] \Rightarrow F_{xz}(\varepsilon) > 1 - \varepsilon; \tag{III^H}$$

and we generalized it by using additive generators (see, e.g., [55], [59], [60]):

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{ such that } [f \circ F_{xy}(\delta) < \delta, f \circ F_{yz}(\delta) < \delta] \Rightarrow f \circ F_{xz}(\varepsilon) < \varepsilon. \tag{III^f}$$

space and $\mathbf{K}(x,y)=1-a,\ \mathbf{K}(y,z)=1-b$, while $\mathbf{K}(x,z)=1-T(a,b)$. Thus we see that $\mathbf{K}(x,z)\le\mathbf{K}(x,y)+\mathbf{K}(y,z)\Leftrightarrow T(a,b)\ge a+b-1$.

Remark 2.1.4. Let $(X,\mathcal{F},T)$ be as in the above example and suppose that $T(a,b)<a+b-1$. Therefore $0<a,b<1$ and there exists $p>1$ such that $((1-a)^{\frac{1}{p}}+(1-b)^{\frac{1}{p}})^p>1-T(a,b)$. Thus $(1-a)^{\frac{1}{p}}+(1-b)^{\frac{1}{p}}>(1-T(a,b))^{\frac{1}{p}}$ and we see that $\mathbf{K}_p$, given by $\mathbf{K}_p(u,v)=\sup\{t\mid t^p\le 1-F_{uv}(t)\}$, is verifying the triangle inequality. This shows that the more general formulae proposed in [55, 59] can give metrics in many situations:

Let us consider the family M of all functions $\mu:[0,\infty]\longrightarrow[0,\infty]$ with the properties (M_0) $\mu(t)=0\Leftrightarrow t=0$, (M_1) μ is continuous and (M_2) $\mu(t+s)\ge\mu(t)+\mu(s),\ \forall t,s\ge 0$. Notice that for each $\mu\in M$ there exists $t_\mu>0$ such that $\mu:[0,t_\mu)\longrightarrow[0,\infty)$ is strictly increasing and invertible. If we set

$$\mathbf{K}^\mu(x,y)=\sup\{t\mid t\ge 0,\ \mu(t)\le 1-F_{xy}(t)\}, \tag{1^μ}$$

then $\mathbf{K}^\mu$ **is a semi-metric**, and

$$\mathbf{K}^\mu(x,y)<\delta\Leftrightarrow F_{xy}(\delta)>1-\mu(\delta), \tag{2^μ}$$

so that $\mathbf{K}^\mu$ generates $\mathcal{T}_F$ and $\mathcal{U}_F$. This motivates the following slight generalization of (III^1).

Definition 2.1.5. *A FSM-space $(X,\mathcal{F})$ is called an* **FM-space of type M** *if and only if*

$$[F_{xy}(t)>1-\mu(t),F_{yz}(s)>1-\mu(s)]\Rightarrow F_{xz}(t+s)>1-\mu(t+s). \tag{III^μ}$$

Remark 2.1.6. The triangle inequality (III^μ) can be useful and appropriate in many cases. For example, if $(X,\mathcal{F})$ verifies (III^1) - that is (III^μ) for $\mu(t)=t$ - then $\widetilde{\mathcal{F}}$, defined by $F_{xy}\circ\mu$, is a fuzzy semi-metric and $\widetilde{F}_{xy}(\delta)>1-\mu(\delta)\Leftrightarrow F_{xy}(\mu(\delta))>1-\mu(\delta)$. The formula (1^μ) leads to $\mathbf{K}^\mu(x,y)=\mu^{-1}(\mathbf{K}(x,y))$. For spaces of random variables (see Example 2.1.2 (ii)), $\widetilde{F}_{xy}(t)=P(|x-y|<\mu(t))=P(\mu^{-1}(|x-y|)<t)$ and $\mu^{-1}\circ\mathbf{K}$ is a metric for the convergence in probability. Generally, one has

Theorem 2.1.7. *Let $(X,\mathcal{F})$ be a FM-space of type M, for which the triangle inequality (III^μ) holds. Then $\boldsymbol{K}^\mu$, defined by (1^μ), is a metric on X which generates $\mathcal{T}_F$ and $\mathcal{U}_F$.*

Corollary 2.1.8. *Let T be a t-norm such that $T(a,b) \geq T^{\mu}(a,b) := \max\{1-\mu[\mu^{-1}(1-a)+\mu^{-1}(1-b)],0\}$. Then $\boldsymbol{K}^{\mu}$ is a metric for every fuzzy Menger space $(X,\mathcal{F},T)$.*

Remark 2.1.9. Since μ is super-additive, then $\mu(\mu^{-1}(\alpha)+\mu^{-1}(\beta)) \geq \alpha+\beta$ and $T^{\mu}(a,b) \leq \max(a+b-1,0) = T_L(a,b)$. Therefore our results are slightly more general than their counterparts from [22].

T. L. Hicks introduced in [21] the following contraction condition on PM-spaces,

$$t>0, F_{pq}(t) > 1-t \Rightarrow F_{ApAq}(Lt) > 1-Lt \quad [L \in (0,1)] \qquad (C)$$

and proved that every such an A on a complete Menger space $(X,\mathcal{F},Min)$ has a unique fixed point. In [55, 58] we proved the above result **for every t-norm for which** $\sup_{a<1} T(a,a) = 1$ and the proof was given directly by the method of successive approximations. We also observed in [55, 58] that the idea of Hicks can be applied for a larger class of t-norms, namely for $T \geq T_m = Max(Sum-1,0)$. This idea was used in [4] for ϕ-probabilistic contractions:

$$t>0, F_{pq}(t) > 1-t \Rightarrow F_{ApAq}(\phi(t)) > 1-\phi(t) \qquad (C_{\phi})$$

where $\phi \in \Phi$, the family of all functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ for which (Φ_0) ϕ is strictly increasing, (Φ_1) ϕ is right continuous and (Φ_2) $\lim\limits_{n\to\infty} \phi^n(t) = 0$ $\forall t \geq 0$. By using the idea from [48], we can extend all these results to FSM-spaces:

Definition 2.1.10. *Let $(X,\mathcal{F})$ be a FSM-space. We say that $A : X \to X$ is a $\Phi - M$* **contraction** *if there exist $\phi \in \Phi$ and $\mu \in M$ such that*

$$F_{xy}(t) > 1-\mu(t) \Rightarrow F_{AxAy}(\phi(t)) > 1-\mu\circ\phi(t) \qquad (\phi\mu - C)$$

For a concrete pair (ϕ,μ) we use the term $\phi-\mu$ **contraction**.

Example 2.1.11. Let A be a contraction of Hicks type, consider the fuzzy semi-metric $\widetilde{\mathcal{F}}$ defined by $\widetilde{F}_{xy} = F_{xy}\circ\mu$ and suppose that $t_{\mu} = \infty$. Then $\widetilde{\phi} := \mu^{-1}\circ\phi\circ\mu \in \Phi$ and A verifies $(\widetilde{\phi}\mu - c)$ **for every** μ.

By using $\mathbf{K}^{\mu}$, one can prove the following two fixed point theorems, slightly extending results from [48] and [22].

Theorem 2.1.12. *Let $(X,\mathcal{F})$ be a complete FM-space of type M, for which the triangle inequality (III^{μ}) holds. Then every $\phi-\mu$ contraction has a unique fixed point which can be obtained by successive approximations.*

Theorem 2.1.13. *Let $(X, \mathcal{F}, T)$ be a complete Menger space, for which $T \geq T^{\mu}$. Then every $\phi - \mu$ contraction on X has a unique fixed point.*

Remark 2.1.14. Notice that $T^{\mu}(a, b) \leq \max(a + b - 1, 0) = T_L(a, b)$.
If we set $f(s) = 1 - s$, then (C_{φ}) can be slightly generalized as

$$f \circ F_{pq}(t) < t \Rightarrow f \circ F_{ApAq}(\varphi(t)) < \varphi(t) \qquad (C_{\varphi}^{f})$$

and we can extend our result from [59]:

Theorem 2.1.15. *Let $(X, \mathcal{F}, T)$ be a complete (fuzzy) Menger space such that $T \geq T_f$ (and $f(0) < \infty$). Then every mapping $A : X \to X$ which satisfies, for some $m \in M$* and $\varphi \in \Phi$, the following condition

$$f \circ F_{pq}(t) < m(t) \Rightarrow f \circ F_{ApAq}(\varphi(t)) < m(\varphi(t)), \qquad (C_{\varphi,m}^{f})$$

has a unique fixed point which is the limit of successive approximations.
The **proof** uses the fact that

$$d(p, q) = \mathbf{K}_f^m(p, q) := sup\{t, m(t) \leq f \circ F_{pq}(t)\},$$

gives a complete metric on X (see Theorem 4.2.A below). Moreover, $d(Ap, Aq) \leq \varphi(d(p, q))$, that is A is a (classical) φ-contraction.

We can also extend to fuzzy Menger spaces an idea from [20] and [64]. Let $(X, \mathcal{F})$ be a given FSM-space and $A : X \to X$ a fixed mapping.

Definition 2.1.16. *A is called a **generalized C-contraction of Krasnoselski type** if for each pair of real numbers (a, b), with $0 < a < b$, there exists $L_{ab} \in (0, 1)$ such that the following implication holds:*

$$a \leq 1 - F_{pq}(a) \ \& \ 1 - F_{pq}(b+) \leq b \ \& \ F_{pq}(t) > 1 - t \Rightarrow F_{AxAy}(L_{ab}t) > 1 - L_{ab}t \qquad (C_{ab})$$

Theorem 2.1.17. *Every generalized C-contraction on a complete fuzzy Menger space $(X, \mathcal{F}, T)$, where $T \geq T_L$, has a unique fixed point, which is globally attractive.*

Proof. As in [20], one shows that A is a *Krasnoselski contraction* [33] in the *complete metric space* $(X, \mathbf{K})$:

$$\mathbf{K}(Ap, Aq) \leq L_{ab}\mathbf{K}(p, q), \text{ if } \mathbf{K}(p, q) \in [a, b].$$

2.2. **Applications to weak Hicks C-contractions.** We will emphasize two recent results of D. Miheţ, also relaxing the contraction condition of Hicks type, whose proofs can be obtained by using the Ky Fan type **(semi)metric**.

Definition 2.2.1. Let $(X, \mathcal{F})$ be a FSM-space. A self-mapping A of X is called a *weak Hicks-contraction* iff there exists $L \in (0,1)$ such that for all $p, q \in X$ the following implication holds

$$(w-H) \;\; t \in (0,1), \; F_{pq}(t) > 1-t \Rightarrow F_{ApAq}(Lt) > 1-Lt;$$

Theorem 2.2.2([40]). *Let $(X, \mathcal{F}, T)$ be a complete fuzzy Menger space with $T \geq T_L$ and let us suppose that $A : X \to X$ is a weak Hicks-contraction with the property that $F_{qAq}(t) > 0$ for some $q \in X$ and some $t \in (0,1)$. Then A has a fixed point.*

Theorem 2.2.3. ([42]). *Let T be a t-norm with $sup_{a<1} T(a,a) = 1$ and $A : X \to X$ be a weak Hicks-contraction in a complete fuzzy Menger space $(X, \mathcal{F}, T)$. Then A has a fixed point iff there exists $x \in X$ such that $D_{O(A,x)}(1) > 0$.*

The **proof** follows easily by using the **semimetric** of Ky Fan. Firstly, as we know,

a) $\mathbf{K}(p,q) < \eta \Leftrightarrow F_{pq}(\eta) > 1-\eta, \forall \eta < 1$;

Moreover,

b) $\mathbf{K}(Ap, Aq) \leq \mathbf{K}(p,q) \leq 1, \forall p, q \in X$, and A is uniformly continuous;

c) $\mathbf{K}(p,q) < 1 \Rightarrow \mathbf{K}(Ap, Aq) \leq L\mathbf{K}(p,q)$.

We have to emphasize that $\mathbf{K}$ does not necessarily satisfy the triangle inequality, so that it is *a semimetric* only. But we can see that

$$D_{O(A,x)}(1) > 0 \Rightarrow \exists \delta < 1, D_{O(A,x)}(\delta) > 1-\delta \Rightarrow \mathbf{K}(p,q) < \delta, \forall p, q \in O(A,x).$$

Therefore A is **K-strictly contractive on the bounded set** $O(A,x)$. Actually, as it is easily seen,

$$\mathbf{K}(A^n(x), A^{n+s}(x)) \leq L^n \mathbf{K}(x, A^s(x)) \leq L^n \delta.$$

Therefore (see the equivalence a) above) $(A^n(x))_{n \geq 0}$ is an $\mathcal{F}-$Cauchy sequence, so convergent etc.

Example 2.2.4. Let us consider the discrete Menger space X under T_M determined by the mapping $(x,y) \to \varepsilon_1, \forall x \neq y$ on a set containing at least two elements. Clearly, **any mapping** $A : X \to X$ is a weak Hicks-contraction and the constant mappings are (the only) Hicks C-contractions. One can construct different kinds (having in mind the fixed point set) of weak Hicks-contraction mappings (It is clear that only the eventually constant sequences are Cauchy/convergent).

3. Extended metrics and Sehgal-type contractions

V. M. Sehgal introduced a natural class of probabilistic *B-contractions* on Menger spaces, as mappings A for which there exists $L \in (0,1)$ such that

$$F_{ApAq}(Lt) \geq F_{pq}(t),\ \forall p,q,t, \tag{BC_L}$$

and proved a partial analogue of the contraction principle (see [74]). H. Sherwood proved that a B-contraction on a given Menger space **either** has a globally attractive fixed point **or** has no fixed points, and constructed *fixed point free* probabilistic B-contractions on complete Menger spaces (see [76] or Theorem 3.2.2.1 below). In the case of fuzzy Menger spaces the situation is much more complex.

3.1. **Fundamental examples II.** As we have seen, $X = [0, \infty)$ is a complete fuzzy Menger space under the *Archimedean* triangular norm T_P, if one sets $F_{xx} := \varepsilon_0$ and $F_{xy}(t) := \frac{\min(x,y)}{\max(x,y)}, \forall t \in (0,\infty), \forall x,y \geq 0, x \neq y$.

1. If $Ax := x^{1-signx}$, then A is a B-contraction with **two fixed points**. Clearly $A^n 0 \to 0 = A0$, and $A^n x \to 1 = A1$, for each $x > 0$.

2. The mapping $x \to Ax := Lx$, where $L \in (0,1)$, is a B-contraction with the **unique fixed point** 0 and $A^n x \to 0$ **only** for $x = 0$.

3. $X_0 = (0,\infty)$ is also a complete fuzzy Menger space under the t-norm T_P. Clearly $x \to Ax := Lx$, where $L \in (0,1)$, is a B-contraction and **has no fixed point**. It is worth noting that $F_{pAp}(t) = L,\ \forall t > 0,\ \forall p \in X_0,\ \mathbf{K}(x,y) = \frac{|x-y|}{\max(x,y)}$ and $\mathbf{K}(Ax,Ay) = \mathbf{K}(x,y), \forall x,y \in (0,\infty)$. Thus the probabilistic B-contraction $x \to Lx$ is a $\mathbf{K}$-isometry and has no fixed point.

3.2. **Methods of type Maia.**

3.2.1. *A proof for the theorem of Hadžić-Istrăţescu.* Let $\mathcal{F} : X \times X \to \Delta^+$ be a fuzzy metric, such that $(S, \mathcal{F}, T_M)$ is a fuzzy Menger space, and consider an arbitrary fixed element F of D^+.

Lemma 3.2.1.1. *The function d_F, defined on $X \times X$ by*

$$d_F(p,q) = inf\{a > 0, F_{pq}(at) \geq F(t)\ ,\ \forall t \in R\},$$

is an extended metric on X (as usually, $\inf \emptyset = +\infty$*). Moreover, the d_{F}– topology is stronger than the $\mathcal{F}$-topology and every d_F-Cauchy sequence is an $\mathcal{F}$-Cauchy sequence.*

Lemma 3.2.1.2. *If $A : X \to X$ is a B-contraction on the fuzzy Menger space $(X, \mathcal{F}, T_M)$, then A is a strict contraction on (X, d_F).*

Theorem 3.2.1.3. *Let $(X, \mathcal{F}, T_M)$ be a complete fuzzy Menger space and suppose that $A : X \to X$ is a probabilistic B-contraction. Then A has a fixed point iff there exists some $p \in X$ such that $F_{pAp} \in D^+$.*
Proof. Choose $F = F_{pAp} \in D^+$ and apply the above lemmas.

Example 3.2.1.4. For the mapping from Example 3.1.1,

$$d_F(A^n p, A^{n+1} p) = \infty, \forall p \in X_0, \forall F \in D^+.$$

Remark 3.2.1.5. One can prove that the above result remains true in fuzzy Menger spaces under t-norms of Hadžić type[3] and generalized B-contractions of type Krasnoselski (see Theorem 2.9 from [20], for the case of Menger spaces).

3.2.2. *Other types of B-contractions in fuzzy Menger spaces.* The family of h-t-norms is the largest class of continuous t-norms with the property that the contraction principle holds for any complete Menger space $(X, \mathcal{F}, T)$ and any Sehgal B-contraction ([51, 52, 58, 61, 64, 66]). Consequently, for other t-norms one has to impose additional conditions **either** on the probabilistic contraction, **or** on the probabilistic metric (space). Results in this direction are largely presented in [18], Chapter 3.

[3] For example, in the continuous case, using a sequence b_n which is strictly increasing to 1 and such that $T(b_n, b_n) = b_n$, it easy to verify that $r_n(x,y) := \inf\{t, F_{xy}(t) \geq b_n\}$ defines a countable family of ecarts which generates the $\mathcal{F}$-uniformity. If we suppose that $r_n(x,y) < \varepsilon$, then $F_{xy}(\varepsilon) \geq b_n$ and, by the contraction condition (BC_L), we see that $F_{AxAy}(L\varepsilon) \geq b_n$, which says that $r_n(Ax, Ay) \leq L\varepsilon$. Therefore $r_n(Ax, Ay) \leq L r_n(x,y), \forall x, y \in X, \forall n$. From the hypothesis $F_{xAx} \in D^+$ we can easily see that for each n there exists $c_n < \infty$ such that $r_n(x, Ax) \leq c_n$. Now we can apply either Monna's theorem ([43], Théorème 1) or the direct method as in [2] and obtain the existence of a fixed point.

In [63], we introduced the notion of **B-contraction of type** r (a parameter in $[0,1]$), by the following condition:

$$(PC_{rL}) \qquad F_{AxAy}(L^r t) \geq \frac{F_{xy}(t)}{F_{xy}(t) + L^{1-r}(1 - F_{xy}(t))}, \ \forall t > 0, \ \forall x, y \in X,$$

for some $L \in (0,1)$. Notice that for $r = 1$ we obtain B-contractions of type Sehgal on Menger spaces and fuzzy Menger spaces. Since in our definition $F_{xy}(t)$ *is not necessarily* $(0,1)$*-valued,* the case $r = 0$ includes properly the notion of **fuzzy contraction** on strong fuzzy metric spaces, as defined in [10]:

$$(FC_L) \qquad \frac{1}{F_{AxAy}(t)} - 1 \leq L(\frac{1}{F_{xy}(t)} - 1), \ \forall x, y \in X, \forall t > 0.$$

Clearly, every B-contraction of type r having the Lipschitz constant L is a B-contraction of type 1 with the Lipschitz constant L^r.

In [65] (see also [66, 67]), we considered the class of **strict B-contractions,** having the property that, for a $\lambda \in (0,1)$,

$$F_{AxAy}(\lambda t) \geq \frac{F_{xy}(t)}{F_{xy}(t) + \lambda(1 - F_{xy}(t))}, \forall x, y \in X, \forall t > 0. \qquad (PSC_\lambda)$$

Actually, the following results hold.

Theorem 3.2.2.1. *Let* A *be a B-contraction of type* r *on a complete fuzzy Menger space* $(X, \mathcal{F}, T)$.

1. *Suppose that* $\sup_{a<1} T(a,a) = 1$. *Then*

(1.i) *If* $r \in (0,1]$, A *has a fixed point iff* ***there exists*** $\mathbf{x} \in \mathbf{X}$ *such that* $\lim_{t\to\infty} \inf_{p\geq 0} F_{xA^px} > 0$.

(1.ii) *If* $r = 0$, A *has a fixed point iff* **there exists** $\mathbf{x} \in \mathbf{X}$ *such that* $\inf_{p\geq 0} F_{xA^px}(t) > 0, \forall t > 0$.

2. *Suppose that* $T \geq T_L$. *Then a B-contraction of type* $r \in (0,1)$ *has a fixed point iff* $F_{xAx}(t) > 0$ for **some** $x \in X$ **and some** $t > 0$.

3. *Let* T *be of Hadžić-type and* $r > 0$. *Then* A *has a fixed point iff* $F_{xAx} \in D^+$ *for* **some** $x \in X$.

The proofs have essentially used the following metric-like mappings $\mathbf{K}_g$ on a fuzzy semi-metric space $(X, \mathcal{F})$: $\mathbf{K}_g(x,y) = \sup\{t | t \leq g \circ F_{xy}(t)\}$. Since $\mathbf{K}_g(x,y) < \delta \Leftrightarrow F_{xy}(\delta) > g^{-1}(\delta\ ), \forall \delta \in (0, g(0))$, for every fuzzy semimetric $\mathcal{F}$ and every additive generator g, then

a) $\mathbf{K}_g$*generates the* $\mathcal{F}$*-topology and the semiuniformity* $\mathcal{U}_{\mathcal{F}}$*, that is* $\mathbf{K}_g$ *is an extended* **semimetric***;*

b) X is $\mathcal{F}-$complete iff it is $\mathbf{K}_g-$complete;

c) $\mathbf{K}_g$ is an extended **metric** *on every fuzzy Menger space under a t-norm $T \geq T_g$.*

d) $\mathbf{K}_g(x, y) = \infty$ *iff* $g(0) = \infty$ *and* $F_{xy}(t) = 0$ *for every* t. *If* $g(0) = 1$, $\mathbf{K}_g$ *is a* $[0, 1]-$*valued semimetric.*

Remark 3.2.2.2. As we have already seen (see Fundamental examples II), by setting

$$F_{xy}(t) = \frac{\min(x, y)}{\max(x, y)}, \forall t \in (0, \infty), \forall x, y \geq 0, x \neq y,$$

$X = [0, \infty)$ becomes a complete fuzzy Menger space under the triangular norm T_P. Simple calculations show that

(i) $\mathbf{K}(x, y) = \frac{|x-y|}{\max\{x,y\}}$ and$\mathbf{K}_{g_P}(x, y) = \log \frac{\max\{x,y\}}{\min\{x,y\}}$ give (extended) metrics.

(ii) For $g_1(s) = \frac{1}{s} - 1$ we obtain $\mathbf{K}_{g_1}(x, y) = \frac{|x-y|}{\min\{x,y\}}$, which gives only an extended semimetric, and

$$\mathbf{K}(x, y) \leq \mathbf{K}_{g_P}(x, y) \leq \mathbf{K}_{g_1}(x, y).$$

(iii) For the mapping $Ax := ax$, where $a > 0$ is given, $\mathbf{K}\left(Ax, Ay\right) = \mathbf{K}\left(x, y\right), \forall x, y \in (0, \infty)$. Hence $x \rightarrow Ax$ is a deterministic isometry as well as a B-contraction of type 1 without fixed points.

(iv) For $Ax := a + x$, we have $\mathbf{K}\left(Ax, Ay\right) = \frac{|x-y|}{\max(a+x,a+y)} \leq \mathbf{K}\left(x, y\right)$, and $lim_{y \to \infty} \frac{\mathbf{K}(Ax,Ay)}{\mathbf{K}(x,y)} = 1$.

Remark 3.2.2.3. In [66], by remarking that the condition (PSC_λ) can be rewritten as $g_1 \circ F_{AxAy}(\lambda t) \leq \lambda g_1 \circ F_{xy}(t)$, where $g_1(s) := \frac{1}{s} - 1$ generates the (strict) t-norm given, for $a + b > 0$, by $T_1(a, b) := \frac{ab}{a+b-ab} \geq T_P(ab) \geq T_L(a, b)$, we have proven the following more general fixed point result by the method of Maia [34] applied to $\mathbf{K}_g$ and $\mathbf{K}_h$:

Theorem 3.2.2.4. *Suppose the next conditions (i)-(iii) are verified for a selfmapping A of a complete fuzzy Menger space $(X, \mathcal{F}, T)$ and the additive generators g, h:*

(i) $g\left(a\right) \geq h\left(a\right)$ and $T(a, b) \geq T_h(a, b), \forall a, b \in \left[0, 1\right];$

(ii) *A is a* **strict g-contraction,** *that is, for some fixed $\lambda \in \left(0, 1\right)$,*

$$F_{AxAy}(\lambda t) \geq g^{(-1)} \circ (\lambda g) \circ F_{xy}(t), \forall t > 0, \forall x, y \in X; \qquad (PSC_\lambda^g)$$

(iii) $F_{zAz}(u) > 0$ *for* **some** $z \in X$ *and* **some** $u > 0$.

Then A has a fixed point.

3.3. **Methods of type Luxemburg-Margoliz.**

3.3.1. *Power-exponential weights and fixed points in FM-spaces.* By using the following nonnegative functions, we can measure the distance between the maximal element ε_0 and the elements of Δ^+. Let k be a (fixed) positive real number.

Lemma 3.3.1.1. *The mapping* $\delta_k : \Delta_+ \to \mathbf{R}_+$, *given by*

$$\delta_k(F) := \sup_{x>0}\{x^k[1 - F(x)]e^{-x}\}, \tag{3.3.1.1}$$

has the following properties:

(i) $\delta_k(F) = 0 \Leftrightarrow F = \varepsilon_0$;

(ii) If $F_1 \leq F_2$, then $\delta_k(F_1) \geq \delta_k(F_2)$;

(iii) $\delta_k(\lambda \circ F) \leq \lambda^k \delta_k(F), \forall \lambda \geq 1$;

Let $\beta = \mathbf{K}(F) = \sup\{t \mid t \leq 1 - F(t)\}$ *define the écart of Lévy-Ky Fan. Then*

(iv) $\beta^{k+1}e^{-\beta} \leq \delta_k(F) \leq \max\{\beta^k, \beta k^k e^{-k}\}$;

(v) $\delta_k(F_n) \to 0 \Leftrightarrow F_n(x) \to 1$, *for each* $x > 0$.

Theorem 3.3.1.2. *Let* $(X, \mathcal{F})$ *be a fuzzy metric space and*

$$e_k(p, q) := \delta_k(F_{pq}) = \sup_{x>0} x^k[1 - F_{pq}(x)]e^{-x}, \forall p, q \in X \tag{3.3.1.2}$$

Then

1^0 e_k *is* **a semi-metric** *that generates the semi-uniformity* $\mathcal{U}_F$;

2^0 *If* $(X, \mathcal{F}, T_L)$ *is a fuzzy Menger space, then*

$$(p, q) \to \theta_k(p, q) := \{e_k(p, q)\}^{\frac{1}{k+1}} \tag{3.3.1.3}$$

gives **a metric** *on X. Moreover,* $(X, \mathcal{F})$ *is complete if and only if* (X, θ_k) *is complete.*

Proof. 1^0 follows from Lemma 3.3.1.1. and the definitions. For 2^0, notice that $(X, \mathcal{F}, T_L)$ is a fuzzy Menger space iff the following inequality holds:

$$1 - F_{pq}(x) \leq 1 - F_{pr}(tx) + 1 - F_{rq}[(1 - t)x], \forall p, q, r \in X, \forall x \in \mathbf{R}, \forall t \in [0, 1]. \tag{3.3.1.4}$$

Therefore, for each $x > 0$,

$$x^k[1 - F_{pq}(x)]e^{-x} \leq \frac{1}{t^k}e_k(p, r) + \frac{1}{(1 - t)^k}e_k(r, q), \forall t \in (0, 1).$$

This implies the inequality

$$e_k(p,q) \leq \frac{1}{t^k} e_k(p,r) + \frac{1}{(1-t)^k} e_k(r,q), \forall t \in (0,1)$$

and we easily see that

$$\{e_k(p,q)\}^{\frac{1}{k+1}} \leq \{e_k(p,r)\}^{\frac{1}{k+1}} + \{e_k(r,q)\}^{\frac{1}{k+1}}$$

that is θ_k verifies the triangle inequality. The last part follows from the inequality (iv) of Lemma 3.3.1.1.

Remark 3.3.1.3. The above proof shows that in the case $(X, \mathcal{F}, T_L)$ is nonArchimedean, e_k itself **is a metric**, generating the $\mathcal{F}$ - uniformity.

θ_k can be successfully used to prove a rather general fixed point result:

Theorem 3.3.1.4. *Let* $(X, \mathcal{F}, T)$ *be a complete fuzzy Menger space such that* $T \geq T_L$. *If* $A : X \to X$ *is a B-contraction, that is*

$$F_{ApAq}(x) \geq F_{pq}(\frac{x}{L}), \forall x \in \mathbf{R} \tag{3.3.1.5}$$

for some $L \in (0,1)$ *and all pairs* $(p,q) \in X \times X$, *then* **the following are equivalent**

(3.3.1.6.1) *A has a fixed point*

(3.3.1.6.2) *There exist* $p \in X$ *and* $k \in (0, \infty)$ *such that*

$$E_k(p) := \sup_{x>0}\{x^k[1 - F_{pAp}(x)]\} < \infty \tag{3.3.1.6}$$

Proof. The implication (3.3.1.6.1) $\Rightarrow$ (3.3.1.6.2) is obvious:

$$p = Ap \ \Rightarrow \ F_{pAp}(x) = 1, \ \forall x > 0 \Rightarrow \ E_k(p) = 0.$$

If $E_k(p) < \infty$ for some $p \in X$ and $k \in (0, \infty)$, then we see that $\delta_k(F_{pAp}) \leq E_k(p)$. Using the inequality (3.3.1.5) we get

$$\begin{aligned} x^k[1 - F_{ApA^2p}(x)]e^{-x} &\leq x^k[1 - F_{pAp}(\frac{x}{L})]e^{-x} \\ &= L^k\{(\frac{x}{L})^k[1 - F_{pAp}(\frac{x}{L})]\}e^{-x} \leq L^k E_k(p). \end{aligned}$$

This shows that

$$\theta(Ap, A^2p) \leq L^{\frac{k}{k+1}}(E_k(p))^{\frac{1}{k+1}} \tag{3.3.1.7}$$

By applying (3.3.1.7) for A^n, which verifies (3.3.1.5) with the Lipschitz constant L^n, we obtain

$$\sum_{n=0}^{\infty} \theta_k(A^n p, A^{n+1}p) \leq \{\sum_{n=0}^{\infty}(L^{\frac{k}{k+1}})^n\}\{E_k(p)\}^{\frac{1}{k+1}} < \infty. \tag{3.3.1.8}$$

Therefore $(A^n p)_{n\geq 0}$ is a Cauchy sequence in the complete metric space (X, θ_k), thus it converges to some point $p_* \in X$. Since (5) implies also the continuity of A, then p_* is a fixed point.

Remark 3.3.1.5. **a)** Generally, A is not contractive relatively to θ_k or e_k.

b) The supremum in (3.3.1.6) may be infinite for certain values of k or for different points in X.

c) Clearly, the condition (3.3.1.6) is verified whenever $F_{pAp}(t_p) = 1$ for some $t_p > 0$.

3.3.2. *Metrics of mean type and a fixed point principle.* The concrete formulae proposed by us in [62, 64] can also be applied to fuzzy metrics:

Lemma 3.3.2.1. 1. *The function* ρ_0, *defined by*

$$\rho_0(p,q) = \int_0^1 (1 - F_{pq}(t))dt, \quad \forall\ (p,q) \in X \times X, \tag{3.3.2.1}$$

is a semi-metric on X, which generates $\mathcal{U}_{\mathcal{F}}$. *Moreover,*

$$\mathbf{K}^2 \leq \rho_0 \leq 2\mathbf{K} - \mathbf{K}^2. \tag{3.3.2.2}$$

2. *If* $(X, \mathcal{F}, T)$ *is a fuzzy Menger space and* $T \geq T_L$, *then the mapping defined by*

$$R_0(p,q) = \left\{ \int_0^1 [1 - F_{pq}(t)]\, dt \right\}^{\frac{1}{2}}, \qquad \forall p,q \in X \times X \tag{3.3.2.3}$$

is ***a metric*** *generating the strong uniformity* $\mathcal{U}_{\mathcal{F}}$. *Moreover,* $\mathbf{K}(p,q) \leq R_0(p,q) \leq \sqrt{2\,\mathbf{K}(p,q)}$, $\forall p,q \in X$ *so that* $(X, \mathcal{F}, T)$ *is complete iff* (X, R_o)*is complete.*

Now, let λ be fixed in [0,1] and define

$$R_\lambda(p,q) := \left(\int_0^1 \frac{1 - F_{pq}(t)}{(1+t)^\lambda} dt \right)^{\frac{1}{2}}, \qquad \forall p,q \in X. \tag{3.3.2.3$_\lambda$}$$

Theorem 3.3.2.2. *Let* $(X, \mathcal{F}, T)$ *be a fuzzy Menger space with* $T \geq T_L$. *Then*

(i) R_λ *is a metric, for each* $\lambda \in [0,1]$.

(ii) $0 \leq \lambda < \mu \leq 1 \Rightarrow \frac{1}{\sqrt{2}} R_0(p,q) \leq R_1(p,q) \leq R_\mu(p,q) \leq R_\lambda(p,q) \leq R_0(p,q)$, $\forall p,q \in X$.

(iii) R_λ *generates the strong* $\mathcal{F}$*-uniformity on X.*

(iv) $(X, \mathcal{F}, T)$ *is complete iff* (X, R_λ) *is complete for some* $\lambda \in [0,1]$.

By using the above metric-like functions and the alternative of fixed point, one can obtain the following fixed point result.

Theorem 3.3.2.3. *Let A be a B-contraction on a complete fuzzy Menger space $(X, \mathcal{F}, T)$.*

(i) *For every $\lambda \in [0,1]$,*

$$R_\lambda(Ap, Aq) \leq L^{\frac{1-\lambda}{2}} \left\{ \int_0^{\frac{1}{L}} \frac{1 - F_{pq}(t)}{(1+x)^\lambda} dt \right\}^{\frac{1}{2}}.$$

(ii) *If the t-norm T is stronger than T_L, then the following statements are equivalent:*

1^0 *A has a fixed point.*

2^0 *There exist $p \in X$ and $\lambda \in [0,1)$ such that*

$$E_{pAp}^{(\lambda)} := \int_0^\infty \frac{1 - F_{pAp}(t)}{(1+t)^\lambda} dx < \infty.$$

Notice that, as in [62, 64], the above method can be applied to any Archimedean t-norm T.

4. General formulae for distances

In the sequel, we will make use of continuous operations S on $\overline{\mathbf{R}}_+ = [0, \infty]$ having, at least, the following properties: 1° $S(x,y) = S(y,x)$, 2° $S(x,y) \leq S(x,z)$ if $y \leq z$ and 3° $S(0,0) = 0$ and $S(x,y) \geq \max\{x,y\}$. The most important examples are the well-known operations S_p : $S_p(a,b) = (a^p + b^p)^{1/p}$, for $0 < p < \infty$, and $S_\infty(a,b) = Max(a,b)$. Other examples can be obtained using representation theorems for continuous Archimedean semigroups: There exists a **generator s** : $\overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ such that (a) $s(0) = 0$, (b) s is continuous, (c) s is strictly increasing and

$$S(a,b) = S_s(a, b := s^{(-1)}(s(a) + s(b))),$$

where $s^{(-1)}(b) := s^{-1}[Min(b, s(\infty))]$.

We say that $\lambda : [0,\infty] \to [0,\infty]$ is an element of $\Lambda(S)$ iff (Λ-0): $\lambda(t) = 0 \Leftrightarrow t = 0$, ($\Lambda$-l): λ is continuous and non-decreasing and (Λ-2): $\lambda \circ S(a,b) \leq \lambda(a) + \lambda(b)$ for all $a, b \in \overline{R}_+$. Notice that, for λ strictly increasing, the condition (Λ-2) is equivalent to the fact that S is weaker than S_λ. It is clear that

$\Lambda := \Lambda(S_1)$ is a family of sub-additive functions, a fact also true for every $S \geq S_1$.

Let $M(S)$ denote the family of all applications $\mu : [0, \infty] \to [0, \infty]$, possessing the following three properties: (M-0): $\mu(t) = 0 \Leftrightarrow t = 0$; $\mu(\infty) = \infty$, (M-1): μ is continuous and increasing, and (M-2): $S(\mu(a), \mu(b)) \leq \mu \circ S(a, b)$ for all $a, b \in \overline{\mathbf{R}}_+$. Notice that $\mu^{-1} \in \Lambda(S)$ if $\mu \in M(S)$ and $S \leq S_1$. Clearly $M(S_1) = M$.

4.1. **A general formula of type Fréchet.** J. F. C. Kingman introduced in [30] the first (deterministic) metric for Wald spaces. By means of so called T-conjugate transforms, the Kingman's formula was extended to Menger spaces under a continuous Archimedean t-norm T, in terms of its *multiplicative generators* (see [45] or [72], p. 131). Our idea from [49, 50], *to use additive generators*, can be successfully applied to fuzzy Menger spaces in a very general setting.

Lemma 4.1.1. *Let $\lambda \in \Lambda(S)$ and $\mu \in M(S)$ be fixed. For any $t \geq 0$ and $F \in \Delta^+$, set*

$$d_t(F) = \lambda(t) + f \circ F \circ \mu(t). \tag{4.1.1}$$

Then

(1) $d_t(\varepsilon_0) = \lambda(t)$ for all $t > 0$;

(2) $F = \varepsilon_0 \Leftrightarrow \inf\limits_{t \geq 0} d_t(F) = 0$;

(3) $d_{S(s,t)}(F) \leq d_s(G) + d_t(H)$, provided

$$F \circ (\mu(s), \mu(t)) \geq T_f(G(\mu(s)), H(\mu(t))).$$

Theorem 4.1.A. *Let $(X, \mathcal{F}, T)$ be a fuzzy S-Menger space with the t-norm stronger than T_f and consider the function $\rho = \mathbf{F}_f^{\lambda,\mu}$ defined by*

$$\mathbf{F}_f^{\lambda,\mu}(x, y) = \inf_{t>0}\{\lambda(t) + f \circ F_{xy} \circ \mu(t)\}, \forall x, y \in X. \tag{4.1.2}$$

Then $\mathbf{F}_f^{\lambda,\mu}$ is an extended metric which generates the uniformity $\mathcal{U}_{\mathcal{F}}$.

Proof. Using the triangle axiom with $T \geq T_f$ and Lemma 4.1.1. we obtain the inequalities $d_{S(s,t)}(F_{xz}) \leq d_s(F_{xy}) + d_t(F_{yz})$ and $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. For any $\delta < f(0)$,

$$\rho(x, y) < \delta \Leftrightarrow \exists\, t > 0 : \lambda(t) + f \circ F_{xy} \circ \mu(t) < \delta \Rightarrow F_{xy} \circ \mu(t) > f^{-1}(\delta)\ \&\ \lambda(t) < \delta.$$

Now, for any given $\varepsilon > 0$ and $p \in (0,1)$, choose $\delta > 0$ such that $f^{-1}(\delta) \geq 1-p$ and $\delta < \lambda \circ \mu^{-1}(\varepsilon)$. Therefore $\{(x,y) : \rho(x,y) < \delta\} \subset \{(x,y) : F_{xy}(\varepsilon) > 1-p\}$. On the other hand, by the continuity of λ and f, there exist $t > 0$ and $p \in (0,1)$ such that $f(1-p) + \lambda(t) < \delta$. If $\varepsilon = \mu(t)$, then $\{(x,y) : F_{xy}(\varepsilon) > 1-p\} \subset \{(x,y) : \rho(x,y) < \delta\}$.

4.1.1. *Some particular cases and applications.* **Corollary 4.1.B.** *Let f be an additive generator of $T_f \leq T$ and, for a given fuzzy Menger space $(X, \mathcal{F}, T)$, consider the mapping defined by*

$$\mathbf{F}_f(p,q) = \inf_{t>0}\{t + f \circ F_{pq}(t)\}, p, q \in X. \tag{4.1.3}$$

Then

(i) $\mathbf{F}_f$ *is an extended metric on X. $\boldsymbol{F}_f(p,q) < \infty$ if $F_{pq} \in D^+$ or T is nonstrict*;

(ii) *The uniformity generated by $\boldsymbol{F}_f$ is the $\mathcal{F}$-uniformity*;

(iii) *If a is a positive real number, then $\boldsymbol{F}_f^a$ defined by*

$$\mathbf{F}_f^a(p,q) = \inf_{t>0}\{at + f \circ F_{pq}(t)\} \tag{4.1.3a}$$

has the properties (i)-(ii);

(iv) *For each $a \in (0,1]$ one has*

$$a\boldsymbol{F}_f \leq \boldsymbol{F}_{af} \leq \boldsymbol{F}_f \tag{4.1.4}$$

and so all $\boldsymbol{F}_f$ are uniformly equivalent.

We only notice that $\mathbf{F}_f(p,q) = \inf_{t\geq 0}\{t + f \circ F_{pq}(t)\}$, that $g = \frac{1}{a}f$ is also an additive generator of T and, for $a \leq 1$, $a\mathbf{F}_f(p,q) = \inf_{t>0}\{at + af \circ F_{pq}(t)\} \leq \inf_{t>0}\{t + af \circ F_{pq}(t) = \mathbf{F}_{af}(p,q) \leq \inf_{t>0}\{t + f \circ F_{pq}(t)\} = \mathbf{F}_f(p,q)$; moreover, for every additive generator g of T there exists $a \in (0,1]$ such that $g = af$ or $f = ag$.

The case of T_L, with the additive generator $t \to 1-t$, gives the **Fréchet-type metric**:

Corollary 4.1.C. *If $(X, \mathcal{F}, T_L)$ is a fuzzy Menger space, then the function $\boldsymbol{F}$ defined by*

$$\boldsymbol{F}(p,q) = \inf_{t>0}\{t + 1 - F_{pq}(t)\} \tag{4.1.5}$$

is a metric on X. In particular, ***F*** *metrizes the topology of the convergence in probability for every E-space.*

As a matter of fact, this result can be used for every Archimedean t-norm:

Corollary 4.1.D. *Let $(X, \mathcal{F}, T)$ be a fuzzy Menger space under a continuous Archimedean t-norm T. Then there exists an* ***increasing bijection*** *$h : [0,1] \to [0,1]$ such that the function $\boldsymbol{F}_{Lh}$ is a metric on X:*

$$\mathbf{F}_{Lh}(p,q) = \inf_{t>0}\{t + 1 - h \circ F_{pq}(t)\}. \tag{4.1.6}$$

The **proof** is simple: To each continuous Archimedean t-norm T there corresponds an h as in the theorem such that $T(a,b) \geq h^{-1}(T_L(h(a), h(b)) = T_{Lh}(a,b)$. Thus $(X, h \circ \mathcal{F}, T_L)$ is a fuzzy Menger space and we can apply (4.1.5) for $h \circ \mathcal{F}$. Notice that T_{Lh} has the additive generator $f_{Lh} = f_L \circ h = 1 - h$.

Remark 4.1.2. On a fuzzy Menger space $(X, \mathcal{F}, Min)$, one obtains metrics for **every** f in (4.1.3) and for **every** increasing bijection h in (4.1.6). Applied to an arbitrary FM-space, (4.1.5) gives a symmetric positively defined mapping. Although it does not necessarily verify the triangle inequality, $\mathbf{F}$ generates the $\mathcal{F}-$uniformity on X. The same is true for $\mathbf{F}_f$ from (4.1.3). The above results, giving sufficient conditions for the corresponding functions to verify the triangle inequality, are best possible in the following sense; for each t-norm T' weaker than the Archimedean t-norm T there exists a Menger space $(X, \mathcal{F}, T')$ such that ρ_f given by (4.1.3) is a metric for **no additive generator** f of T: For $X = \{p, q, r\}$ of three distinct points, choose $a, b \in (0,1)$ such that $T'(a,b) < T(a,b)$ and set $F_{pr}(t) = \begin{cases} 0, t \leq a \\ 1, t > a \end{cases}$, $F_{rq}(t) = \begin{cases} 0, t \leq b \\ 1, t > b \end{cases}$ and $F_{pq}(t) = \begin{cases} 0, t \leq T'(a,b) \\ 1, t > T_1(a,b) \end{cases}$. Then $(X, \mathcal{F}, T')$ is a Menger space and $\rho_f(p,r) = f(a)$, $\rho_f(r,q) = f(b)$, $\rho_f(p,q) = f(T'(a,b))$.

Remark 4.1.3. Like in [50], [56], a family of extended metrics of Lévy type on the set of distance distribution functions can be obtained: Let T be a left-continuous t-norm and consider the corresponding t-function τ_T ([75],[72]). Then, as in [26], there exists a mapping $F_T : \Delta^+ \times \Delta^+ \to \Delta^+$,

$$F_T(F,G) = \sup\{H \in \Delta^+, \tau_T(F,H) \leq G, \tau_T(G,H) \leq F\}$$

such that (Δ^+, F_T, T) is a fuzzy Menger space. Then, using (4.1.3) for $T \geq T_f$, we obtain an extended metric on Δ^+, which is **a metric** if $f(0) < \infty$:

$$\mathbf{L}_f(F,G) = \inf_{t>0}\{t + f \circ F_T(F,G)(t)\}.$$

4.2. **A general formula of type Ky Fan.** We can also obtain a comprehensive class of extended metrics of type Ky Fan. For $\nu \in M := M(S_1),\ \mu \in M(S)$, and f an additive generator, let us consider the following formula:

$$\mathbf{K}_f^{\nu,\mu}(x,y) = sup\{t,\ t \geq 0,\ \nu(t) \leq f \circ F_{xy} \circ \mu(t)\}. \tag{4.2.1}$$

Theorem 4.2.A. *If $(X, \mathcal{F}, T)$ is a fuzzy S-Menger space, where $S \leq S_1$ and $T \geq T_f$, then $r = \boldsymbol{K}_f^{\nu,\mu}$ is an extended metric on X and $\mathcal{U}_r \equiv \mathcal{U}_{\mathcal{F}}$.*
Proof. From (4.2.1) it is clear, for $\nu(\varepsilon) < f(0)$, that

$$\mathbf{K}_f^{\nu,\mu}(x,y) < \varepsilon \Leftrightarrow F_{xy}(\mu(\varepsilon)) > f^{-1}(\nu(\varepsilon)), \tag{4.2.2}$$

which also implies the last affirmation. Suppose that $r(x,y) < \varepsilon$ and $r(y,z) < \delta$. Then we have $f \circ F_{xy} \circ \mu(\varepsilon) < v(\varepsilon), \quad f \circ F_{yz} \circ \mu(\delta) < v(\delta)$. Since $\nu \in M, \mu \in M(S)$ and $\mathcal{F}$ verifies the triangle inequality, one can write successively

$$f \circ F_{xz} \circ \mu \circ S_1(\varepsilon,\delta) \leq f \circ F_{xz} \circ \mu \circ S(\varepsilon,\delta) \leq f \circ F_{xz} \circ S(\mu(\varepsilon),\mu(\delta)) \leq$$

$$\leq f \circ F_{xz}(\mu(\varepsilon)) + f \circ F_{yz}(\mu(\delta)) < v(\varepsilon) + v(\delta) \leq v(S_1(\varepsilon,\delta)).$$

Hence $r(x,z) < S_1(\varepsilon,\delta) = \varepsilon + \delta$. Therefore $r(x,z) \leq r(x,y) + r(y,z)$.

Example 4.2.1. (1) In every (extended) metric space (X,d), considered as a (fuzzy) Menger space $(X, \mathcal{E}_d, Min)$, $\mathbf{K}_f^{\nu,\mu}(x,y) = min\{\mu^{-1}(d), \nu^{-1}(f(0))\}$ gives an equivalent (extended) metric.
(2) Applied to $X(\Omega, \mathcal{K}, P)$, (4.2.2) leads to a whole family of metrics for the convergence in probability:

$$\mathbf{K}^{\nu,\mu}(x,y) = \sup\{t : \nu(t) \leq P(|x-y| \geq \mu(t))\}.$$

References

[1] A. T. Bharucha-Reid, *Fixed point theorems in probabilistic analysis*, Bull. Amer. Math Soc., **82**(1976), 761-657.

[2] G. L. Jr. Cain, R. H. Kasriel, *Fixed and periodic points of local contractions on PM-spaces*, Math. Systems Theory, **9**(1975-76), 289-297.

[3] S. S. Chang, Y. J. Cho, S. M. Kang, *Nonlinear Operator Theory In PM-Spaces*, Nova Sc., 2001.

[4] Gh. Constantin, *On some classes of contraction mappings in Menger Spaces*, in: Seminarul de Teoria Probalităţilor şi Aplicaţii (STPA), Univ. Timişoara, Nr. 76 (1985)

[5] Gh. Constantin, I. Istrăţescu, *Elements of Probabilistic Analysis with Applications*, Kluwer, 1989.

[6] A. George, P. Veeramani, *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets and Systems, **90**(1997), 365-368.

[7] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, **27**(1988), 385-489.

[8] V. Gregori, S. Romaguera, *Some properties of FM-spaces*, Fuzzy Sets and Systems, **115**(2000), 485-489.

[9] V. Gregori, S. Romaguera, *On completion of FM-spaces*, Fuzzy Sets and Systems, **130**(2002), 399-404.

[10] V. Gregori, A. Sapena, *On fixed point theorems in FM-spaces*, Fuzzy Sets and Systems, **125**(2002), 245-253.

[11] O. Hadžić, *On the* (ε, λ)*-topology of probabilistic locally convex spaces*, Glasnik Mat., **13**(1978), 61-66.

[12] O. Hadžić, M. Budinčević, *A class of t-norms in the fixed point theory on PM-spaces*, Review of Research, Faculty of Science, Univ. Novi Sad, **9**(1979), 37-41.

[13] O. Hadžić, *A generalization of the contraction principle in PM spaces*, Review of Research, Faculty of Science, Univ. Novi Sad, **10**(1980), 13-21.

[14] O. Hadžić, *Fixed point theory in topological vector spaces,* Novi Sad, 1984.

[15] O. Hadžić, *Some fixed point theorems in PM-spaces*, Review of Research, Faculty of Science, Univ. Novi Sad, **15**(1985), 23-35.

[16] O. Hadžić, *On* (n, f, g) *- locally contractions in PM-spaces*, Review of Research, Faculty of Science, Univ. Novi Sad, **22**(1992), 1-10.

[17] O. Hadžić, *Fixed Point Theory in PM-Spaces,* Novi Sad, 1995.

[18] O. Hadžić, E. Pap, *Fixed Point Theory in probabilistic spaces*, Kluwer Academic Publ., 2001.

[19] O. Hadžić, E. Pap, *A fixed point theorem for multivalued mappings in probabilistic metric spaces and an application in fuzzy metric spaces*, Fuzzy Sets and Systems, **127**(2002), 333-344.

[20] O. Hadžić, E. Pap, V. Radu, *Generalized contraction mapping principles in probabilistic metric spaces*, Acta Math. Hungar., **101**(2003), 131-148.

[21] T. L. Hicks, , *Fixed point theory in probabilistic metric spaces,* Review of Research, Faculty of Science, Univ. Novi Sad, **13**(1983), 63-72.

[22] T. L. Hicks, *Fixed point theory in PM-spaces II* Math. Japonica, **43**(1996), 487-493.

[23] T. L. Hicks, P. L. Sharma, *Probabilistic metric structures. Topological classification*, Review of Research, Faculty of Science, Univ. Novi Sad, **14**(1984), 43-50.

[24] T. L. Hicks, B. E. Rhoades, *Fixed point theory in symmetric spaces with applications to probabilistic spaces*, Nonlinear Analysis, **36**(1999), 331-344.

[25] U. Höhle, *Probabilistische Topologien*, Man. Math., **26**(1978), 223-245.

[26] U. Höhle, *Probabilistiche Metriken auf der Menge der nichtnegativen Verteilungsfunctionen*, Aequationes Math., **18**(1978), 345-356.

[27] I. Istrăţescu, *On some fixed point theorems in general Menger spaces*, Boll. U. M. I., **5**(1976), 95-100.

[28] J. Jachymski, J. Matkowski, T. Swiatkowski, *Nonlinear contractions on semimetric spaces*, Journal of Applied Analysis, **1**(1995), 125-134.

[29] O. Kaleva, O. Seikalla, *On fuzzy metric spaces*, Fuzzy Sets and Systems, **12**(1984), 215-229.

[30] J. F. C. Kingman, *A metric for Wald Spaces*, J. London Math. Soc., **39**(1964), 129-130.

[31] E. P. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Kluwer Acad. Publ., Dordrecht, 2000.

[32] O. Kramosil, J. Michalek, *Fuzzy metric and SM-spaces*, Kybernetica (Praha), **11**(1975), 326-334.

[33] M. A. Krasnoselski, et al., *Approximate solutions of operator equations*, Moscow, 1969.

[34] M. Maia, *Un'osservazione sulle contrazioni metriche*, Sem. Mat. U. Padova, **40**(1968), 139-143.

[35] B. Margolis, *On some fixed points theorems in generalized complete metric spaces*, Bull. Amer. Math. Soc., **74**(1968), 275-282.

[36] K. Menger, *Statistical metrics*, Proc. Nat. Academy Sci. U. S. A., **28**(1942), 535-537.

[37] D. Miheţ, *A type of contraction mappings in PM-spaces*, in Seminarul de Teoria Probabilităţilor şi Aplicaţii (STPA), West Univ. of Timişoara, **114**(1994).

[38] D. Miheţ, *The triangle inequlity and fixed points in PM - spaces*, Ph.D. Thesis (1997), University of Timişoara, Surveys Lectures Notes and Monographs Series on Probability Statistics and Applied Mathematics, **4**(2001).

[39] D. Miheţ, *Erratum to "A class of Sehgal's contractions in probabilistic metric spaces"* [An. Univ. Timişoara Ser. Mat.-Inform., **37**(1999), 105-108], An. Univ. Timişoara Ser. Mat.-Inform., **39**(2001), 155-156.

[40] D. Miheţ, *A note on Hicks type contractions on Generalized Menger Spaces*, in: Seminarul de Teoria Probabilităţilor şi Aplicaţii, West Univ. of Timişoara, **133**(2002).

[41] D. Miheţ, *Fixed point theorems for multivalued contractions of Hicks type in probabilistic metric spaces*, Scientiae Mathematicae Japonicae - Online Version, **10-42**(2004).

[42] D. Miheţ, *A fixed point theorem for weak-Hicks contractions*, Ind. J. Pure Appl. Math., (to appear).

[43] A. Monna, *Sur un théorème de M. Luxemburg concernant les points fixes d'une classe d'application d'un espace mtrique dans lui-même*, Ned. Ak. Wet. Proc., **64**(1961), 89-96.

[44] B. Morrel, J. Nagata, *SM-spaces as related to topological spaces*, Gen. Top. Appl., **9**(1978), 233-237.

[45] R. Moynihan, B. Schweizer, *Betweenes relations in PM-spaces*, Pacific J. Math., **81**(1979), 175-196.

[46] E. Pap, O. Hadžić, R. Mesiar, *A fixed point theorem in probabilistic metric spaces and applications*, J. Math. Anal. Appl., **202**(1996) 431-440.

[47] E. Părău, V. Radu, *Some remarks on Tardiff's fixed point theorem on Menger spaces*, Portugaliae Math., **54**(1997), 431-440.

[48] E. Părău, V. Radu, *On the triangle inquality and probabilistic contractions in PM-spaces*, Analele Univ. Bucureşti, Matematică, **XLVIII**(1998), 57-64.

[49] V. Radu, *On some metrics for the (ε, λ)-topology in PM-spaces*, in Seminarul de Teoria Probabilităţilor şi Aplicaţii, Univ. Timişoara, **61**(1982).

[50] V. Radu, *On a family of metrics for the distribution functions*, in: Proceedings of the Seventh Conference on Probability Theory, Marius Iosifescu (Ed.), Editura Academiei RSR, Bucureşti, 1984, 487-492.

[51] V. Radu, *On the t-norms of Hadžić-type and fixed points in PM-spaces*, Review of Research, Faculty of Science, Univ. Novi Sad, **13**(1983), 81-85.

[52] V. Radu, *A remark on contractions in Menger spaces*, in Seminarul de Teoria Probabilităţilor şi Aplicaţii, Univ. din Timişoara, **64**(1983).

[53] V. Radu, *On the t-norms with the fixed point property*, in: Seminarul de Teoria Probabilităţilor şi Aplicaţii, Univ. din Timişoara, **72**(1984).

[54] V. Radu, *On the contraction principle in Menger spaces*, Analele Univ. Timişoara, **XXII**(1984), 83-88.

[55] V. Radu, *On some contraction - type mappings in Menger spaces*, Analele Univ. Timişoara, **XXIII**(1985), 61-65.

[56] V. Radu, *On obtaing Levi's metric by probabilistic metric*, Analele Univ. Timişoara, **XXIV**(1986), 60-65 .

[57] V. Radu, *A family of deterministic metrics on Menger spaces*, in Seminarul de Teoria Probabilităţilor şi Aplicaţii, Univ. Timişoara, **78**(1986).

[58] V. Radu, *Some fixed point theorems in PM-Spaces*, Lecture Notes Math., **1233**(1987), 125-133.

[59] V. Radu, *Deterministic metrics on Menger spaces and applications to fixed point theorems*, Babeş-Bolyai University, Fac. of Math. and Physics, Research Seminar, **2**(1988), 163-166.

[60] V. Radu, *Lectures on Probabilistic Analysis*, University of Timişoara, Surveys Lectures Notes and Monographs Series on Probability Statistics and Applied Mathematics, vol. 2, 1994.

[61] V. Radu, *Triangular norms and Menger spaces with the fixed point property for probabilistic contractions*, Analele Univ. Timişoara, Seria Matematică-Informatică, **37**(1999), 123-132.

[62] V. Radu, *Some metrics on Menger spaces and a characterization of probabilistic contractions having a fixed point*, in Fixed Point Theory and Applications, Y. J. Cho, J. K. Kim, S. M. Kang (Eds.), Nova Science Publ., 2001, 53-60.

[63] V. Radu, *Probabilistic contractions on fuzzy Menger spaces*, Analele Univ. Bucuresti, **LI** (2002) (Special issue: The 5th Conference of SPSR, Bucureşti, 22-23 February, 2002), 63-72.

[64] V. Radu, *Ideas and methods in fixed point theory for probabilistic contractions*, Seminar on Fixed Point Theory Cluj-Napoca, **3**(2002), 73-98.

[65] V. Radu, *Some Remarks on the Probabilistic Contractions on Fuzzy Menger Spaces*, Automat. Comput. Appl. Math., **11**(2002), 104-110.

[66] V. Radu, *Equicontinuous iterates of t-norms and applications to random normed and fuzzy Menger spaces*, ITERATION THEORY (ECIT '02), J. Sousa Ramos, D. Gronau, C. Mira, L. Reich, A. N. Sharkovsky (Eds.), Grazer Math. Ber., **346**(2004), 323-350.

[67] V. Radu, *A fixed point theorem for probabilistic contractions on fuzzy Menger spaces*, Comm. 6th Conference of SPSR, Bucureşti, February, 2003.

[68] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[69] A. Sapena, *A Contribution to the study of fuzzy metric spaces*, Applied General Topology, Universidad Politecnica de Valencia, **1**(2000), 1-14.

[70] B. Schweizer, H. Sherwood, R. M. Tardiff, *Contractions on PM-spaces: Examples and Counterexamples*, Stochastica, **XII-1**(1988), 5-17.

[71] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific Journal Math., **10**(1960), 314-334.

[72] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North Holland, 1983.

[73] B. Schweizer, A. Sklar, E. Thorp, *The metrization of SM spaces*, Pacific Journal Math., **10**(1960), 673-675.

[74] V. M. Sehgal, A. T. Bharucha-Reid, *Fixed points of contraction mappings in PM spaces*, Math. Systems Theory, **6**(1972), 97-100.

[75] A. N. Šerstnev, *On the probabilistic generalization of metric spaces*(in Russian), Kazan Gos. Univ. Uch. Zap., **124**(1964), 3-11.

[76] H. Sherwood, *Complete probabilistic metric spaces*, Z. Wahr. Verw. Geb., **20**(1971), 117-128.

[77] R. M. Tardiff, *Contraction maps on PM-Spaces*, Journal Math. Anal. Appl., **165**(1992), 517-523.