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EXISTENCE AND LOCALIZATION RESULTS FOR THE NONLINEAR WAVE EQUATION

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Abstract. Existence and localization results for the nonlinear wave equation are established by Krasnoselskii's compression-expansion fixed point theorem in cones. The main idea is to handle two equivalent operator forms of the wave equation, one of fixed point type giving the operator to which Krasnoselskii's theorem applies and an other one of coincidence type for the localization of a solution. In this way, the compression-expansion technique is extended from scalar equations to abstract equations, specifically to partial differential equations.
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1. Introduction

Krasnoselskii's compression-expansion fixed point theorem in cones [8] is one of the most significant results of nonlinear functional analysis.

Theorem 1.1 (Krasnoselskii)**.** *Let $(E, |.|)$ be a normed linear space, $C \subset E$ a proper wedge and $N : C \to C$ a completely continuous map. Assume that for some numbers ρ and R with $0 < \rho < R$, one of the following conditions is satisfied:*

(a) $|N(x)| \leq |x|$ *for* $|x| = \rho$ *and* $|N(x)| \geq |x|$ *for* $|x| = R$,
(b) $|N(x)| \geq |x|$ *for* $|x| = \rho$ *and* $|N(x)| \leq |x|$ *for* $|x| = R$.

Then N has a fixed point x with $\rho \leq |x| \leq R$.

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This result as well as similar others are simple applications of the index (see Amann [2], Deimling [3] and Granas and Dugundji [6, p 325]) and can be proved elementarily using the notion of an essential map (see O'Regan and Precup [13], Precup [14] and Simon and Volkmann [16]).

Krasnoselskii type theorems in cones have been successfully applied to discuss existence, localization and multiplicity of nonnegative solutions for two-point boundary value problems, see Agarwal, Meehan, O'Regan and Precup [1], Erbe, Hu and Wang [4], Erbe and Wang [5], Guo, Lakshmikantham and Liu [7], Lan and Webb [9], Lian, Wong and Yeh [10], Meehan and O'Regan [12] and O'Regan and Precup [13]. All these applications are based on upper and lower inequalities for the appropriate Green's functions.

Similar inequalities for boundary value problems to partial differential equations are not known and Krasnoselskii's Theorem appears quite unapplicable to this type of problems. The main goal of this paper is to explain how this difficulty can be overcome when treating partial differential equations.

We finish this introductory section by some notations. For a bounded and open set $\Omega \subset \mathbf{R}^n$, $1 \leq p < \infty$ and $0 < T < \infty$, we consider the space $L^p(\Omega)$ with norm $|u|_p = \left(\int_\Omega |u(x)|^p\, dx\right)^{1/p}$ and the space $C([0,T]; L^p(\Omega))$ with norm $|.|_{\infty,p}$ defined by

$$|u|_{\infty,p} = \max_{t\in[0,T]} |u(t)|_p\,.$$

The space $H^{-1}(\Omega)$ with norm $|.|_{-1}$ is the dual of the Sobolev space $H_0^1(\Omega)$, the notation $L^p(\Omega; \mathbf{R}_+)$ stands for the set of nonnegative functions of $L^p(\Omega)$ and $H^{-1}(\Omega; \mathbf{R}_+)$ is the set of all $u \in H^{-1}(\Omega)$ whose values (u, v) on all nonnegative functions $v \in H_0^1(\Omega)$ are nonnegative. Also we use the notation $|u|_{\infty,-1}$ to denote the norm $|u|_{\infty,-1} = \max_{t\in[0,T]} |u(t)|_{-1}$ on $C\left([0,T]; H^{-1}(\Omega)\right)$. We recall that $H_0^1(\Omega) \subset L^p(\Omega)$ and $L^q(\Omega) \subset H^{-1}(\Omega)$ (with continuous imbeddings) for $1 \leq p \leq 2^* = \frac{2n}{n-2}$ and $q \geq (2^*)' = \frac{2n}{n+2}$ if $n \geq 3$ and for all $p, q \geq 1$ if $n = 1$ or $n = 2$.

2. Main existence and localization result

We shall discuss the mixed Cauchy–Dirichlet problem for the nonlinear wave equation

$$\begin{cases} -u''(t) + \Delta u(t) - mu(t) = F(u)(t), \quad t \in [0,T] \\ u(0) = u(T) = 0 \\ u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^2\left([0,T]; H^{-1}(\Omega)\right). \end{cases} \tag{2.1}$$

Here $0 < T < \infty$, $\Omega \subset \mathbf{R}^n$ is a bounded open subset, $m > -\lambda_1$ (λ_1 is the first eigenvalue corresponding to $-\Delta$ and to the homogenous Dirichlet boundary condition) and F is a map from $C\left([0,T]; H_0^1(\Omega)\right)$ to $C\left([0,T]; H^{-1}(\Omega)\right)$.

Let $A : D(A) \to C\left([0,T]; H^{-1}(\Omega)\right)$ be given by

$$(Au)(t) = -u''(t).$$

Here $D(A) = \left\{u \in C^2\left([0,T]; H^{-1}(\Omega)\right) : u(0) = u(T) = 0\right\}$. Clearly A is invertible and

$$\left(A^{-1}v\right)(t) = \int_0^T g(t,s)\, v(s)\, ds, \quad v \in C\left([0,T]; H^{-1}(\Omega)\right)$$

where g is the Green's function of the differential operator A with respect to the boundary condition $u(0) = u(T) = 0$, i.e.

$$g(t,s) = \begin{cases} \frac{s(T-t)}{T}, & 0 \le s \le t \le T \\ \frac{t(T-s)}{T}, & 0 \le t \le s \le T. \end{cases}$$

Notice that for every subinterval $[a,b]$ of $[0,T]$, $0 < a < b < T$, g satisfies the following upper and lower inequalities

$$\begin{aligned} g(t,s) &\le g(s,s) \quad \text{for } t \in [0,T] \text{ and } s \in [0,T] \\ k_{a,b}\, g(s,s) &\le g(t,s) \quad \text{for } t \in [a,b] \text{ and } s \in [0,T]. \end{aligned} \tag{2.2}$$

Here $k_{a,b} = \min\left\{\frac{a}{T}, \frac{T-b}{T}\right\}$. Obviously $0 < k_{a,b} < 1$. In what follows we shall also use the notation

$$g_{a,b}^* = \max_{t \in [0,T]} \int_a^b g(t,s)\, ds.$$

Clearly $g_{a,b}^* \ge k_{a,b} \int_a^b g(s,s)\, ds > 0$. Also note that $\int_0^T g(s,s)\, ds = \frac{T^2}{6}$.

Let $B : H_0^1(\Omega) \to H^{-1}(\Omega)$ be defined by

$$Bu = -\Delta u + m\, u, \quad u \in H_0^1(\Omega).$$

Since $m > -\lambda_1$, B is invertible and its inverse B^{-1} is a linear continuous and positive (by the maximum principle) operator. We make the convention that for a function $u \in C\left([0,T]; H_0^1(\Omega)\right)$, Bu is the function in $C\left([0,T]; H^{-1}(\Omega)\right)$

defined by $(Bu)(t) = Bu(t)$ for $t \in [0,T]$. An analogue convention is made for B^{-1}.

Basic theory on the non-homogenous linear wave equation (see Lions and Magenes [11] and Precup [15]) guarantees that the operator $A - B$ from $C([0,T];\, H_0^1(\Omega)) \cap D(A)$ to $C\left([0,T];H^{-1}(\Omega)\right)$ is invertible and its inverse $(A-B)^{-1}$ is a linear operator, completely continuous from $C\left([0,T];H^{-1}(\Omega)\right)$ to $C([0,T];\, L^p(\Omega))$ for $(2^*)' \leq p < 2^*$ if $n \geq 3$ and any $p \geq 1$ if $n = 1$ or $n = 2$.

One can check that the following equality is true

$$\left(B^{-1} - A^{-1}\right)^{-1} = (A-B)^{-1} BA$$

for operators acting from $C^2\left([0,T];H_0^1(\Omega)\right)$ to $C\left([0,T];H_0^1(\Omega)\right) \cap C^2([0,T];$ $H^{-1}(\Omega))$.

Now solving (2.1) is equivalent to the problem

$$(A-B)\,u = F(u)\,,\quad u \in C\left([0,T];H_0^1(\Omega)\right) \cap D(A) \tag{2.3}$$

which can be written under the form

$$u = (A-B)^{-1} F(u) \tag{2.4}$$

or equivalently as

$$\left(B^{-1} - A^{-1}\right) u = A^{-1}B^{-1}F(u)\,. \tag{2.5}$$

Under suitable conditions on F, the complete continuity of $(A-B)^{-1}$ implies that the nonlinear operator $N := (A-B)^{-1} F$ associated to the right hand side of equation (2.4) is completely continuous. Hence equation (2.4) gives us the operator to which Krasnoselskii's Theorem applies. On the other hand, the upper and lower inequalities (2.2) for the Green's kernel in A^{-1} make equation (2.5) useful for the localization of a solution of (2.4) in a conical shell of the form

$$0 < \rho \leq \left|\left(B^{-1} - A^{-1}\right) u\right|_{\infty,q} \leq R.$$

More exactly, we have the following result on the existence and the localization of a solution of (2.1).

Theorem 2.1. *Let $(2^*)' \leq p < 2^*$, $1 \leq q \leq 2^*$ if $n \geq 3$ and $p \geq 1$, $q \geq 1$ if $n = 1$ or $n = 2$. Let C be the cone of $C([0,T];L^p(\Omega))$ given by*

$$C = \left\{u \in C([0,T];L^p(\Omega)) : u = (A-B)^{-1} v,\ v \in C\left([0,T];H^{-1}(\Omega;\mathbf{R}_+)\right)\right\}.$$

1^0 *Assume that the following two conditions are satisfied:*

(h1) $F : C \to C\left([0,T]\,; H^{-1}(\Omega; \mathbf{R}_+)\right)$ *is continuous and sends bounded sets into bounded sets;*

(h2) *there exists* $\alpha > 0$ *such that*

$$\left|B^{-1}F(u)(t)\right|_q \leq \frac{6\alpha}{T^2}, \ \ t \in [0,T]$$

for every $u \in C$ *with* $\|u\| := \left|\left(B^{-1} - A^{-1}\right)u\right|_{\infty,q} = \alpha.$

Then (2.1) *has at least one solution* $u \in C$ *with* $\|u\| \leq \alpha$.

2^0 *Assume that* (h1), (h2) *and the following additional condition are satisfied:*

(h3) *there exists an interval* $[a,b]$ *with* $0 < a < b < T$, *a map* $\phi : C \to H^{-1}(\Omega; \mathbf{R}_+)$ *and a number* $\beta > 0$, $\beta \neq \alpha$, *such that*

$$\phi(u) \leq F(u)(t)\,, \ \ t \in [a,b] \tag{2.6}$$

and

$$\left|B^{-1}\phi(u)\right|_q \geq \frac{\beta}{g^*_{a,b}} \tag{2.7}$$

for all $u \in C$ *with* $\|u\| = \beta$.

Then (2.1) *has at least one solution* $u \in C$ *with*

$$\min\{\alpha,\beta\} \leq \|u\| \leq \max\{\alpha,\beta\}\,.$$

Proof. We shall apply Theorem 1.1 to the space $E = C([0,T]\,; L^p(\Omega))$ with norm $\|u\|$ and to the operator $N = (A-B)^{-1}F$. From (h1) we have for every $u \in C$ that $F(u) \in C\left([0,T]\,; H^{-1}(\Omega; \mathbf{R}_+)\right)$. It follows that $(A-B)^{-1}F(u) \in C$, that is $N(u) \in C$. Also (h1) and the complete continuity of $(A-B)^{-1}$ guarantee that $N : C \to C$ is completely continuous.

Let $u \in C$ with $\|u\| = \alpha$. Since $N(u) = \left(B^{-1} - A^{-1}\right)^{-1} A^{-1}B^{-1}F(u)$, we have

$$\|N(u)\| = \left|A^{-1}B^{-1}F(u)\right|_{\infty,q}.$$

On the other hand, in virtue of (2.2) one has

$$\begin{aligned} 0 \leq A^{-1}B^{-1}F(u)(t) &= \int_0^T g(t,s)\, B^{-1}F(u)(s)\, ds \\ &\leq \int_0^T g(s,s)\, B^{-1}F(u)(s)\, ds. \end{aligned}$$

Then, since the norm $|.|_q$ is monotone,

$$\begin{aligned}\left|A^{-1}B^{-1}F(u)(t)\right|_q &\leq \int_0^T g(s,s)\left|B^{-1}F(u)(s)\right|_q ds \\ &\leq \frac{6\alpha}{T^2}\int_0^T g(s,s)\,ds = \alpha.\end{aligned}$$

Hence

$$\|N(u)\| \leq \alpha = \|u\|. \tag{2.8}$$

1^0 Inequality (2.8) clearly guarantees that the boundary condition $u \neq \lambda N(u)$ holds for all $\lambda \in (0,1)$ and $u \in \partial U$, where $U := \{u \in C : \|u\| < \alpha\}$. Thus the conclusion follows from the Leray-Schauder Fixed Point Theorem (see Granas and Dugundji [6, p 123]).

2^0 Under assumptions (h1)-(h3), condition (a) or (b) holds (with $\rho = \min\{\alpha,\beta\}$ and $R = \max\{\alpha,\beta\}$) if $\alpha < \beta$ or $\beta < \alpha$, respectively. To show this, assume $u \in C$ and $\|u\| = \beta$. Let $t^* \in [0,T]$ with $g_{a,b}^* = \int_a^b g(t^*,s)\,ds$. Using (2.6), we obtain

$$\begin{aligned}\left(B^{-1}-A^{-1}\right)N(u)(t^*) &= A^{-1}B^{-1}F(u)(t^*) \\ &= \int_0^T g(t^*,s)\,B^{-1}F(u)(s)\,ds \\ &\geq \int_a^b g(t^*,s)\,B^{-1}F(u)(s)\,ds \\ &\geq g_{a,b}^* B^{-1}\phi(u).\end{aligned}$$

Then, by (2.7), we deduce that

$$\left|\left(B^{-1}-A^{-1}\right)N(u)(t^*)\right|_q \geq g_{a,b}^*\left|B^{-1}\phi(u)\right|_q \geq \beta.$$

It follows that

$$\|N(u)\| \geq \beta = \|u\|.$$

This together with (2.8) shows that (a) or (b) holds if $\alpha < \beta$, respectively $\beta < \alpha$. Thus Krasnoselskii's Theorem applies. □

Remark 2.1. For each $u \in C$ and all $t \in [0,T]$ and $t' \in [a,b]$, we have

$$0 \leq k_{a,b}\left(B^{-1}-A^{-1}\right)u(t) \leq \left(B^{-1}-A^{-1}\right)u(t'). \tag{2.9}$$

Indeed, if $u=(A-B)^{-1}v$ with $v\in C\left([0,T];H^{-1}(\Omega;\mathbf{R}_{+})\right)$, then

$$\begin{aligned}
k_{a,b}\left(B^{-1}-A^{-1}\right)u(t) &= k_{a,b}A^{-1}B^{-1}v(t)=k_{a,b}\int_0^T g(t,s)\,B^{-1}v(s)\,ds\\
&\leq k_{a,b}\int_0^T g(s,s)\,B^{-1}v(s)\,ds\\
&\leq \int_0^T g(t',s)\,B^{-1}v(s)\,ds\\
&= A^{-1}B^{-1}v(t')=\left(B^{-1}-A^{-1}\right)u(t').
\end{aligned}$$

Also, the positivity of A^{-1} and B^{-1} guarantees $A^{-1}B^{-1}v(t)\geq 0$. Hence

$$\left(B^{-1}-A^{-1}\right)u(t)\geq 0,\quad t\in[0,T].$$

Remark 2.2. The norms $|.|_{\infty,p}$ and $\|.\|$ are equivalent on $C([0,T];L^p(\Omega))$.

Indeed, if for any $u\in C([0,T];L^p(\Omega))$ we let $v=\left(B^{-1}-A^{-1}\right)u$, then

$$\left|\left(B^{-1}-A^{-1}\right)^{-1}\right|^{-1}|u|_{\infty,p}\leq\|u\|=|v|_{\infty,q}\leq\left|\left(B^{-1}-A^{-1}\right)\right||u|_{\infty,p}.$$

Thus

$$c_0|u|_{\infty,p}\leq\|u\|\leq c_1|u|_{\infty,p},\quad u\in C([0,T];L^p(\Omega)) \tag{2.10}$$

where

$$c_0=\left|\left(B^{-1}-A^{-1}\right)^{-1}\right|^{-1}\ \text{ and }\ c_1=\left|\left(B^{-1}-A^{-1}\right)\right|.$$

Here c_0^{-1} is the norm of operator $\left(B^{-1}-A^{-1}\right)^{-1}$ acting from $L^q(\Omega)$ to $L^p(\Omega)$, while c_1 is the norm of operator $\left(B^{-1}-A^{-1}\right)$ from $L^p(\Omega)$ to $L^q(\Omega)$.

Remark 2.3. A sufficient condition for (h2) is that

$$\lim_{|u|_{\infty,p}\to 0}\frac{|F(u)|_{\infty,-1}}{|u|_{\infty,p}}=0\quad\textit{or}\quad\lim_{|u|_{\infty,p}\to\infty}\frac{|F(u)|_{\infty,-1}}{|u|_{\infty,p}}=0. \tag{2.11}$$

Indeed, under condition (2.11), for every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that if $|u|_{\infty,p}\leq\delta$ (respectively, $|u|_{\infty,p}\geq\delta$), then $|F(u)|_{\infty,-1}\leq\varepsilon|u|_{\infty,p}$. Consequently, if c is a constant with $\left|B^{-1}w\right|_{L^q(\Omega)}\leq c|w|_{H^{-1}(\Omega)}$, then

$$\left|B^{-1}F(u)\right|_{\infty,q}\leq c|F(u)|_{\infty,-1}\leq c\varepsilon|u|_{\infty,p}\leq c\varepsilon c_0^{-1}\|u\|.$$

Thus (h2) holds for $\varepsilon=\varepsilon_0:=\frac{6c_0}{T^2c}$ and any $\alpha>0$ with $\alpha\leq c_0\delta(\varepsilon_0)$ (respectively, $\alpha\geq c_1\delta(\varepsilon_0)$).

Remark 2.4. A sufficient condition for (2.7) is that

$$\lim_{|u|_{\infty,p}\to 0} \frac{\left|B^{-1}\phi(u)\right|_q}{|u|_{\infty,p}} = \infty \quad or \quad \lim_{|u|_{\infty,p}\to \infty} \frac{\left|B^{-1}\phi(u)\right|_q}{|u|_{\infty,p}} = \infty.$$

Indeed, under this condition, for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $|u|_{\infty,p} \leq \delta$ (respectively, $|u|_{\infty,p} \geq \delta$), then $\left|B^{-1}\phi(u)\right|_q \geq \varepsilon |u|_{\infty,p}$, hence

$$\left|B^{-1}\phi(u)\right|_q \geq \varepsilon c_1^{-1} \|u\|.$$

Thus (2.7) holds for $\varepsilon = \varepsilon_0 := \frac{c_1}{g^*_{a,b}}$ and any $\beta > 0$, $\beta \neq \alpha$ with $\beta \leq c_0\delta(\varepsilon_0)$ (respectively, $\beta \geq c_1\delta(\varepsilon_0)$).

Remark 2.5. A sufficient condition for (h3) is to exist an interval $[a,b]$ with $0 < a < b < T$, a continuous nondecreasing function $f : R_+ \to R_+$ and a number $\beta > 0$, $\beta \neq \alpha$, such that $f(v) = f \circ v \in H^{-1}(\Omega; \mathbf{R}_+)$ for every $v \in L^q(\Omega; \mathbf{R}_+)$,

$$F(u)(t) \geq f\left(\left(B^{-1} - A^{-1}\right)u(t)\right), \quad t \in [a,b], \ u \in C \tag{2.12}$$

and

$$\inf\left\{\left|B^{-1}f(k_{a,b}v)\right|_q : v \in L^q(\Omega; \mathbf{R}_+), \ |v|_q = \beta\right\} \geq \frac{\beta}{g^*_{a,b}}. \tag{2.13}$$

Indeed, if $u \in C$, then (2.9) is true and (2.12) and the monotonicity of f imply

$$F(u)(t) \geq f\left(k_{a,b}\left(B^{-1} - A^{-1}\right)u(t_u)\right), \quad t \in [a,b],$$

where $t_u \in [0,T]$ and $\left|\left(B^{-1} - A^{-1}\right)u(t_u)\right|_q = \|u\|$. Hence (2.6) holds with $\phi(u) = f\left(k_{a,b}\left(B^{-1} - A^{-1}\right)u(t_u)\right)$. Now (2.13) guarantees (2.7).

3. Applications

We shall specialize Theorem 2.1 to discuss the existence and the localization of solutions for nonlinear problems of the form

$$\begin{cases} -u''(t) + \Delta u(t) - mu(t) = h\left(\left|\left(B^{-1} - A^{-1}\right)u(t)\right|_q\right) f(t, u(t)) \quad \text{on } [0,T] \\ u(0) = u(T) = 0 \\ u \in C\left([0,T]; H^1_0(\Omega)\right) \cap C^2\left([0,T]; H^{-1}(\Omega)\right). \end{cases} \tag{3.1}$$

Theorem 3.1. *Let $h : \mathbf{R}_+ \to \mathbf{R}_+$ be continuous and nondecreasing and $f : [0,T] \times \mathbf{R} \to \mathbf{R}_+$ be continuous and*

$$f(t,u) \le c + d\,|u|^{\gamma}, \quad t \in [0,T], \; u \in \mathbf{R} \tag{3.2}$$

for some $c, d > 0$, $1 \le \gamma < 2^* - 1 = \frac{n+2}{n-2}$ if $n \ge 3$ and $1 \le \gamma < \infty$ if $n = 1$ or $n = 2$. Let $|B^{-1}|$ denotes the norm of operator B^{-1} from $L^r(\Omega)$ to $L^q(\Omega)$ where $r = (2^*)'$, $q = 2^*$ if $n \ge 3$ and $r \ge 1$, $q \ge 1$ in case that $n = 1$ or $n = 2$. Denote $c^* = c\,|1|_r$ and let $p = \gamma r$. Assume that there exists $\alpha > 0$ such that

$$|B^{-1}| \left(c^* + d c_0^{-\gamma} \alpha^{\gamma}\right) \frac{T^2}{6} \le \frac{\alpha}{h(\alpha)}. \tag{3.3}$$

Then problem (3.1) has at least one solution u with

$$\|u\| = \left|\left(B^{-1} - A^{-1}\right) u\right|_{\infty,q} \le \alpha.$$

If in addition there exists an interval $[a,b]$ with $0 < a < b < T$, a number σ with

$$0 < \sigma \le f(t,u), \quad t \in [a,b], \; u \in \mathbf{R} \tag{3.4}$$

and a number $\beta > 0$, $\beta \ne \alpha$, such that

$$\frac{\sigma\, g^*_{a,b}\, |\varphi_1|_q}{(\lambda_1 + m)\, |\varphi_1|_\infty} \ge \frac{\beta}{h(k_{a,b}\beta)} \tag{3.5}$$

then (3.1) has at least one solution u with

$$\min\{\alpha, \beta\} \le \|u\| \le \max\{\alpha, \beta\}.$$

Proof. We shall apply Theorem 2.1. Notice that for $n \ge 3$, from $1 \le \gamma < 2^* - 1 = \frac{2^*}{(2^*)'}$, we have $(2^*)' \le p = \gamma (2^*)' < 2^*$.

Let $F : C([0,T]; L^p(\Omega)) \to C([0,T]; L^r(\Omega; \mathbf{R}_+))$ be given by

$$F(u)(t) = h\left(\left|\left(B^{-1} - A^{-1}\right) u(t)\right|_q\right) f(t, u(t)).$$

From (3.2) we immediately see that

$$|f(t, u(t))|_r \le c^* + d\,|u(t)|_p^{\gamma}, \quad t \in [0,T]. \tag{3.6}$$

Here $c^* = c\,|1|_r$. Then, also using the monotonicity of h, we deduce

$$|F(u)(t)|_r \le h(\|u\|) \left(c^* + d\,|u(t)|_p^{\gamma}\right), \quad t \in [0,T]. \tag{3.7}$$

Consequently, F sends bounded subsets of $C([0,T];L^p(\Omega))$ into bounded sets of $C([0,T];L^r(\Omega))$. Also the continuity of f and h guarantees that F is continuous. Thus (h1) holds.

Next, using (3.6) and (2.10) we obtain

$$\begin{aligned}\left|B^{-1}F(u)(t)\right|_q &= h\left(\left|\left(B^{-1}-A^{-1}\right)u(t)\right|_q\right)\left|B^{-1}f(t,u(t))\right|_q\\ &\leq h(\|u\|)\left|B^{-1}\right|\left(c^*+d\,|u(t)|_p^{\gamma}\right)\\ &\leq h(\|u\|)\left|B^{-1}\right|\left(c^*+dc_0^{-\gamma}\|u\|^{\gamma}\right).\end{aligned}$$

This together with (3.3) proves (h2). Now the existence of a solution u with $\|u\|\leq\alpha$ is guarantees by Theorem 2.1 1^0.

Next we prove that the additional assumption guarantees (h3). Indeed, according to (2.9), (2.6) holds with $\phi(u)=\sigma h(k_{a,b}\|u\|)$. It remains to check (2.7). It is well known that any eigenfunction of $-\Delta$ is bounded on Ω, so $0<\varphi_1(x)\leq|\varphi_1|_\infty<\infty$ for all $x\in\Omega$. Then

$$0<\frac{1}{|\varphi_1|_\infty}\varphi_1(x)\leq 1,\quad x\in\Omega.$$

This together with $-\Delta\varphi_1+m\varphi_1=(\lambda_1+m)\varphi_1$ and the positivity of B^{-1}, guarantees that

$$B^{-1}1\geq\frac{1}{|\varphi_1|_\infty}B^{-1}\varphi_1=\frac{1}{|\varphi_1|_\infty(\lambda_1+m)}\varphi_1.$$

Hence

$$\left|B^{-1}1\right|_q\geq\frac{1}{|\varphi_1|_\infty(\lambda_1+m)}|\varphi_1|_q.$$

As a result

$$\left|B^{-1}\phi(u)\right|_q=\phi(u)\left|B^{-1}1\right|_q\geq\sigma h(k_{a,b}\|u\|)\frac{1}{|\varphi_1|_\infty(\lambda_1+m)}|\varphi_1|_q.$$

This together with (3.5) guarantees (2.7).

Thus all the assumptions of Theorem 2.1 are satisfied. □

Remark 3.1. *Multiple solutions to problem* (3.1) *are guaranteed by Theorem* 3.1 *if f and h satisfy all the assumptions for several disjoint intervals* $[\alpha,\beta]$ *(or* $[\beta,\alpha]$*).*

In particular, for $h \equiv 1$, Theorem 3.1 yields the following existence and localization result for the problem

$$\begin{cases} -u''(t) + \Delta u(t) - mu(t) = f(t, u(t)), \quad t \in [0, T] \\ u(0) = u(T) = 0 \\ u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^2\left([0,T]; H^{-1}(\Omega)\right). \end{cases} \tag{3.8}$$

Corollary 3.2. *Let $f : [0,T] \times \mathbf{R} \to \mathbf{R}_+$ be continuous and*

$$f(t,u) \le c + d\,|u|^{\gamma}, \quad t \in [0,T], \ u \in \mathbf{R} \tag{3.9}$$

for some $c, d > 0$, $1 \le \gamma < 2^* - 1$ if $n \ge 3$ and $1 \le \gamma < \infty$ if $n = 1$ or $n = 2$. Let $\left|B^{-1}\right|$ denotes the norm of operator B^{-1} from $L^r(\Omega)$ to $L^q(\Omega)$ where $r = (2^*)'$, $q = 2^*$ if $n \ge 3$ and $r \ge 1$, $q \ge 1$ in case that $n = 1$ or $n = 2$. Let $c^* = c\,|1|_r$ and $p = \gamma r$. Assume that there exists $\alpha > 0$ such that

$$\left|B^{-1}\right| \left(c^* + dc_0^{-\gamma}\alpha^{\gamma}\right) \frac{T^2}{6} \le \alpha. \tag{3.10}$$

Then problem (3.8) has at least one solution u with

$$\|u\| = \left|\left(B^{-1} - A^{-1}\right) u\right|_{\infty,q} \le \alpha.$$

If in addition there exists an interval $[a,b]$ with $0 < a < b < T$ and a number σ with

$$0 < \sigma \le f(t,u), \quad t \in [a,b], \ u \in \mathbf{R}, \tag{3.11}$$

then $\|u\| \ge \beta$, where

$$\beta := \frac{\sigma\, g_{a,b}^* \left|\varphi_1\right|_q}{(\lambda_1 + m)\left|\varphi_1\right|_\infty}.$$

Proof. The result is a direct consequence of Theorem 3.1 for $h \equiv 1$. Notice that $\beta < \alpha$. Indeed, if α satisfies (3.10), then

$$\alpha \ge \frac{T^2}{6}\left|B^{-1}\right|\left(c^* + d\,c_0^{-\gamma}\alpha^{\gamma}\right) \ge \frac{T^2}{6}\left|B^{-1}\right| c^* = \left|B^{-1}\right| c\,|1|_r \int_0^T g(s,s)\,ds. \tag{3.12}$$

On the other hand, from $B\varphi_1 = (\lambda_1 + m)\varphi_1$ we have

$$\frac{1}{\lambda_1 + m}\left|\varphi_1\right|_q = \left|B^{-1}\varphi_1\right|_q \le \left|B^{-1}\right| \left|\varphi_1\right|_r.$$

Hence

$$\left|B^{-1}\right| \ge \frac{\left|\varphi_1\right|_q}{(\lambda_1 + m)\left|\varphi_1\right|_r} \ge \frac{\left|\varphi_1\right|_q}{(\lambda_1 + m)\,|1|_r \left|\varphi_1\right|_\infty}. \tag{3.13}$$

Now (3.12) and (3.13) imply

$$\alpha \geq \frac{c\,|\varphi_1|_q}{(\lambda_1 + m)\,|\varphi_1|_\infty} \int_0^T g\,(s,s)\,ds. \tag{3.14}$$

Conditions (3.9) and (3.11) yield in particular $\sigma \leq f\,(t,0) \leq c$ for $t \in [a,b]$. Hence $\sigma \leq c$. Also, taking into account

$$g_{a,b}^* = \int_a^b g\,(t^*,s)\,ds \leq \int_a^b g\,(s,s)\,ds < \int_0^T g\,(s,s)\,ds,$$

we deduce from (3.14) that $\alpha > \beta$ as claimed. □

Remark 3.2. *The existence result in Corollary* 3.2 *also follows from the Schauder fixed point theorem, while the localization part can also be established by showing that any solution u satisfies $\|u\| \geq \beta$.*

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