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SHIFT POINTS AND THE FIXED POINT PROPERTY FOR PRODUCTS

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Abstract. Characterizations of when countable Cartesian products have the fixed property are given in terms of shift points.

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1. Introduction

A map is a continuous function; X always denotes a nonempty topological space (sometimes satisfying conditions that we specify); X^∞ denotes the countably infinite Cartesian product of X with itself with the product topology.

This paper began when we observed the following consequence of 3.10 of [4]: Let X be a compact metric absolute retract, and let $f : X^\infty \to X^\infty$ be a map; then, for each positive integer n, there is a point $p^n = (p_i^n)_{i=1}^\infty \in X^\infty$ such that f shifts p^n to the left n coordinates; that is,

$$f(p_1^n, p_2^n, \ldots, p_n^n, \ldots) = (p_{n+1}^n, p_{n+2}^n, \ldots).$$

We became interested in determining the points of X^∞ that are limits as $n \to \infty$ of points that are shifted n coordinates to the left under various maps $f : X^\infty \to X^\infty$. At first we thought that whether or not all points are such limits always depended on the map, not on the space; then we discovered that for any topological space X, all points of X^∞ are such limits for every map $f : X^\infty \to X^\infty$ if and only if X^∞ has the fixed point property. We prove this theorem together with two other characterizations of the fixed point property for X^∞ (one of which we use to formulate a corollary). Our theorem generalizes and substantially strengthens the result for absolute retracts that we stated above. In addition, the proof of our theorem is elementary; in particular, the proof does not use CE-maps (which were used in [4]).

2. Notation and terminology

For each $n = 1, 2, \ldots, \sigma_n : X^\infty \to X^\infty$ is the map that shifts coordinates to the left n places; that is,

$$\sigma_n((x_i)_{i=1}^\infty) = (x_i)_{i=n+1}^\infty, \text{ all } (x_i)_{i=1}^\infty \in X^\infty.$$

Let $f : X^\infty \to X^\infty$ be a map. For any $n = 1, 2, \ldots$, we call a point y in X^∞ an n-shift point for f provided that $f(y) = \sigma_n(y)$. We call a point p in X^∞ an infinite shift point for f provided that there is a sequence $\{y^n\}_{n=1}^\infty$ in X^∞ such that $\{y^n\}_{n=1}^\infty$ converges to p and y^n is an n-shift point for f for each n (i.e., $f(y^n) = \sigma_n(y^n)$ for each n).

3. Characterization theorem

Theorem. *For any nonempty topological space X, the following four conditions are equivalent:*

(1) for some integer $n_0 \geq 1$, every map of X^∞ into X^∞ has an n_0-shift point;

(2) for each integer $n \geq 1$, every map of X^∞ into X^∞ has an n-shift point;

(3) every point of X^∞ is an infinite shift point for every map of X^∞ to X^∞;

(4) X^∞ has the fixed point property.

Proof. It is obvious that (3) implies (2) and that (2) implies (1). We prove that (1) implies (4) and that (4) implies (3).

Let $f : X^\infty \to X^\infty$ be a given map.

Assume (1). Then the map $f \circ \sigma_{n_0} : X^\infty \to X^\infty$ has an n_0-shift point p. This means $(f \circ \sigma_{n_0})(p) = \sigma_{n_0}(p)$. Therefore, $\sigma_{n_0}(p)$ is a fixed point of f. This proves (1) implies (4).

Assume (4). Fix a point $p = (p_i)_{i=1}^\infty \in X^\infty$. Convergence in X^∞ is coordinatewise convergence; thus, to prove (3), it suffices to prove that for each $n = 1, 2, \dots$, there is a point $y^n = (y_i^n)_{i=1}^\infty$ in X^∞ such that $y_i^n = p_i$ for all $i \leq n$ and $f(y^n) = \sigma_n(y^n)$.

Fix n. Define a map $\varphi : X^\infty \to X^\infty$ by

$$\varphi(x) = f(p_1, \dots, p_n, x_1, x_2, \dots), \text{ all } x = (x_i)_{i=1}^\infty \in X^\infty.$$

By (4), $\varphi(q) = q$ for some point $q = (q_i)_{i=1}^\infty \in X^\infty$. Let

$$y^n = (p_1, \dots, p_n, q_1, q_2, \dots).$$

Then $y_i^n = p_i$ for all $i \leq n$ and

$$f(y^n) = \varphi(q) = q = \sigma_n(y^n).$$

This proves (4) implies (3). $\square$

Corollary. *If X is a nonempty topological space such that for some integer $n_0 \geq 1$, every map of X^∞ into X^∞ has an n_0-shift point, then X has the fixed point property.*

Proof. By our Theorem, X^∞ has the fixed point property. Therefore, since X (as the Cartesian product $X \times \{p\} \times \{p\} \times \dots$, where p is a given point of X) is a retract of X^∞, X has the fixed point property. $\square$

The following example shows that the converse of the Corollary is false even for compact metric spaces.

Example. There are compact metric spaces X with the fixed point property such that for each integer $n \geq 1$, there is a map $f_n : X^\infty \to X^\infty$ that has no n-shift point. This is seen by applying our Theorem to R. J. Knill's example B in [2]: Knill proved that B (which is compact and metric) has the fixed point property and that the Cartesian product $B \times [0, 1]$ does not have the fixed point property. Since B contains arcs, $B \times B$ retracts onto $B \times arc$; hence, $B \times B$ does not have the fixed point property. Thus, B^∞ does not have the fixed point property. Therefore, by our Theorem, there is a map $f_n : B^\infty \to B^\infty$ with no n-shift point for each integer $n \geq 1$. (An example of a noncompact metric space X with the fixed point property for which it is easy to verify that $X \times X$ does not have the fixed point property is in [1, Example 2].)

If X is a compact metric space such that every finite Cartesian product of X with itself has the fixed point property, then X^∞ has the fixed point property (apply 21.3 of [3, p.182]). However, we do not know the answer to the following:

Question. Is there a metric space X such that every finite Cartesian product of X with itself has the fixed point property but such that X^∞ does not have the fixed point property?

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