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THE METRIC PROJECTION AND ITS APPLICATIONS TO SOLVING VARIATIONAL INEQUALITIES IN BANACH SPACES

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Abstract. In this paper, we study the properties of the metric projection operator (the nearest point projection operator) and its continuity. Then we use it to solve variational inequalities in general Banach spaces and to approximate the solutions in uniformly convex and uniformly smooth Banach spaces.

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1. Preliminaries

Let (X, d) be a metric space and let K be a nonempty subset of X. For every $x \in X$, the distance between point x and the set K is denoted by $d(x, K)$ and is defined by the following minimum equation

$$d(x, K) = \inf_{y \in K} d(x, y).$$

The metric projection operator P_K defined on X is a mapping from X to 2^K:

$$P_K(x) = \{z \in K : \ d(x, z) = d(x, K)\}, \text{ for all } x \in X.$$

If $P_K(x) \neq \emptyset$, for every $x \in X$, then K is called *proximal.* If $P_K(x)$ is a singleton, for every $x \in X$, then K is said to be a *Chebyshev* set.

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In a special case, let $(B, \|\cdot\|)$ be a Banach space with the topological dual space B^*, and let $\langle\varphi, x\rangle$ denote the duality pairing of B^* and B, where $\varphi \in B^*$ and $x \in B$. Let K be a subset of B. The metric projection operator (the nearest point projection) $P_K : B \to 2^K$ has been used in many areas of mathematics such as: theory of optimization and approximation, fixed point theory. It is interesting to know what conditions to imply that a subset K is a proximal set, or, furthermore, is a Chebyshev set in a given Banach space. For example, in [9], Goeble and Reich proved the following theorem.

Theorem A. [9] *Every closed convex subset of a uniformly convex Banach space is a Chebyshev set.*

In [14], the present author studied the characteristics of the operator P_K based on the properties of Banach spaces. A non-proximal set example was given. We list the example and some properties below.

Example. Let $B = l_1$. It is known that l_1 is a non-reflexive Banach space with dual space l_∞. For any positive integer n, let $e_n \in l_1$ such that its nth entry is $(n+1)/n$ and all other entries are 0. Let $K = \overline{co}\{e_1, e_2, \ldots, e_n, \ldots\}$. Then K is a closed convex subset of l_1 and is not proximal.

From the above example, we see that the reflexive condition of a Banach space is required for a closed convex subset to be proximal. In fact, this condition is sufficient. It is described by the following theorem.

Theorem B. [14] *Let $(B, \|\cdot\|)$ be a Banach space. Then B is reflexive if and only if every nonempty closed convex subset $K \subset B$ is a proximal set.*

In order for a closed convex subset to be a Chebyshev set, we need stronger conditions on the Banach space.

Theorem C. [14] *Let $(B, \|\cdot\|)$ be a reflexive Banach space. Then B is strictly convex if and only if every nonempty closed convex subset $K \subset B$ is a Chebyshev set.*

Noting that every uniformly convex Banach space is reflexive and strictly convex, we see that Theorem C implies Theorem A immediately. Furthermore, as a special case of uniformly convex and uniformly smooth Banach space, every nonempty closed convex subset of a Hilbert space is a Chebyshev set.

From Theorem C, we have that in a reflexive Banach and strictly convex Banach space, every nonempty closed convex subset K is a Chebyshev set. Can we prove that in some Banach spaces, a nonempty subset is a Chebyshev set if and only if it is closed and convex? This is an open problem. Even in a special case of infinite-dimensional Hilbert space, it is an open problem too.

In 1934 L. N. H. Bunt proved that each Chebyshev set in a finite-dimensional Hilbert space must be convex (see [5]). From this result, we see that in a finite-dimensional Hilbert space, a nonempty subset is a Chebyshev set if and only if it is closed and convex.

In [12], G. G. Johnson gave an example: there exists an incomplete inner product space (not Hilbert space) that has a Chebyshev set which is not convex (M. Jiang completed the proof in 1993). Is there an infinity-dimensional Hilbert space that has a Chebyshev set which is not convex? As addressed above, it is unknown.

2. Continuity

In case B is a Hilbert space, it is known that for any closed convex subset K, the metric projection operator $P_K : B \to K$ is single valued and is not only continuous but also nonexpansive. In general Banach spaces, the nonexpansive property does not hold. Fortunately, there exist some continuity properties and some inequalities for some special Banach spaces. We need to recall some definitions before we study the continuity property of the operator P_K.

A Banach space B is uniformly convex if and only if its modulus of convexity δ satisfies the following inequality

$$\delta(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \ \|x\| = 1, \ \|y\| = 1, \ \|x-y\| = \varepsilon\right\} > 0,$$

for all $\varepsilon \in (0, 2]$.

It follows that δ is a strictly increasing, convex and continuous function from $(0, 2]$ to $[0, 1]$, and it is known that $\dfrac{\delta(\varepsilon)}{\varepsilon}$ is nondecreasing on $(0, 2]$.

A Banach space B is uniformly smooth if and only if its modulus of smoothness denoted by $\rho(\tau)$ satisfies the following

$$\rho(\tau) = \sup\left\{\frac{1}{2}\|x+y\| + \frac{1}{2}\|x-y\| - 1 : \ \|x\| = 1, \ \|y\| \le \tau\right\} > 0,$$

for all $\tau \in (0, \infty)$.

It can be shown that ρ is a convex and continuous function from $[0, \infty)$ to $[0, \infty)$ with the properties that $\dfrac{\rho(\tau)}{\tau}$ is nondecreasing, $\rho(\tau) \le \tau$, for all $\tau \ge 0$,

$\lim\limits_{\tau\to 0^+} \dfrac{\rho(\tau)}{\tau} = 0$, and $\rho(0) = 0$. For the details of the properties of δ and ρ, the reader is referred to [6], [18] and [20]. In [9], Goeble and Reich proved the following result.

Theorem D. ([9]) *The nearest point projection on to a closed convex subset of a uniformly convex Banach space is continuous.*

Noting again that every uniformly convex Banach space is reflexive and strictly convex, in [14], the present author extended the above result to more broad Banach spaces.

Theorem E. ([14]) *Let $(B, \|\cdot\|)$ be a reflexive and strictly convex Banach space and $K \subset B$ a nonempty convex subset. Then $P_K : B \to K$ is continuous.*

Since uniformly convex and uniformly smooth Banach spaces are reflexive and strictly convex, the above theorem implies that if $(B, \|\cdot\|)$ is a uniformly convex and uniformly smooth Banach space, then every nonempty closed convex subset $K \subset B$ is a Chebyshev set. In case B is a uniformly convex and uniformly smooth Banach space, the continuity property of the metric projection operator P_K can be given by Theorem B. In [5], Goebel directly proved the continuity. Furthermore, in 1992, Roach and Xu proved some inequalities (see [21] and [22]).

Theorem F. (Xu and Roach [22]). *Let M be a convex Chebyshev set of a uniformly convex and uniformly smooth Banach space X and $P : X \to M$ be the metric projection. Then for every x, y in X*

$$\|P(x) - P(y)\| \le \|x - y\| + 4(\|x - P(y)\| \vee \|P(x) - y\|)\cdot$$

$$\cdot\delta^{-1}\left(C_1\psi\left(\frac{\|x - y\|}{\|x - P(y)\| \vee \|y - P(x)\|}\right)\right),$$

where C_1 is a fixed constant and ψ is defined by

$$\psi(t) = \int_0^t \frac{\rho(s)}{s} ds.$$

Theorem G. (Xu and Roach [22]). *Let M be a convex Chebyshev set of a uniformly convex and uniformly smooth Banach space X and $P : X \to M$ be the metric projection. Then* *(i) P is a Lipschitz continuous mod M; namely, there exists a constant $k > 0$ such that*

$$\|P(x) - z\| \leq k\|x - z\|, \text{ for any } x \in X \text{ and } z \in M,$$

(ii) P is uniformly continuous on every bounded subset of X and, furthermore, there exist positive constants k_r for every $B_r = \{x \in X : \|x\| \leq r\}$ such that

$$\|P(x) - P(y)\| \leq \|x - y\| + k_r \delta^{-1}(\psi(\|x - y\|)), \text{ for any } x, y \in B_r,$$

where ψ is defined by Theorem F.

3. Existence of solutions of variational inequalities

Let B be a general Banach space, the normalized duality mapping $J : B \to 2^{B^*}$ is defined by

$$J(x) = \{j(x) \in B^* : \langle j(x), x\rangle = \|j(x)\|\|x\| = \|x\|^2 = \|j(x)\|^2\}.$$

Clearly $\|j(x)\|$ is the B^*-norm of $j(x)$ and $\|x\|$ is the B-norm of x. It is known that if B is uniformly convex and uniformly smooth, then J is single valued, strictly monotone, homogeneous, continuous and uniformly continuous operator on each bounded set. Furthermore, J is the identity in Hilbert spaces; i.e. $J = I$.

Let $(B, \|\cdot\|)$ be a Banach space and K a subset of B. Let $f : K \to B$ be a mapping. The variational inequality defined by the mapping F and the set K is:

$$VI(F, K) : \text{ find } x_* \in K, \text{ and } j(F(x_*)) \in J(F(x_*)) \text{ such that}$$

$$\langle j(F(x_*)), y - x_*\rangle \geq 0, \text{ for every } y \in K. \tag{1}$$

It is known that J is a single valued mapping if B^* is strictly convex. Hence if B^* is strictly convex, then the above definition can be restated as follows: the variational inequality defined by the mapping F and the set K is:

$$VI(F, K) : \text{ find } x_* \in K, \text{ such that } \langle J(F(x_*)), y - x_*\rangle \geq 0, \text{ for every } y \in K. \tag{2}$$

In 1994, Alber [1] introduced the generalized projections $\pi_K : B^* \to K$ and $\Pi_K : B \to K$ that are generalizations of metric projection P_K from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and studied their properties in detail. In [2]-[4], Alber presented some important properties of the generalized projections, such as continuity, some inequalities and some applications to approximate solutions of variational inequalities and J. von-Neumann intersection problem in a uniformly convex and uniformly smooth Banach space. In this paper, we use the metric projection P_k instead of π_K. The continuity property of the operator P_K stated in Theorem E can be used for studying the existence of solutions of variational inequalities. The inequalities given in Theorem F and G provide tools for approximating the solutions.

The following theorem provides a tool to solve a variational inequality by finding a fixed point of a certain operator.

Theorem H. (Li [14]) *Let $(B, \|\cdot\|)$ be a reflexive and smooth Banach space and $K \subset B$ a nonempty closed convex subset. For any given $x \in B$, $x_0 \in P_K(x)$ if and only if*

$$\langle J(x - x_0), x_0 - y\rangle \geq 0, \text{ for all } y \in K.$$

By using Theorem H, similar to the proof of Theorem 8 in [14], we can prove the following theorem.

Theorem 1. *Let $(B, \|\cdot\|)$ be a reflexive, strictly convex and smooth Banach space and $K \subset B$ a nonempty closed convex subset. Let $F : K \to B$ be a mapping. Then an element $x_* \in K$ is a solution of $VI(F, K)$ if and only if $x_* = P_K(x_* - \alpha F(x_*))$, for any $\alpha > 0$.*

Proof. Since B is reflexive, strictly convex and smooth, so is B^*. Noting that both J and P_K are single valued in this case, from Theorem H, $x_* = P_K(x_* - \alpha F(x_*))$, if and only if

$$\langle J((x_* - \alpha F(x_*)) - x_*), x_* - y\rangle \geq 0, \text{ for all } y \in K,$$

that is

$$\langle J(-\alpha F(x_*)), x_* - y\rangle \geq 0, \text{ for all } y \in K.$$

Since J is homogeneous and $\alpha > 0$, it is equivalent to the inequality

$$\langle J(F(x_*)), y - x_*\rangle \geq 0, \text{ for all } y \in K.$$

This theorem is proved.

Remark. From the proof of Theorem 1, we see that for an element $x_* \in K$, if there exists a number $\alpha > 0$ such that $x_* \in P_K(x_* - \alpha F(x_*))$, then x_* is a solution of $VI(F, K)$.

The Fan-KKM theorem, Leray-Schauder Alternative Theorems and the concept of exceptional family of element (EFE) play important roles for studying the existence of solutions of variational inequalities.

KKM mapping. Let K be a nonempty subset of a linear space X. A set-valued mapping $G : K \to 2^X$ is said to be a KKM mapping if for any finite subset $\{y_1, y_2, \dots, y_n\}$ of K, we have

$$co\{y_1, y_2, \dots, y_n\} \subseteq \bigcup_{i=1}^{n} G(y_i),$$

where $co\{y_1, y_2, \dots, y_n\}$ denotes the convex hull of $\{y_1, y_2, \dots, y_n\}$.

Fan-KKM Theorem. *Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $G : K \to 2^X$ be a KKM mapping with closed values. If there exists a nonempty compact convex subset D of C such that $\bigcap_{y \in D} G(y)$ is contained in a compact subset of K, then $\bigcap_{y \in K} G(y) \neq \emptyset$.*

The Fan-KKM Theorem has another version.

Fan-KKM Theorem. *Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $G : K \to 2^X$ be a KKM mapping with closed values. If there exists a point $y_0 \in K$ such that $G(y_0)$ is contained in a compact subset of K, then $\bigcap_{y \in K} G(y) \neq 0$.*

Theorem 2. *Let K be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space B. Let $F : K \to B$ be a continuous mapping. If there exists an element $y_0 \in K$ such that the subset of K*

$$\{x \in K : \ \|x - P_K(x - F(x))\| \leq \|y_0 - P_K(x - F(x))\|\} \tag{3}$$

is compact, then the variational inequality (2) has at least one solution.

Proof. From Theorem 1, we only need to prove that the following equation has a solution

$$x = P_K(x - F(x)).$$

Define $G : K \to 2^K$ as follows:

$$G(y) = \{x \in K : \ \|x - P_K(x - F(x))\| \leq \|y - P_K(x - F(x))\|\}.$$

It is clear that for all $y \in K$, we have $y \in G(y)$. From the continuity condition of F and the continuity property of P_K (Theorem E), it yields that for every $y \in K$, $G(y)$ is nonempty and closed.

Next we prove that the map $G : K \to 2^K$ is a KKM map in K. Let n be an arbitrary positive integer. For any $y_1, y_2, \dots, y_n \in K$ and $0 < \lambda_1, \lambda_2, \dots, \lambda_n \leq 1$, such that $\sum_{i=1}^{n} \lambda_i = 1$, let $v = \sum_{i=1}^{n} \lambda_i y_i$. We have

$$\begin{aligned}\|v - P_K(v - F(v))\| &= \left\| \sum_{i=1}^{n} \lambda_i y_i - P_K(v - F(v)) \right\| \\ &\leq \sum_{i=1}^{n} \lambda_i \|y_i - P_K(v - F(v))\| \\ &\leq \max_{1 \leq i \leq n} \{\|y_i - P_K(v - F(v))\|\}.\end{aligned}$$

Hence there is at least one number $j = 1, 2, \dots, n$, such that

$$\|v - P_K(v - F(v))\| \leq \|y_j - P_K(v - F(v))\|,$$

i.e., $v \in G(y_j)$. We obtain $v = \sum_{i=1}^{n} \lambda_i y_i \in \bigcup_{i=1}^{n} G(y_i)$. Thus K is a KKM mapping.

Condition (3) implies that $G(y_0)$ is compact. From the Fan-KKM Theorem, we have $\bigcap_{y \in K} G(y) \neq \emptyset$. Then there exist at least one element $x_* \in \bigcap_{y \in K} G(y)$, that is,

$$\|x_* - P_K(x_* - F(x_*))\| \leq \|y - P_K(x_* - F(x_*))\|, \text{ for all } y \in K.$$

Taking $y = P_K(x_* - F(x_*)$, we obtain $\|x_* - P_K(x_* - F(x_*))\| = 0$, that is,

$$x_* = P_K(x_* - F(x_*)).$$

This theorem is proved.

Corollary 3. *Let K be a nonempty compact convex subset of a reflexive, strictly convex and smooth Banach space B. Let $F : K \to B$ be a continuous mapping. Then the variational inequality (2) has a solution.*

Proof. This corollary follows immediately from Theorem 2.

Corollary 4. *Let B be a uniformly convex and uniformly smooth Banach space and K a closed convex subset of B. Let $F : K \to B$ be a continuous* *mapping. If there exists an element* $y_0 \in K$ *such that the subset of* K

$$\{x \in K: \ \|x - P_K(x - F(x))\| \leq k\|y_0 - (x - F(x))\|\} \tag{3'}$$

is compact, where k *is the constant given in Theorem F, then the variational inequality (2) has a solution.*

Proof. Since B is uniformly convex and uniformly smooth, from Theorem C, K is a Chebyshev set. Since $y_0 \in K$, applying Theorem G, we get

$$\|y_0 - P_K(x - F(x))\| \leq k\|y_0 - (x - F(x))\|, \text{ for all } x \in B.$$

It implies

$$\{x \in K: \ \|x - P_K(x - F(x))\| \leq \|y_0 - P_K(x - F(x))\|\}$$

$$\subseteq \{x \in K: \ \|x - P_K(x - F(x))\| \leq k\|y_0 - (x - F(x))\|\}.$$

Therefore the set $\{x \in K: \ \|x - P_K(x - F(x))\| \leq \|y_0 - P_K(x - F(x))\|\}$ is compact and the corollary is achieved immediately by following Theorem 2.

Leray-Schauder Alternative. *Let* X *be a closed subset of a locally convex space* E *such that* $0 \in \text{int}(X)$ *and* $f: X \to E$ *a compact u.s.c. set-valued mapping with non-empty compact contractible values. If* f *is fixed point free, then it satisfies the following Leray-Schauder condition:*

there exists $(\lambda_*, x_*) \in (0, 1) \times \partial X$ *such that* $x_* \in \lambda_* f(x_*)$.

We recall that a mapping $T: B \to B$ is said to be completely continuous if T is continuous and for any bounded set $D \subset B$, $T(D)$ is relatively compact. A mapping $F: B \to B$ is said to be a completely continuous field if F has a representation $F(x) = x - T(x)$, for all $x \in B$, where $T: B \to B$ is a completely continuous mapping.

Theorem 4. *Let* $(B, \|\cdot\|)$ *be a reflexive, strictly convex and smooth Banach space,* $K \subset B$ *a closed convex cone and* $F: K \to B$ *a completely continuous field with the representation* $F(x) = x - T(x)$*. Then* F *has at least one of the following two properties:*

(i) the problem $VI(F, K)$ *has a solution;*

(ii) for all $r > 0$*, there exist* $\lambda_r \in (0, 1)$ *and* $x_r \in K$ *satisfying* $\|x_r\| = r$*. Let* $F_r(x) = x - \lambda_r T(x)$*. Then* x_r *is a solution of the problem* $VI(F_r, K)$.

Proof. From Theorem 1, the problem $VI(F, K)$ has a solution if and only if the following mapping

$$\phi_K(x) = P_K(x - F(x)) = P_K(T(x)), \text{ for all } x \in K,$$

has a fixed point. If it has a fixed point, it is clearly in K.

Hence if $\phi_K(x)$ has a fixed point, the problem $VI(F,K)$ has a solution. Therefore the proof is completed.

Assume that the problem $VI(F,K)$ has no solution. Obviously the mapping ϕ_K is fixed point free. Define a mapping ϕ from B to K as follows:

$$\phi(x) = \phi_K(P_K(x)) = P_K(P_K(x) - F(P_K(x))) = P_K(T(P_K(x))),$$

for any $x \in B$.

Let $\mathcal{P}(\phi_K), \mathcal{P}(\phi)$ denote the sets of fixed points of ϕ_K and ϕ respectively. It is clear that $\mathcal{P}(\phi_K), \mathcal{P}(\phi)$ are subsets of K. Since $P_K(x) = x$, for all $x \in K$, we obtain $\phi|_K = \phi_K$. Therefore $\mathcal{P}(\phi_K) = \mathcal{P}(\phi)$. Then the hypothesis that the mapping ϕ_K has no fixed point in K implies $\mathcal{P}(\phi) = \emptyset$.

The continuity property of P_K and the completely continuous condition on T imply that the operator ϕ is continuous and compact from B to K.

For any $r > 0$, we define a closed convex set

$$D_r = \{x \in B : \ \|x\| \leq r\}.$$

It is clear that the set D_r has a non-empty interior and $0 \in \text{int}(D_r)$.

The property that the mapping ϕ has no fixed point in K implies that the mapping ϕ has no fixed point in D_r, for any $r > 0$. As ϕ is restricted to D_r, applying the Leray-Schauder type alternative, we have that there exist $x_r \in \partial D_r$ and $\lambda_r \in (0,1)$ such that

$$x_r = \lambda_r \phi(x_r) = \lambda_r P_K(T(P_K(x_r))),$$

that is

$$\frac{1}{\lambda_r} x_r = \phi(x_r) = P_K(T(P_K(x_r))).$$

Since $P_K(T(P_K(x_r))) \in K$ and K is a cone, we have $x_r \in K$. Then we obtain $P_K(x_r) = x_r$. Therefore,

$$\frac{1}{\lambda_r} x_r = \phi(x_r) = P_K(T(x_r)). \tag{4}$$

From Theorem H, we have

$$\left\langle J\left((T(x_r)) - \frac{1}{\lambda_r} x_r\right), \frac{1}{\lambda_r} x_r - y \right\rangle \geq 0, \text{ for all } y \in K. \tag{5}$$

Since J is homogeneous and K is a cone, (5) is equivalent to

$$\langle J(\lambda_r T(x_r)) - x_r), x_r - y\rangle \geq 0, \text{ for all } y \in K, \tag{6}$$

that is,

$$\langle J(x_r - \lambda_r T(x_r)), y - x_r \geq 0, \text{ for all } y \in K. \tag{7}$$

It implies that x_r is a solution of $VI(F_r, K)$. From $x_r \in \partial D_r$, we have $\|x_r\| = r$, for all $r > 0$. This theorem is proved.

Remark. If the set $\{x_r\}$ satisfies condition (ii) of Theorem 4, is said to be an exceptional family of elements of F with respect to K.

From the proof of Theorem 4, we can explain the conclusion of Theorem 4 as follows.

Corollary 5. *Let $(B, \|\cdot\|)$ be a reflexive, strictly convex and smooth Banach space, $K \subset B$ a closed convex cone and $F : K \to B$ a completely continuous field with the representation $F(x) = x - T(x)$. Then F has at least one of the following two properties:*

(i) the mapping $P_K \circ T$ has a fixed point (an eigenvector with eigenvalue 1);

(ii) for all $r > 0$, $P_K \circ T$ has an eigenvector $x_r \in K$ satisfying $\|x_r\| = r$ with eigenvalue $\mu_r > 1$.

Proof. Let λ_r and x_r be given in the proof of Theorem 4 and let $\mu_r = \dfrac{1}{\lambda_r} > 1$, for all $r > 0$. The corollary follows immediately from (4) in the proof of Theorem 4.

Comments. Let $(B, \|\cdot\|)$ be a Banach space and K a subset of B. Let $f : K \to B^*$ be a mapping. We may define another type of variational inequality defined by the mapping f and the set K:

$$VI(f, K) : \text{ find } x_* \in K, \text{ such that } \langle f(x_*), y - x_* \rangle \geq 0, \text{ for every } y \in K. \tag{2'}$$

It is known that $J^* : B^* \to B$ and $J : B \to B^*$ are single valued mappings if B is reflexive, strictly convex and smooth. We define a mapping $F : K \to B$ by

$$F(x) = J^*(f(x)), \text{ for every } x \in K.$$

Noting $J \circ J^* = I^*$, we see that to solve the variational inequality $VI(f, K)$ defined by (2') is equivalent to solve the variational inequality $VI(F, K)$ defined by (2).

4. Approximations of solutions of variational inequalities

In this section, we always assume that B is a uniformly convex and uniformly smooth Banach space. Applying Theorems F and G, the solutions of variational inequalities can be approximated by some iterated sequences. We list some results obtained by the present author and Rhoades. For the details of the results, the reader is referred to [15].

Theorem 6. ([15]) *Let $(B, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space and K a nonempty closed convex subset of B. Let $F : K \to B$ be a continuous mapping. Suppose $VI(F, K)$ has a solution $x_* \in K$ and F satisfies the following condition*

$$\|x - x_* - (F(x) - F(x_*))\| + k_r\delta^{-1}(\rho\|x - x_* - (F(x) - F(x_*))\|) \leq \|x - x_*\|, \text{ for every } x \in K, \tag{8}$$

where k_r is a positive constant given in Theorem G that depends on the bounded subset K. For any $x_0 \in K$, we define the Mann type iteration scheme as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_K(x_n - F(x_n)), \quad n = 1, 2, 3, \ldots \tag{9}$$

where $\{\alpha_n\}$ satisfies conditions

(a) $0 \leq \alpha_n \leq 1$, *for all* n,

(b) $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Then there exists a subsequence $\{n(i)\} \subseteq \{n\}$ such that $\{x_{n(i)}\}$ converges to a solution x' of $VI(F, K)$.

Theorem 7. ([15]) *Let B, K, F be given as in Theorem 6. If the inequality (8) holds for all solutions of $VI(F, K)$, then the sequence $\{x_n\}$ defined by (9) converges to a solution x' of the $VI(F, K)$ problem.*

In the case K is unbounded, for example if K is a closed convex cone, we have the following theorem for our estimation.

Theorem 8. ([15]) *Let $(B, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space and K a nonempty closed convex subset of B. Let $F : K \to B$ be a continuous mapping such that the $VI(F, K)$ problem has a solution $x_* \in K$. If there exist positive constants κ and λ satisfying the following conditions*

(i) $\|x - x_* - (F(x) - F(x_*))\| \leq \kappa\|x - x_*\|$, *for every* $x \in K$;

(ii) $t^{-1}\delta^{-1}(t) \leq \lambda$, *for all* t;

(iii) $(\kappa + 4C_1\kappa\lambda) < 1$, *where C_1 is the constant given in Theorem F,*

then the sequence $\{x_n\}$ defined by (9) converges to the solution x_* of the $VI(F, K)$ problem.

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