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AN EXISTENCE AND UNIQUENESS RESULT FOR NONLINEAR INTEGRAL EQUATIONS

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Abstract. Let X be a Banach space, $\{S(s) \mid s \geq 0\}$ a C_0-semigroup of contractions on X and ν a positive measure. We study the solvability of the nonlinear equation

$$u \in \mathcal{C}([a,b];X), \quad F\left(t, u(t), \int_a^t S(t-s)d(u\nu)(s)\right) = f(t), \quad t \in [a,b].$$

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1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{C}([a,b];X)$ the Banach space of all continuous vector-valued functions defined on $[a,b]$ endowed with the supremum norm $\|\cdot\|_{\mathcal{C}}$. Consider $\mathcal{L}(X)$ the Banach space of all continuous linear operators on X and $\{S(s) \mid s \geq 0\}$ a C_0-semigroup of contractions on X which is continuous from $(0,\infty)$ to $\mathcal{L}(X)$ in the uniform operator topology. Denote by $\mathcal{M}_{[a,b]}$ the σ-algebra of the Lebesgue measurable sets which are contained in $[a,b]$. Consider ν a positive measure which is defined on $\mathcal{M}_{[a,b]}$. Suppose that ν is different from the null measure and also that $\nu(\{s\}) = 0$ for each

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$s \in [a, b]$. If $u \in \mathcal{C}([a, b]; X)$ then $u\nu$ is the measure defined by u as density and by ν as base. We are looking for $u \in \mathcal{C}([a, b]; X)$ such that

$$F\left(t, u(t), \int_a^t S(t-s)d(u\nu)(s)\right) = f(t), \quad t \in [a, b]. \tag{1}$$

We assume that $F : [a, b] \times X^2 \to X$ is a continuous nonlinear function and also that $f \in \mathcal{C}([a, b]; X)$. Consider $t \in [a, b]$. As we shall see in the next part, the integral $\int_a^t S(t-s)d(u\nu)(s)$ is a kind of the Riemann-Stieltjes integral on $[a, t]$. In fact, making the notation $g_u(s) = u\nu([a, s])$, $s \in [a, b]$ we see that g_u is a vector-valued function of bounded variation and further, our integral is a particular case of the Riemann-Stieltjes integral of an operator-valued function with respect to a vector-valued function of bounded variation (see [6], p.206 and [7], p.3184). More precisely

$$\int_a^t S(t-s)d(u\nu)(s) = \int_a^t S(t-s)dg_u(s),$$

where $\int_a^t S(t-s)dg_u(s)$ is the Riemann-Stieltjes integral on $[a, t]$ of the operator-valued function $s \in [a, t] \to S(t-s) \in \mathcal{L}(X)$ with respect to the function $g_u \in BV([a, b]; X)$ (see [6], p.206 and [7], p.3184). We denoted by $BV([a, b]; X)$ the set of all vector-valued functions of bounded variation on $[a, b]$.

The study of equation (1) has been suggested to us by the following Cauchy problem which can be found in [6], p.269:

$$\begin{cases} du = \{Au\}dt + dh_u, \\ u(a) = \xi. \end{cases} \tag{2}$$

The above Cauchy problem is equivalent by the next equation

$$u(t) = S(t-a)\xi + \int_a^t S(t-s)dh_u(s). \tag{3}$$

In the context of (2) and (3), $A : D(A) \subset X \to X$ is the infinitesimal generator of the C_0-semigroup of contractions $\{S(s) \mid s \geq 0\}$, $\xi \in D(A)$ and $h_u = H(u)$ where $H : \mathcal{C}([a, b]; X) \to BV([a, b]; X)$.

We adapt some steps of the technique proposed by Adriana Buică in [2]. So, using the theory of nearness between two operators and the Banach Fixed Point Theorem we obtain an existence and uniqueness result for equation (1).

2. Preliminaries

Consider $t \in (a,b]$ and $u \in \mathcal{C}([a,b];X)$. Let $\mathcal{P}[a,t]$ be the set of all partitions of the interval $[a,t]$. Consider $\mathcal{P} \in \mathcal{P}[a,t]$ and let us denote $\mathcal{P} : a = t_0 < t_1 < \cdots < t_n = t$. We also denote by $\lambda(\mathcal{P})$ the norm of the partition $\mathcal{P}$. We recall that $\lambda(\mathcal{P}) = \max\{t_i - t_{i-1} \mid i = 1,2,\ldots,n\}$. Consider $\tau_i \in [t_{i-1},\ t_i], i = 1,2,\ldots,n$. The measure defined by u as density and by ν as base is a measure of bounded variation and its variation on $[a,b]$ satisfies

$$\mathrm{Var}(u\nu,[a,b]) = \sup_{\mathcal{P}\in\mathcal{P}[a,b]} \sum_{i=1}^{n} \|u\nu([t_{i-1},t_i])\| =$$

$$= \sup_{\mathcal{P}\in\mathcal{P}[a,b]} \sum_{i=1}^{n} \| \int_{t_{i-1}}^{t_i} u(s)d\nu(s)\| \le \nu([a,b])\|u\|_{\mathcal{C}}.$$

If $g_u(s) = u\nu([a,s]),\ s \in [a,b]$, we easily prove that $g_u \in BV([a,b],X) \cap \mathcal{C}([a,b];X)$. Indeed the variation of g_u on $[a,b]$ is given by

$$\mathrm{Var}(g_u;[a,b]) = \sup_{\mathcal{P}\in\mathcal{P}_{[a,b]}} \sum_{i=1}^{n} \|g_u(t_i) - g_u(t_{i-1})\| =$$

$$= \sup_{\mathcal{P}\in\mathcal{P}_{[a,b]}} \sum_{i=1}^{n} \|u\nu([t_{i-1},t_i])\| = \mathrm{Var}(u\nu,[a,b]).$$

The assumptions regarding the measure ν yield that $g_u \in \mathcal{C}([a,b];X)$. Next, using the properties of a C_0-semigroup of contractions, the continuity of the function $s \in [a,t] \to S(t-s)u(s) \in X$ follows.

The integral $\int_a^t S(t-s)d(u\nu)(s)$ is a kind of Riemann-Stieltjes integral and it is defined in the following way

$$\int_a^t S(t-s)d(u\nu)(s) = \lim_{\substack{\lambda(\mathcal{P})\downarrow 0 \\ \mathcal{P}\in\mathcal{P}[a,t]}} \sum_{i=1}^{n} S(t-\tau_i)(u\nu([t_{i-1},t_i])).$$

The limit on the right-hand side exists in the norm topology of X because the semigroup $\{S(s) \mid s \geq 0\}$ is continuous from $(0, \infty)$ to $\mathcal{L}(X)$ in the uniform operator topology (see [6], p.208, Theorem 9.1.1 and [7], p.3186, Theorem 2.1). We notice that

$$\int_a^t S(t-s)d(u\nu)(s) = \int_a^t S(t-s)dg_u(s).$$

Therefore, taking into account that $g_u \in \mathcal{C}([a,b];X)$ and using Corollary 9.2.1 from [6], p.213 or Corollary 3.1 from [7], p.3190, it follows that the integral $\displaystyle\int_a^t S(t-s)d(u\nu)(s)$ defines a continuous function as function of $t \in [a,b]$.

Proposition 2.1. *The following equality holds*

$$\int_a^t S(t-s)d(u\nu)(s) = \int_a^t S(t-s)u(s)d\nu(s).$$

Proof. First let us remind that

$$\int_a^t S(t-s)u(s)d\nu(s) = \lim_{\substack{\lambda(\mathcal{P}) \downarrow 0 \\ \mathcal{P} \in \mathcal{P}[a,t]}} \sum_{i=1}^{n} \nu[t_{i-1}, t_i] S(t-\tau_i)(u(\tau_i)). \tag{4}$$

Consider $\mathcal{P} \in \mathcal{P}[a,t]$, $\mathcal{P} : a = t_0 < t_1 < \cdots < t_n = t$ and $\tau_i \in [t_{i-1}, t_i]$, $i = 1, 2, \cdots, n$. There are obvious the following equalities

$$\sum_{i=1}^{n} S(t-\tau_i)(u\nu([t_{i-1},t_i])) = \sum_{i=1}^{n} S(t-\tau_i)\left(\int_{t_{i-1}}^{t_i} u(s)d\nu(s)\right) =$$
$$= \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} S(t-\tau_i)u(s)d\nu(s) =$$
$$= \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} [S(t-\tau_i)u(s) - S(t-s)u(s)]d\nu(s) +$$
$$+ \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} S(t-s)u(s)d\nu(s) = \tag{5}$$
$$= \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} [S(t-\tau_i)u(s) - S(t-s)u(s)]d\nu(s) +$$
$$+ \int_a^t S(t-s)u(s)d\nu(s).$$

We shall prove that

$$\lim_{\substack{\lambda(\mathcal{P})\downarrow 0\\ \mathcal{P}\in\mathcal{P}[a,t]}} \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} [S(t-\tau_i)u(s) - S(t-s)u(s)]d\nu(s) = 0.$$

Consider $\varepsilon > 0$. We easily obtain

$$\|S(t-\tau_i)u(s) - S(t-s)u(s)\| \le \|S(t-\tau_i)u(s) - S(t-\tau_i)u(\tau_i)\| +$$
$$+ \|S(t-\tau_i)u(\tau_i) - S(t-s)u(s)\| \le \tag{6}$$
$$\le \|u(s) - u(\tau_i)\| + \|S(t-\tau_i)u(\tau_i) - S(t-s)u(s)\|.$$

From the uniform continuity of the function $s \in [a,t] \to u(s) \in X$ we find $\delta_1 > 0$ such that for $\mathcal{P} \in \mathcal{P}[a,t]$ with $\lambda(\mathcal{P}) < \delta_1$ and $s \in [t_{i-1}, t_i]$ we get

$$\|u(s) - u(\tau_i)\| < \frac{\varepsilon}{2\nu([a,b])}, \quad i = 1, 2, \ldots, n. \tag{7}$$

From the uniform continuity of the function $s \in [a,t] \to S(t-s)u(s) \in X$ we find $\delta_2 > 0$ such that for $\mathcal{P} \in \mathcal{P}[a,t]$ with $\lambda(\mathcal{P}) < \delta_2$ and $s \in [t_{i-1}, t_i]$ we get

$$\|S(t-\tau_i)u(\tau_i) - S(t-s)u(s)\| < \frac{\varepsilon}{2\nu([a,b])}, \quad i = 1, 2, \ldots, n. \tag{8}$$

Let us denote $\delta = \min\{\delta_1, \delta_2\}$. Taking into account (6), (7) and (8) we can see that for each $\mathcal{P} \in \mathcal{P}[a,t]$ such that $\lambda(\mathcal{P}) < \delta$ we get

$$\begin{aligned} &\|\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} [S(t-\tau_i)u(s) - S(t-s)u(s)] d\nu(s)\| \le \\ &\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|S(t-\tau_i)u(s) - S(t-s)u(s)\| d\nu(s) \le \varepsilon. \end{aligned} \tag{9}$$

This completes the proof. □

3. The Main Result

We prove the main result of the paper, Theorem 3.1, which is an existence and uniqueness result for equation (1).
Consider

- $T : \mathcal{C}([a,b];X) \to \mathcal{C}([a,b];X), \quad Tu(t) = \displaystyle\int_a^t S(t-s)u(s)d\nu(s);$
- $\mathcal{C}_T([a,b];X) = \{Tu \mid u \in \mathcal{C}([a,b];X)\}.$

We prove that $\mathcal{C}_T([a,b];X)$ is a Banach space and that T is a continuous linear operator and a bijective map between $\mathcal{C}([a,b];X)$ and $\mathcal{C}_T([a,b];X)$.

Proposition 3.1. *The mapping* $T : \mathcal{C}([a,b];X) \to \mathcal{C}([a,b];X)$

$$Tu(t) = \int_a^t S(t-s)u(s)d\nu(s)$$

is an injective continuous linear operator.

Proof. Since $\|Tu\|_{\mathcal{C}} \le \nu([a,b])\|u\|_{\mathcal{C}}$ for every $u \in \mathcal{C}([a,b];X)$ we conclude that T is continuous. Consider $u \in \mathcal{C}([a,b];X)$ and suppose that $Tu(t) = 0$ for every $t \in [a,b]$. This means

$$\int_a^t S(t-s)u(s)d\nu(s) = 0, \quad t \in [a,b].$$

We shall prove that u is the null function. To this aim, consider $t \in (a, b]$ and $\varepsilon > 0$ such that $t - \varepsilon > a$. We have

$$S(\varepsilon) \int_a^{t-\varepsilon} S(t - \varepsilon - s)u(s)d\nu(s) = 0,$$

hence

$$\int_a^{t-\varepsilon} S(t - s)u(s)d\nu(s) = 0.$$

Now, consider the function $h(s) = S(t - s)u(s),\ s \in [a, t]$. It is easy to see that

$$\int_{t-\varepsilon}^{t} h(s)d\nu(s) = 0, \quad \text{for every } \varepsilon > 0 \text{ such that } t - \varepsilon > a.$$

Therefore, $0 \in \bigcap_{\varepsilon>0} \overline{conv}\ h([t - \varepsilon, t])$, where $\overline{conv}\ h([t-\varepsilon, t])$ is the closed convex hull of the set $h([t - \varepsilon, t])$. We shall prove that

$$\bigcap_{\varepsilon>0} \overline{conv}\ h([t - \varepsilon, t]) = \{h(t)\}.$$

It is clear that

$$h(t) \in \bigcap_{\varepsilon>0} \overline{conv}\ h([t - \varepsilon, t]).$$

Suppose that there exists $y \in X,\ y \neq h(t)$ such that

$$y \in \bigcap_{\varepsilon>0} \overline{conv}\ h([t - \varepsilon, t]).$$

By setting $\varepsilon_0 = \|h(t) - y\|$ we obviously have that $\varepsilon_0 > 0$. Using the fact that h is a continuous function we can choose $\delta > 0$ such that $t - \delta > a$ and for $s \in [a, t], \mid s - t \mid < \delta$ it is true the inequality $\|h(s) - h(t)\| < \dfrac{\varepsilon_0}{4}$. Since $y \in \overline{conv}\ h([t - \delta, t])$, it follows that there exists a sequence (y_n) such that $y_n \in conv\ h[t-\delta, t], n \in \mathbb{N}$ and $y = \lim y_n$. Hence, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ it is true the inequality $\|y_n - y\| < \dfrac{\varepsilon_0}{4}$. From $y_{n_0} \in conv\ h([t-\delta, t])$ it results that there exist the sets $\{y^i_{n_0} \mid i = 1, 2, \ldots, p\} \subset h([t - \delta, t]),\ \{\alpha_i \mid i = 1, 2, \ldots, p\} \subset [0, 1]$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_p = 1$ and

$$y_{n_0} = \alpha_1 y^1_{n_0} + \alpha_2 y^2_{n_0} + \cdots + \alpha_p y^p_{n_0}.$$

Further, consider $t^i_{n_0} \in [t - \delta, t]$ such that $y^i_{n_0} = h(t^i_{n_0}),\ i = 1, 2, \ldots, p$. This means

$$y_{n_0} = \alpha_1 h(t^1_{n_0}) + \alpha_2 h(t^2_{n_0}) + \cdots + \alpha_p h(t^p_{n_0}).$$

From the continuity of h, the following inequalities hold

$$\|h(t^i_{n_0}) - h(t)\| < \frac{\varepsilon_0}{4}, \quad i = 1, 2, \dots, p.$$

We also see that

$$\|h(t) - y_{n_0}\| =$$

$$= \|\alpha_1(h(t) - h(t^1_{n_0})) + \alpha_2(h(t) - h(t^2_{n_0})) + \cdots + \alpha_p(h(t) - h(t^p_{n_0}))\| \leq$$

$$\leq \alpha_1\|h(t) - h(t^1_{n_0})\| + \alpha_2\|h(t) - h(t^2_{n_0})\| + \cdots + \alpha_p\|h(t) - h(t^p_{n_0})\| < \frac{\varepsilon_0}{4}.$$

Therefore

$$\|h(t) - y\| \leq \|h(t) - y_{n_0}\| + \|y_{n_0} - y\| < \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{2}. \tag{10}$$

But $\|h(t) - y\| = \varepsilon_0$ and this contradicts (10). The consequence is $h(t) = 0$ so $S(t-t)u(t) = 0$ and further $u(t) = 0$. Taking into account that t was arbitrary it results that $u(t) = 0$ for every $t \in (a, b]$ and from the continuity of u we obtain $u \equiv 0$. □

Corollary 3.1. $\mathcal{C}_T([a, b]; X) = \{Tu \mid u \in \mathcal{C}([a, b]; X)\}$ *is a Banach space and the mapping*

$$T : \mathcal{C}([a, b]; X) \to \mathcal{C}_T([a, b]; X), \quad Tu(t) = \int_a^t S(t-s)u(s)d\nu(s)$$

is a bijective continuous linear operator.

Proof. Since T is continuous it results that T has closed graph, hence $\mathcal{C}_T([a, b]; X)$ is a closed subspace of $\mathcal{C}([a, b]; X)$. Therefore, $\mathcal{C}_T([a, b]; X)$ is a Banach space. From a corollary of the Open Mapping Theorem we conclude that $T^{-1} \in \mathcal{L}(\mathcal{C}_T([a, b]; X), \mathcal{C}([a, b]; X))$, where $\mathcal{L}(\mathcal{C}_T([a, b]; X), \mathcal{C}([a, b]; X))$ is the Banach space of all continuous linear operators between $\mathcal{C}_T([a, b]; X)$ and $\mathcal{C}([a, b]; X)$. □

Consider $m > 0$ such that $m\|u\|_{\mathcal{C}} \leq \|Tu\|_{\mathcal{C}}$ for all $u \in \mathcal{C}([a, b]; X)$.
For $w \in \mathcal{C}([a, b]; X)$ we define the mapping

$$A_w : \mathcal{C}([a, b]; X) \to \mathcal{C}([a, b]; X), \quad A_w(u)(t) = F(t, w(t), Tu(t)), \quad t \in [a, b].$$

Theorem 3.1. *Suppose that*
(i) For every $(w, u) \in \mathcal{C}^2([a, b]; X)$ *we have* $A_w(u) \in \mathcal{C}_T([a, b]; X)$*;*
(ii) There exist $\alpha > 0$ *and* $\beta \in (0, 1)$ *such that for all* $u, v, w \in \mathcal{C}([a, b]; X)$ *we have*

$$\|Tu - Tv - \alpha(A_w u - A_w v)\|_{\mathcal{C}} \le \beta \|Tu - Tv\|_{\mathcal{C}};$$

(iii) For every $w_1, w_2, u \in \mathcal{C}([a, b]; X)$

$$\|A_{w_1} u - A_{w_2} u\|_{\mathcal{C}} \le \|w_1 - w_2\|_{\mathcal{C}}.$$

If $m > \dfrac{\alpha}{1-\beta}$ *then equation (1) has a unique solution.*

Proof. Consider $w \in \mathcal{C}([a, b]; X)$. We shall prove that the mapping

$$A_w : \mathcal{C}([a, b]; X) \to \mathcal{C}_T([a, b]; X), \quad A_w(u)(t) = F(t, w(t), Tu(t)), \quad t \in [a, b],$$

is bijective. If $u, v \in \mathcal{C}([a, b]; X)$ and $A_w u = A_w v$ it results $\|Tu - Tv\|_{\mathcal{C}} \le \beta \|Tu - Tv\|_{\mathcal{C}}$, so $Tu = Tv$ and this means $u = v$. Let us denote $B = T - \alpha A_w$. We have the inequality $\|Bu - Bv\|_{\mathcal{C}} \le \beta \|Tu - Tv\|_{\mathcal{C}}$. From the hypothesis (ii), the continuity of A_w follows. Consider $g \in \mathcal{C}_T([a, b]; X)$. We shall see that the equation $A_w u = g$ has a solution $u \in \mathcal{C}([a, b]; X)$. Let u_0 be in $\mathcal{C}([a, b]; X)$. We define the sequence $(u_n), u_n \in \mathcal{C}([a, b]; X), n \in \mathbb{N}$ as follows

$$Tu_{n+1} = Tu_n - \alpha A_w u_n + \alpha g, \quad n \in \mathbb{N}. \tag{11}$$

For every $n \in \mathbb{N}^*$ we obtain the following inequalities

$$\|Tu_{n+1} - Tu_n\|_{\mathcal{C}} = \|Bu_n - Bu_{n-1}\|_{\mathcal{C}} \le$$

$$\le \beta \|Tu_n - Tu_{n-1}\|_{\mathcal{C}} \le \cdots \le \beta^n \|Tu_1 - Tu_0\|_{\mathcal{C}}$$

and further, for $p \in \mathbb{N}^*$ it follows

$$\|Tu_{n+p} - Tu_n\|_{\mathcal{C}} \le \beta^n \frac{1-\beta^p}{1-\beta} \|Tu_1 - Tu_0\|_{\mathcal{C}} < \beta^n \frac{1}{1-\beta} \|Tu_1 - Tu_0\|_{\mathcal{C}}.$$

Hence (Tu_n) is a Cauchy sequence in the Banach space $\mathcal{C}_T([a, b]; X)$. Therefore there exists $y \in \mathcal{C}_T([a, b]; X)$ such that $Tu_n \to y$. But T is a bijective mapping, so, there exists $u \in \mathcal{C}([a, b]; X)$ such that $Tu = y$. Taking into account that $u = T^{-1} y$ and that $T^{-1} \in \mathcal{L}(\mathcal{C}_T([a, b]; X), \mathcal{C}([a, b]; X))$ it results that $u_n \to u$. Passing to the limit in (11) we obtain $A_w(u) = g$.
Now, consider the operator

$$U : \mathcal{C}([a, b]; X) \to \mathcal{C}_T([a, b]; X), \quad Uw = Tu_w,$$

where u_w is the solution of the equation $A_w(u) = f$.
Consider also the operator

$$Q : \mathcal{C}([a,b];X) \to \mathcal{C}([a,b];X), \quad Q = T^{-1}U.$$

We prove that Q has a unique fixed point. Indeed, if $w_1, w_2 \in \mathcal{C}([a,b];X)$ then

$$\|T^{-1}Uw_1 - T^{-1}Uw_2\|_{\mathcal{C}} \le \frac{1}{m}\|Tu_{w_1} - Tu_{w_2}\|_{\mathcal{C}} \le$$

$$\le \frac{1}{m}\frac{\alpha}{1-\beta}\|A_{w_1}u_{w_1} - A_{w_1}u_{w_2}\|_{\mathcal{C}} =$$

$$= \frac{1}{m}\frac{\alpha}{1-\beta}\|A_{w_2}u_{w_2} - A_{w_1}u_{w_2}\|_{\mathcal{C}} \le q\|w_1 - w_2\|_{\mathcal{C}},$$

where $q = \dfrac{\alpha}{m(1-\beta)}$. Hence, there exists $w_0 \in \mathcal{C}([a,b];X)$ such that $w_0 = T^{-1}U(w_0)$. But this means that $w_0 = u_{w_0}$ and further $F(t, w_0(t), Tw_0(t)) = f(t),\ t \in [a,b]$, and the proof is complete. □

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