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FIXED POINT THEOREMS OF KRASNOSELSKII TYPE IN A SPACE OF CONTINUOUS FUNCTIONS

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Abstract. A well known result due to Krasnoselskii ensures the existence of a fixed point to an operator $K = A + B$ which is defined on a closed convex and bounded subset of a Banach space X, where A is a contraction operator and B is a compact operator. In the particular case when $X = C\left(J, \mathbb{R}^N\right)$, the condition B is compact can be replaced by a weaker one, of Lipschitz type in an integral form. In the present Note, on a closed convex and bounded subset of the space $X = C\left(J, \mathbb{R}^N\right)$ ($J \subset \mathbb{R}$ being a compact or not compact interval) one considers an operator $K = A + B + C$, where A and B fulfill the conditions mentioned above and C is a compact operator. To this operator K, certain theorems of existence of at least one fixed point are presented and some particular cases are distinguished.
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1. Introduction

Two main results of the fixed point theory are Schauder's theorem and the Banach Contraction Principle. Krasnoselskii combined them into the following result (see [9]).

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Theorem 1.1. *Let M be a closed convex and bounded subset of a Banach space X. Suppose that A and C map M into X such that:*

(1) A is $\alpha-$contraction with $\alpha < 1$;

(2) C is compact operator;

(3) $Ax + Cy \in M, \qquad \forall x,\, y \in M.$

Then there exists $y \in M$ such that

$$y = Ay + Cy. \tag{1.1}$$

(Remark that C is a compact operator if it is continuous and $\overline{CM}$ is a compact subset of X, for every bounded set M.)

This result is captivating and it has many interesting applications. The proof idea consists in the fact that hypothesis (1) ensures the existence and the continuity to operator $(I - A)^{-1}$ (where I denotes the identity operator). Then the solutions to (1.1) coincide with the fixed points to operator $(I - A)^{-1} C$; but the operator $(I - A)^{-1} C : M \to M$ is a compact one and so, due to Schauder's theorem, the existence of a fixed point to this operator is assured.

Krasnoselskii's theorem has known different generalizations and improvements (see, e.g., [5], [6], [15]). In [5], Burton proved the following improvement of Theorem 1.1.

Theorem 1.2. *Let M be a closed convex and bounded subset of a Banach space X. Suppose that the following conditions are fulfilled:*

(i) $A : X \to X$ is contraction with constant $\alpha < 1$;

(ii) $C : M \to X$ is compact;

(iii) $(x = Ax + Cy,\ y \in M) \Longrightarrow (x \in M)$.

Then (1.1) *has solutions.*

In [6] another result of Krasnoselskii type is obtained through the following theorem of Schaefer (see, e.g., [14]):

Theorem 1.3. *Let E be a locally convex space and $H : E \to E$ compact operator. Then either*

(S_1) the equation $x = \lambda Hx$ has a solution for $\lambda = 1$,

or

(S_2) the set $\{x \in E,\ x = \lambda Hx,\ \lambda \in (0,1)\}$ is unbounded.

Using this result, Burton & Kirk (see [6]) proved the following theorem.

Theorem 1.4. *Let X be a Banach space, $A : X \to X$ contraction with constant $\alpha < 1$, and $C : X \to X$ compact operator. Then either*
the equation $x = \lambda A(x/\lambda) + \lambda Cx$ has a solution for $\lambda = 1$
or
the set $\{x \in X,\ x = \lambda A(x/\lambda) + \lambda Cx,\ \lambda \in (0,1)\}$ is unbounded.

The proof is based on the fact that the operator $x \longmapsto \lambda A(x/\lambda)$ is a contraction with constant α and therefore

$$(x = \lambda A(x/\lambda) + \lambda Cx) \iff \left(x = \lambda (I - A)^{-1} Cx\right).$$

The method used by Burton & Kirk is used by Dhage ([7], [8]) in the case when A is a nonlinear contraction or A is a compact operator having the property that there exists a $p \geq 1$ such that A^p is a contraction with constant $\alpha < 1$.

In our Note [1], we obtained results of Krasnoselskii type in the case when X is a Fréchet space and in [2] we deduced results of Krasnoselskii type to an operator of the form $(x, y) \longmapsto (A(x, y), C(x, y))$, where $A(\cdot, y)$ is a contraction with $\alpha < 1$ for every y and $C(x, \cdot)$ is a compact operator for every x.

Although the fixed point theorems are stated in Banach spaces (or Fréchet spaces) X, the more frequent cases we can meet in applications are those when X is a space of continuous functions or integrable functions (respectively locally integrable in the case of the Fréchet spaces). In this particular case, the hypotheses become often more general.

Recently, other results of Krasnoselskii type have been obtained in the particular case $X = C\left(J, \mathbb{R}^N\right)$, where J is a compact or not compact interval. In [15] one proves the existence of a unique solution to equation

$$x = Ax + Bx \tag{1.2}$$

in the case when $X = C\left([0,T], \mathbb{R}^N\right)$, A, $B : X \to X$ satisfy the conditions

$$|(Ax)(t) - (Ay)(t)| \leq \alpha |x(t) - y(t)|,\ \forall t \in [0,T],\ x,\ y \in X, \tag{1.3}$$

where $\alpha \in [0,1)$,

$$|(Bx)(t) - (By)(t)| \leq b \int_0^t |x(s) - y(s)|\, ds,\ \forall t \in [0,T],\ x,\ y \in X, \tag{1.4}$$

where $b > 0$, and $|\cdot|$ denotes a norm in $\mathbb{R}^N$. Remark that in [15] the more restrictive condition $\alpha \in [0, 1/e)$ is used, but it still holds for $\alpha < 1$. The operators A, B, of this type, arise in the study of mathematical models which describe the contact between a deformable body and a foundation. The same problem is independently considered in [10], [11], wherein condition (1.4) is replaced by

$$|(Bx)(t) - (By)(t)| \leq \frac{b}{t^\beta} \int_0^t |x(s) - y(s)|\, ds, \ \forall b > 0, \ t \in [0, T], \tag{1.5}$$

where $\beta \in [0, 1)$.

In [3] this result is extended to the case $X = C\left(\mathbb{R}_+, \mathbb{R}^N\right)$, where $\mathbb{R}_+ := [0, \infty)$; in the same manner the result can be obtained in the case when $X = C\left(\mathbb{R}, \mathbb{R}^N\right)$.

The proof of the results mentioned is based on the fact that, in the conditions stated above, the operator $D = A + B$ is contraction in X, eventually by using a norm convenient and equivalent to the classical norm.

In the present Note we shall research the existence of the fixed points in $X = C\left(J, \mathbb{R}^N\right)$, where J is a compact or not compact interval, to operators of type

$$K = A + B + C, \tag{1.6}$$

where A and B satisfy conditions of type (1.3), (1.5), and C is a compact operator. The first result is proved by using Schauder-Tychonoff theorem and the second by using Burton-Kirk theorem cited. Next we prove a theorem of existence of the fixed points to a product mapping of type $(x, y) \longmapsto (U(x, y), V(x, y))$. The last existence result is proved in the case when K is a multi-valued operator.

Finally, in the last section, we present applications concerning the proof of the existence of the solutions to integral equations with modified argument.

2. Main results

2.1. **General hypotheses and notations.** In the sequel, $|\cdot|$ denotes a norm in $\mathbb{R}^N$.

Let $J \subset \mathbb{R}$ be an interval, where $J = [0, T]$ or $J = \mathbb{R}_+$ (the case $J = \mathbb{R}$ is treated similarly). Set

$$X := C\left(J, \mathbb{R}^N\right) = \left\{x : J \to \mathbb{R}^N, \ x \text{ continuous}\right\}.$$

If $J = [0,T]$, then X is a Banach space equipped with the norm of the uniform convergence on J,

$$\|x\|_\infty := \sup_{t\in J} \{|x(t)|\}. \tag{2.1}$$

A norm equivalent to $\|\cdot\|_\infty$ which will be used is

$$\|x\| = \|x\|_\gamma + \|x\|_h, \tag{2.2}$$

where

$$\|x\|_\gamma = \sup_{0\le t\le\gamma} \{|x(t)|\}, \ \ \|x\|_h = \sup_{\gamma\le t\le T} \left\{e^{-h(t-\gamma)} |x(t)|\right\},$$

$\gamma \in (0,T)$ and $h > 0$ is an arbitrary number.

If $J = \mathbb{R}_+$, then X can be organized as a Fréchet space (i.e., a linear topological metrisable and complete space) endowed with the numerable family of seminorms

$$|x|_n := \sup_{t\in[0,n]} \{|x(t)|\}, \qquad n \in \mathbb{N}\backslash\{0\}. \tag{2.3}$$

The Banach Contraction Principle can be easily extended to an arbitrary Fréchet space.

Definition 2.1. *Let E be a Fréchet space and $M \subset E$ an arbitrary subset. The function $D : M \to E$ is said to be a* **contraction** *if there exists a family of seminorms $\{\|x\|_n\}_{n\in\mathbb{N}\backslash\{0\}}$, equivalent to the initial family (of E) and for every $n \in \mathbb{N}\backslash\{0\}$ there exists $\alpha_n < 1$ such that*

$$\|Dx - Dy\|_n \le \alpha_n \|x - y\|_n, \qquad \forall x,\ y \in M,\ n \in \mathbb{N}\backslash\{0\}. \tag{2.4}$$

Proposition 2.1. *(Banach) Let E be a Fréchet space, $M \subset E$ a closed subset, and $D : M \to M$ a contraction. Then D admits a unique fixed point and the operator $I - D$ admits a continuous inverse.*

The proof of Proposition 2.1 is classical and it uses the successive approximations scheme.

In addition recall that Schauder's fixed point theorem works as well as in Fréchet spaces; in this case it is known as the Schauder-Tychonoff theorem.

In what follows we consider the following functions with the specified properties:

$a : J \to [0, c),\ c < 1,$
$b : J \to \mathbb{R}_+$ bounded function,
$\nu,\ \sigma : J \to J$ continuous functions with $\nu(t) \leq t,\ \sigma(t) \leq t,\ \forall t \in J.$

Let $M \subset X$ be an arbitrary subset. Consider that the operators A, B, $C : M \to X$ fulfil the following hypotheses:

(A) $|(Ax)(t) - (Ay)(t)| \leq a(t) |x(\nu(t)) - y(\nu(t))|$, $\forall t \in J$, $\forall x,\ y \in M$;

(B) $|(Bx)(t) - (By)(t)| \leq \frac{b(t)}{t^\beta} \int_0^t |x(\sigma(s)) - y(\sigma(s))|\, ds$, $\forall t \in J$, $\forall x,\ y \in M$, where $\beta \in [0, 1)$;

(C) $C : M \to X$ is compact operator.

In the case $a(t) \equiv \alpha$, $b(t) \equiv b$, $\nu(t) = \sigma(t) = t,\ J = [0, T]$ hypotheses (A), (B) are identical to (1.3), (1.5).

Set $K := A + B + C$, $D := A + B$.

2.2. **The first existence result.** The first existence result is contained in the following theorem.

Theorem 2.1. *Suppose that:*

i) M is a closed convex and bounded set;

ii) A, B, C fulfil hypotheses (A), (B), (C);

iii) for every x, $y \in M$ one has $Dx + Cy \in M$.

Then K admits at least one fixed point.

Proof. It is enough to prove Theorem 2.1 in the case $J = [0, T]$, since in the case $J = \mathbb{R}_+$ the reason repeats on each compact $[0, n]$, $n \in \mathbb{N} \backslash \{0\}$.

The key of the proof consists in showing that D is contraction; then,

$$(x = Kx) \Longleftrightarrow \left(x = (I - D)^{-1} Cx\right)$$

and so the conclusion follows from an easy application of Schauder-Tychonoff theorem to $(I - D)^{-1} C$ on M.

We shall apply the Banach Contraction Principle. To this aim, we show that D is contraction, i.e. there exists $\delta \in [0, 1)$ such that for any $x, y \in M$,

$$\|Dx - Dy\| \leq \delta \|x - y\|.$$

Let $t \in [0, \gamma]$ be arbitrary. Then we have

$$\begin{aligned} |(Dx)(t) - (Dy)(t)| &\leq a(t)\,|x(\nu(t)) - y(\nu(t))| \\ &\quad + \frac{b(t)}{t^{\beta}} \int_0^t |x(\sigma(s)) - y(\sigma(s))|\, ds \\ &\leq \left(a(t) + b(t)\,\gamma^{1-\beta}\right) \|x - y\|_{\gamma} \end{aligned}$$

and hence

$$\|Dx - Dy\|_{\gamma} \leq \left(c + c_1 \gamma^{1-\beta}\right) \|x - y\|_{\gamma}, \tag{2.5}$$

where $c_1 := \sup_{t \in J} \{b(t)\}$.

Let $t \in [\gamma, T]$ be arbitrary. Then we get

$$\begin{aligned} |(Dx)(t) - (Dy)(t)| &\leq a(t)\,|x(\nu(t)) - y(\nu(t))| \\ &\quad + \frac{b(t)}{t^{\beta}} \left(\int_0^{\gamma} |x(\sigma(s)) - y(\sigma(s))|\, ds \right. \\ &\quad \left. + \int_{\gamma}^{t} |x(\sigma(s)) - y(\sigma(s))|\, e^{-h(\sigma(s)-\gamma)} e^{h(\sigma(s)-\gamma)} ds \right). \end{aligned}$$

After easy estimates, it follows that

$$\begin{aligned} |(Dx)(t) - (Dy)(t)|\, e^{-h(t-\gamma)} &< a(t)\,|x(\nu(t)) - y(\nu(t))|\, e^{-h(t-\gamma)} \\ &\quad + b(t)\,\gamma^{1-\beta} \|x - y\|_{\gamma} + \frac{b(t)}{h} \gamma^{-\beta} \|x - y\|_h \end{aligned}$$

and therefore

$$\begin{aligned} \|Dx - Dy\|_h &\leq c \sup_{t \in [\gamma, T]} \left\{ |x(\nu(t)) - y(\nu(t))|\, e^{-h(t-\gamma)} \right\} \\ &\quad + c_1 \gamma^{1-\beta} \|x - y\|_{\gamma} + \frac{c_1}{h} \gamma^{-\beta} \|x - y\|_h \\ &\leq \left(c + \frac{c_1}{h} \gamma^{-\beta}\right) \|x - y\|_h + c_1 \gamma^{1-\beta} \|x - y\|_{\gamma}. \end{aligned} \tag{2.6}$$

By (2.5) and (2.6) we obtain

$$\|Dx - Dy\| \leq \left(c + 2c_1 \gamma^{1-\beta}\right) \|x - y\|_{\gamma} + \left(c + \frac{c_1}{h} \gamma^{-\beta}\right) \|x - y\|_h. \tag{2.7}$$

Since $c \in [0, 1)$, for $\gamma \in \left(0, \left(\frac{1-c}{2c_1}\right)^{\frac{1}{1-\beta}}\right)$ we deduce $c + 2c_1\gamma^{1-\beta} < 1$ and for $h > \frac{c_1}{1-c}\gamma^{-\beta}$ we deduce $c + \frac{c_1}{h}\gamma^{-\beta} < 1$. Let $\delta := \max\left\{c + 2c_1\gamma^{1-\beta}, c + \frac{c_1}{h}\gamma^{-\beta}\right\}$. It follows that $\delta < 1$ and, since (2.7),

$$\|Dx - Dy\| \leq \delta \left(\|x - y\|_{\gamma} + \|x - y\|_h\right) = \delta \|x - y\|.$$

Hence, D is contraction. □

Remark 2.1. *If* $C \equiv 0$ *we get the results from* [10], [11]. *If* $B \equiv 0$ *the Krasnoselskii's theorem is deduced in this particular case* $C\left(J, \mathbb{R}^N\right)$.

2.3. **The second existence result.** In this subsection we consider the case $M = X$.

Theorem 2.2. *Admit that hypotheses (A), (B), (C) are fulfilled and that*

(D) the set $\{x \in X,\ x = \lambda D(x/\lambda) + \lambda Cx,\ \lambda \in (0,1)\}$ *is bounded.*

Then K *admits at least one fixed point.*

Proof. The proof is reduced to the remark that

$$(x = \lambda D(x/\lambda) + \lambda Cx) \Longleftrightarrow \left(x = \lambda (I - D)^{-1} Cx\right)$$

and to an application of Burton & Kirk Theorem 1.4. □

2.4. **The third existence result.** In this subsection we consider the problem of the existence of the solutions to a system of type

$$\begin{cases} x = A_1(x,y) + B_1(x,y) \\ y = A_2(y) + B_2(x,y), \end{cases} \tag{2.8}$$

where $A_1,\ B_1,\ B_2 : M \to X$, $A_2 : M_2 \to X$, $X = C\left(J, \mathbb{R}^N\right)$, $J = [0,T]$, $M = M_1 \times M_2$, $M_1 \subset X$ is a closed and bounded set, $M_2 \subset X$ is a closed convex and bounded set. (The case $X = C\left(\mathbb{R}_+, \mathbb{R}^N\right)$ is treated similarly.)

Consider the following hypotheses:

1) $|A_1(x_1,y)(t) - A_1(x_2,y)(t)| \leq \alpha(y)|x_1(\nu(t)) - x_2(\nu(t))|$, $\forall (x_1,y)$, $(x_2,y) \in M$, $\forall t \in J$, where $\alpha : M_2 \to [0,1)$, and $A_1(x,y)$ is continuous with respect to y, for every $x \in M_1$;

2) $|B_1(x_1,y)(t) - B_1(x_2,y)(t)| \leq \frac{b(y)}{t^{\beta(y)}} \int_0^t |x_1(\sigma(s)) - x_2(\sigma(s))|\, ds$, $\forall t \in J$, $\forall (x_1,y), (x_2,y) \in M$, where $b : M_2 \to (0,\infty)$, $\beta : M_2 \to [0,1)$, and $B_1(x,y)$ is continuous with respect to y, for every $x \in M_1$;

3) $\|A_2(y_1) - A_2(y_2)\| \leq L\|y_1 - y_2\|$, where $L \in [0,1)$ is constant;

4) $B_2 : M \to X$ is a compact operator;

5) $\forall (x,y) \in M$, $A_1(x,y) + B_1(x,y) \in M_1$;

6) $\forall y \in M_2$, $\forall (x,v) \in M$, $A_2(y) + B_2(x,v) \in M_2$.

We can state and prove the following result.

Theorem 2.3. *Suppose that hypotheses* 1) − 6) *are fulfilled. Then system* (2.8) *admits solutions.*

Proof. Let $y \in M_2$ be arbitrary fixed. By hypotheses 1), 2), an argument similar to the one from the proof of Theorem 2.1 leads us to conclude that the operator

$$x \longmapsto A_1(x, y) + B_1(x, y)$$

is a contraction. From hypothesis 5) we conclude that this operator maps M_1 into M_1. Therefore, by applying the Banach Contraction Principle, we deduce that there exists $g(y) \in M_1$ such that

$$g(y) = A_1(g(y), y) + B_1(g(y), y).$$

First one shows that the mapping $g : M_2 \to M_1$ is continuous. So, the operator

$$y \longmapsto B_2(g(y), y)$$

is continuous. By hypothesis 4) we get that this operator is compact from M_2 to X. Therefore, using hypotheses 3), 6), and applying Krasnoselskii's Theorem 1.1, we conclude that the operator

$$y \longmapsto A_2(y) + B_2(g(y), y)$$

admits fixed points.

The proof of Theorem 2.3 is now complete. □

2.5. **Multivalued operators.** In the excellent book of A. Petruşel [12] the author presents certain theorems of Krasnoselskii type in the case of the multivalued operators (see Theorems 2.8.4, 2.8.5, 2.8.6). Using these results in the particular case of the space $C\left([0, T], \mathbb{R}^N\right)$, we can obtain a fixed point theorem for the sum of three operators. It is necessarily first to make some notations.

So, let $X = C\left([0, T], \mathbb{R}^N\right)$. Set

$$\begin{aligned} \mathcal{P}_{b,cl}(X) &: = \{Y \subset X,\ Y \neq \emptyset,\ Y \text{ bounded and closed}\}, \\ \mathcal{P}_{cp,cv}(X) &: = \{Y \subset X,\ Y \neq \emptyset,\ Y \text{ compact and convex}\}, \\ \mathcal{P}_{b,cl,cv}(X) &: = \mathcal{P}_{b,cl}(X) \cap \mathcal{P}_{cp,cv}(X). \end{aligned}$$

Let $M \subset \mathcal{P}_{cp,cv}(X)$. Consider the operators

$$A,\ B : M \to \mathcal{P}_{b,cl,cv}(X),\ C : M \to \mathcal{P}_{cp,cv}(X).$$

Theorem 2.4. *Suppose that the following hypotheses are fulfilled:*
i) $\forall x_i \in M$, $\forall y_i \in A(x_i)$, $i = \overline{1,2}$, *the following inequality holds:*

$$|y_1(t) - y_2(t)| \le a\,|x_1(t) - x_2(t)|, \qquad t \in [0,T],\ a \in [0,1);$$

ii) $\forall x_i \in M$, $\forall y_i \in B(x_i)$, $i = \overline{1,2}$, *the following inequality holds:*

$$|y_1(t) - y_2(t)| \le \frac{b}{t^\beta}\int_0^t |x_1(s) - x_2(s)|\,ds, \qquad b > 0,\ \beta \in [0,1);$$

iii) C *is l.s.c. and compact operator;*
iv) $A(x) + B(x) + C(y) \subset M, \qquad \forall x \in M,\ y \in M.$
Then, $Fix(A+B+C) \neq \emptyset$.

Proof. Consider on M the metric $d(x,y) := \|x-y\|_\gamma + \|x-y\|_h$.

As in the proof of Theorem 2.1 one can show that the operator $D := A+B$ is contraction with respect to the Hausdorff-Pompeiu metric H, generated by d. By using Theorem 2.8.4 from [12], the conclusion follows. □

Remark 2.2. *Hypothesis iv) can be replaced by the condition of Burton type:*

$$\text{if } y \in D(y) + C(x),\ x \in M,\ \text{then } y \in M.$$

Remark 2.3. *If* $M \subset \mathcal{P}_{b,cl,cv}(X)$ *and* $A,\ B,\ C : M \to \mathcal{P}_{cp,cv}(X)$, *then* C *can be u.s.c. and compact.*

3. Applications

3.1. **The first application.** In this subsection we shall present an application of Theorem 2.2 to establish the existence of solutions to an integral equation with modified argument of type

$$\begin{aligned} x(t) &= f(t) + F(t, x(\nu(t))) + \int_0^{\rho(t)} G(t,s,x(\sigma(s)))\,ds \\ &\quad + \int_0^{\varepsilon(t)} \Phi(t,s)\,\Psi(s, x(\mu(s)))\,ds, \end{aligned} \tag{3.1}$$

admitting that the following hypotheses are fulfilled:

$$f \in C\left(J, \mathbb{R}^N\right); \tag{3.2}$$

$$F : J \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a continuous and bounded function;} \tag{3.3}$$

$$|F(t,x) - F(t,y)| \le a(t)\,|x-y|, \qquad t \in J,\ x,\ y \in \mathbb{R}^N, \tag{3.4}$$

where $a : J \to [0,1)$;

$$G : \Delta \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a continuous function,} \tag{3.5}$$

where $\Delta := \{(s,t) \in J \times J,\ 0 \le s \le t\}$;

$$|G(t,s,x) - G(t,s,y)| \le g(t,s)\,|x-y|, \qquad x,\ y \in \mathbb{R}^N,\ (s,t) \in \Delta; \tag{3.6}$$

$$G(t,s,0) \equiv 0; \tag{3.7}$$

$$\Phi : \Delta \to \mathcal{M}_N(\mathbb{R}) \text{ is a continuous quadratic matrix;} \tag{3.8}$$

$$\Psi : J \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a continuous function;} \tag{3.9}$$

$$|\Psi(t,x)| \le \varphi(t)\,\psi(|x|), \tag{3.10}$$

where $\varphi : J \to \mathbb{R}_+$ is continuous, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and increasing, and ν, σ, $\mu : J \to J$ are continuous functions such that $\nu(t) \le t$, $\sigma(t) \le t$, $\mu(t) \le t$, $\forall t \in J$.

Denote

$$\begin{aligned} (Ax)(t) &:= f(t) + F(t, x(\nu(t))), \\ (Bx)(t) &:= \textstyle\int_0^{\rho(t)} G(t,s,x(\sigma(s)))\,ds, \\ (Cx)(t) &:= \textstyle\int_0^{\varepsilon(t)} \Phi(t,s)\,\Psi(s, x(\mu(s)))\,ds. \end{aligned}$$

It is easily seen that conditions (A), (B), (C) from subsection 2.1 are fulfilled with $\beta = 0$. In order to apply Theorem 2.2 it will be sufficient to check condition (D) from subsection 2.3. To this purpose, it is enough to check it in the case $J = [0,T]$; for the case $J = \mathbb{R}_+$ the reason will repeat on each interval $[0,n]$, $n \in \mathbb{N}\backslash\{0\}$.

So, let $\lambda \in (0,1)$ and $x \in C\left(J, \mathbb{R}^N\right)$, such that

$$\begin{aligned} x(t) \;&=\; \lambda f(t) + \lambda F\left(t, \frac{1}{\lambda} x(\nu(t))\right) + \lambda \int_0^{\rho(t)} G\left(t, s, \frac{1}{\lambda} x(\sigma(s))\right) ds + \\ &\quad + \lambda \int_0^{\varepsilon(t)} \Phi(t,s)\,\Psi(s, x(\mu(s)))\,ds. \end{aligned} \tag{3.11}$$

First, from (3.2) and (3.3) there exists a $C_1 > 0$, such that

$$\left| \lambda f(t) + \lambda F\left(t, \frac{1}{\lambda} x(\nu(t))\right) \right| \le C_1, \qquad \forall t \in [0,T].$$

Next, since

$$|G(t,s,x)| = |G(t,s,x) - G(t,s,0)| \le g(t,s)\,|x|\,,$$

we get

$$\begin{aligned}\left|\lambda\int_0^{\rho(t)} G\left(t,s,\frac{1}{\lambda}x(\sigma(s))\right)ds\right| &\le \int_0^{\rho(t)} g(t,s)\,|x(\sigma(s))|\,ds\\ &\le C_2\int_0^t |x(\sigma(s))|\,ds,\end{aligned}$$

where $C_2 := \sup_{0\le s\le t\le T}\{g(t,s)\}$;

$$\begin{aligned}\lambda\left|\int_0^{\varepsilon(t)} \Phi(t,s)\,\Psi(s,x(\mu(s)))\,ds\right| &\le \int_0^t |\Phi(t,s)|\,\varphi(t)\,\psi(|x(\mu(s))|)\,ds\\ &\le C_3\int_0^t \psi(|x(\mu(s))|)\,ds,\end{aligned}$$

where $C_3 := \sup_{0\le s\le t\le T}\{|\Phi(t,s)|\,\varphi(t)\}$.

In conclusion, we obtain from (3.11)

$$|x(t)| \le C_1 + C_0\int_0^t |x(\sigma(s))|\,ds + C_3\int_0^t \psi(|x(\mu(s))|)\,ds, \tag{3.12}$$

where $C_0 := \max\{C_2, C_3\}$.

We set $w(t) := \sup_{0\le s\le t\le T}\{|x(s)|\}$.

Obviously, w is an increasing function and $|x(t)| \le w(t)\,, \qquad \forall t\in[0,T]\,,$

$$|x(\sigma(s))| \le w(s)\,,\ \ |x(\mu(s))| \le w(s)\,,$$

and therefore

$$\psi(|x(\mu(s))|) \le \psi(w(s))\,. \tag{3.13}$$

Definitively we deduce by (3.12) and (3.13)

$$w(t) \le C_1 + C_0\int_0^t [w(s) + \psi(w(s))]\,ds. \tag{3.14}$$

We denote $u(t) := C_1 + C_0\int_0^t [w(s)+\psi(w(s))]\,ds$.

By relation (3.14) it follows that $w(t) \le u(t)$, hence $|x(t)| \le u(t)$.

But

$$\dot{u}(t) = C_0[w(t) + \psi(w(t))] \le C_0[u(t) + \psi(u(t))]\,,$$

and, since $u(0) = C_1$, we obtain

$$\int_{C_1}^{u(t)} \frac{ds}{s + \psi(s)} \leq C_0, \qquad t \in [0, T]. \tag{3.15}$$

Taking into account relation (3.15) we easily deduce the following corollary.

Corollary 3.1. *Suppose that hypotheses* (3.2) – (3.10) *are fulfilled. Then, if* $\int_{(\cdot)}^{\infty} \frac{ds}{s+\psi(s)} = \infty$, *equation* (3.1) *has at least one solution.*

Remark 3.1. *It is clear that the condition* $\int_{(\cdot)}^{\infty} \frac{ds}{s+\psi(s)} = \infty$ *can be replaced by* $\int_{C_1}^{\infty} \frac{ds}{s+\psi(s)} > C_0$.

3.2. **The second application.** In this subsection we shall present an application of Theorem 2.3 in order to establish the existence of solutions to an integral system with modified argument of type

$$\begin{cases} x(t) = \int_0^t \mathcal{K}(t, s, y(\mu_1(s)))\, x(\nu_1(t))\, ds + \int_0^t \mathcal{F}(t, s, x(\sigma_1(s)), y(\theta_1(s)))\, ds \\ y(t) = \int_0^T \mathcal{G}(t, s, y(\mu_2(s)))\, ds + \int_0^T \mathcal{H}(t, s, x(\sigma_2(s)), y(\theta_2(s)))\, ds, \end{cases} \tag{3.16}$$

where $\mathcal{K} : \Delta \times \mathrm{I\!R}^N \to \mathcal{M}_N(\mathrm{I\!R})$ is continuous and bounded quadratic matrix, $\mathcal{F} : \Delta \times \mathrm{I\!R}^N \times \mathrm{I\!R}^N \to \mathrm{I\!R}^N$, $\mathcal{G}$, $\mathcal{H} : J \times J \times \mathrm{I\!R}^N \to \mathrm{I\!R}^N$ are continuous and bounded functions, such that:

$$|\mathcal{F}(t, s, u_1, v) - \mathcal{F}(t, s, u_2, v)| \leq L_1 |u_1 - u_2|, \tag{3.17}$$

$\forall (t, s, u_1, v), (t, s, u_2, v) \in \Delta \times \mathrm{I\!R}^N \times \mathrm{I\!R}^N$,

$$|\mathcal{G}(t, s, v_1) - \mathcal{G}(t, s, v_2)| \leq L_2 |v_1 - v_2|, \tag{3.18}$$

$\forall (t, s, v_1), (t, s, v_2) \in J \times J \times \mathrm{I\!R}^N$,

$$|\mathcal{H}(t, s, u, v_1) - \mathcal{H}(t, s, u, v_2)| \leq L_3 |v_1 - v_2|, \tag{3.19}$$

$\forall (t, s, u, v_1), (t, s, u, v_2) \in J \times J \times \mathrm{I\!R}^N \times \mathrm{I\!R}^N$, ν_1, σ_1, μ_1, θ_1, σ_2, μ_2, $\theta_2 : J \to J$ are continuous functions such that $\nu_1(t) \leq t$, $\sigma_1(t) \leq t$, $\mu_1(t) \leq t$, $\theta_1(t) \leq t$, $\sigma_2(t) \leq t$, $\mu_2(t) \leq t$, $\theta_2(t) \leq t$, $\forall t \in J$, and L_1, L_2, $L_3 \in [0, 1)$ are constant.

Consider $X = C([0, T], \mathrm{I\!R}^N)$, $J = [0, T]$, $M = \overline{B_\rho} \times \overline{B_\rho}$, $M_1 = M_2 = \overline{B_\rho}$, where

$$\overline{B_\rho} := \{z \in X, \ \|z\| \leq \rho\}, \qquad \rho > 0,$$

and denote

$$\begin{aligned} a &:= \sup_{(t,s,v)\in\Delta\times\mathbb{R}^N} \left\{\left|\mathcal{K}\left(t,s,v\right)\right|\right\}, \\ b &:= \sup_{(t,s,u,v)\in\Delta\times\mathbb{R}^N\times\mathbb{R}^N} \left\{\left|\mathcal{F}\left(t,s,u,v\right)\right|\right\}, \\ c &:= \sup_{(t,s,v)\in J\times J\times\mathbb{R}^N} \left\{\left|\mathcal{G}\left(t,s,v\right)\right|\right\}, \\ d &:= \sup_{(t,s,u,v)\in J\times J\times\mathbb{R}^N\times\mathbb{R}^N} \left\{\left|\mathcal{H}\left(t,s,u,v\right)\right|\right\}. \end{aligned}$$

Let us define the operators A_1, B_1, $B_2 : M \to X$, $A_2 : M_2 \to X$, through

$$\begin{aligned} A_1\left(x,y\right)\left(t\right) &:= \int_0^t \mathcal{K}\left(t,s,y\left(\mu_1\left(s\right)\right)\right) x\left(\nu_1\left(t\right)\right) ds, \\ B_1\left(x,y\right)\left(t\right) &:= \int_0^t \mathcal{F}\left(t,s,x\left(\sigma_1\left(s\right)\right),y\left(\theta_1\left(s\right)\right)\right) ds, \\ A_2\left(y\right)\left(t\right) &:= \int_0^T \mathcal{G}\left(t,s,y\left(\mu_2\left(s\right)\right)\right) ds, \\ B_2\left(x,y\right)\left(t\right) &:= \int_0^T \mathcal{H}\left(t,s,x\left(\sigma_2\left(s\right)\right),y\left(\theta_2\left(s\right)\right)\right) ds, \end{aligned}$$

$\forall\left(x,y\right)\in M$, $\forall t\in J$.

In order to apply Theorem 2.3 it will be sufficient to check hypotheses 1)-6) from subsection 2.4. We shall check them in the case $J=[0,T]$; for the case $J=\mathbb{R}_+$ the reason will repeat on each interval $[0,n]$, $n\in\mathbb{N}\backslash\{0\}$.

Indeed, taking into account relations $(3.17)-(3.19)$, hypotheses 1)-6) are fulfilled with $\alpha\left(y\right)\equiv aT$, $b\left(y\right)\equiv L_1$, $\beta\left(y\right)\equiv 0$, $L=cL_2T$, for $a<1/T$, $c<\min\left\{1/\left(L_2T\right),\rho/T\right\}$, $b<\rho\left(1-aT\right)/T$, and $d<\rho/T-c$.

Then, we deduce the following result.

Corollary 3.2. *Suppose that hypotheses* $(3.17)-(3.19)$ *are fulfilled. Then, if* $a<1/T$, $c<\min\{1/\left(L_2T\right),\ \rho/T\}$, $b<\rho\left(1-aT\right)/T$, *and* $d<\rho/T-c$, *system* (3.16) *admits solutions.*

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