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FIBER PICARD OPERATORS AND CONVEX CONTRACTIONS

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Abstract. In this paper we generalize a fiber Picard operator theorem by replacing the condition of uniform contractions on fibers with the uniform convex contraction condition on fibers.
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1. Introduction

In [15] the author proved the following theorem

Theorem 1.1. *Let (X, d) be a generalized metric space with $d(x, y) \in \mathbb{R}^p_+$, and (Y, ρ) a generalized complete metric space with $\rho(x, y) \in \mathbb{R}^m_+$. Let $A : X \times Y \to X \times Y$ be a continuous operator. If we suppose that:*

a) *$A(x, y) = (B(x), C(x, y))$ for all $x \in X$ and $y \in Y$;*
b) *the operator $B : X \to X$ is a weakly Picard operator;*
c) *there exist a matrix $Q \in M_m(\mathbb{R}_+)$ convergent to zero, such that the operator $C(x, \cdot) : Y \to Y$ is a Q-contraction for all $x \in X$*

then the operator A is a weakly Picard operator. Moreover, if B is Picard operator, then the operator A is a Picard operator too.

In [20] the author proved the following theorem (Theorem 3.1.3):

Theorem 1.2. *Let (X_k, d_k) with $k = \overline{0, q}$ and $q \geq 1$ be some metric spaces and $A_k : X_0 \times X_1 \times ... \times X_k \to X_k$ for $k = \overline{0, q}$ be some operators such that:*

a) *the spaces* (X_k, d_k) *are complete metric spaces for* $k = \overline{1, q}$;
b) *the operator* A_0 *is (weakly) Picard;*
c) *there exist* $\alpha_k \in (0, 1]$ *such that the operators* $A_k(x_0, ..., x_{k-1}, \cdot) : X_k \to X_k$ *is an* α_k*-contraction* $\forall$ $(x_0, x_1, ..., x_{k-1}) \in X_0 \times X_1 \times ... \times X_k$ *and* $k = \overline{1, q}$;
d) *the operators* A_k *are continuous with respect to* $(x_0, x_1, ..., x_{k-1})$ *for all* $x_k \in X_k$ *and* $k = \overline{1, q}$;

then the operator $B_p = (A_0, A_1, ..., A_{p-1}, A_p)$ *is (weakly) Picard operator. Moreover if* A_0 *is a Picard operator and* $F_{A_0} = x_0^*, F_{A_1(x_0^*,\cdot)} = x_1^*, ..., F_{A_p(x_0^*,x_1^*,...,x_{p-1}^*,\cdot)} = x_p^*$*, then* $F_{B_q} = (x_0^*, x_1^*, ..., x_{q-1}^*, x_q^*)$.

We extend the definition of a convex contraction (see [7]) and [2]) to generalized metric spaces:

Definition 1.1. *Let* (X, d) *be a generalized metric space with* $d(x, y) \in \mathbb{R}^n$, $\forall$ $x, y \in X$*. The operator* $T : X \to X$ *is called a convex contraction if it satisfies the condition*

$$d(T^{(p)}(x_1), T^{(p)}(x_2)) \leq \sum_{j=0}^{p-1} \Lambda_j \cdot d(T^{(j)}(x_1), T^{(j)}(x_2)),$$

where $p \in \mathbb{N}^*$, $\Lambda_j \in M_n(\mathbb{R}_+)$ *for* $j = \overline{0, p-1}$ *and* $\sum\limits_{j=0}^{p-1} ||\Lambda_j||_m \leq 1$ *(the symbol* $||\cdot||_m$ *denotes an arbitrary matrix norm on* $M_n(\mathbb{R}_+)$ *subordinated to the vector norm* $||\cdot||_v : \mathbb{R}^n \to \mathbb{R}_+$*).*

In [1] we proved that the convex contractions over a complete metric space are Picard operators, this result was proved by V. Istrăţescu in [7] by completely different methods. In [2] we proved by the same technique that the metric can be replaced by a generalized metric (which has values in $\mathbb{R}^n$). In this paper we prove that the contraction condition from theorem 1.1 and 1.2 can be replaced with their convex analogous.

2. Main result

In this section we prove the following theorem

Theorem 2.1. *Let* (X_k, d_k) *with* $k = \overline{0, q}$ *and* $q \geq 1$ *be some generalized metric spaces and* $A_k : X_0 \times X_1 \times ... \times X_k \to X_k$ *for* $k = \overline{0, q}$ *be some continuous operators such that:*

a) *the spaces* (X_k, d_k) *are generalized complete metric spaces, with* $d_k : X_k \to \mathbb{R}_+^{n_k}$, $n_k \in \mathbb{N}^*$ *for* $k = \overline{1,q}$;
b) *the operator* A_0 *is (weakly) Picard;*
c) *there exist* $p_k \in \mathbb{N}^*$ *and* $\Lambda_{p_k}^{(j)} \in M_{n_k}(\mathbb{R}_+)$ *for* $j = \overline{0, p_k - 1}$ *with the property* $\sum\limits_{j=0}^{p_k-1} ||\Lambda_{p_k}^{(j)}||_{m_k} \leq 1$ *such that the operators*

$$(T_k)(\cdot) = A_k(x_0, ..., x_{k-1}, \cdot) : X_k \to X_k$$

satisfy the following condition

$$d_k(T_k^{(p_k)}(x_{k1}), T_k^{(p_k)}(x_{k2})) \leq \sum_{j=0}^{p_k-1} \Lambda_{p_k}^{(j)} \cdot d_k(T_k^{(j)}(x_{k1}), T_k^{(j)}(x_{k2})),$$

$\forall\ (x_0, x_1, ..., x_{k-1}) \in X_0 \times X_1 \times ... \times X_{k-1}$ *and* $x_{k1}, x_{k2} \in X_k$, $k = \overline{1,q}$;
d) *the operators* A_k *are continuous with respect to* $(x_0, x_1, ..., x_{k-1})$ *for all* $x_k \in X_k$ *and* $k = \overline{1,q}$;

then the operator $B_q = (A_0, A_1, ..., A_{q-1}, A_q)$ *is (weakly) Picard operator. Moreover if* A_0 *is a Picard operator and* $F_{A_0} = x_0^*, F_{A_1(x_0^*,\cdot)} = x_1^*, ..., F_{A_p(x_0^*, x_1^*, ..., x_{q-1}^*, \cdot)} = x_q^*$, *then* $F_{B_q} = (x_0^*, x_1^*, ..., x_{q-1}^*, x_q^*)$.

In order to prove this theorem we need the following lemma

Lemma 2.1. *The matrices* $\Lambda_{ip_k}^{(j)} \in M_{n_k}(\mathbb{R}_+)$ *with* $i = \overline{1, p_k}$ *and* $j = \overline{0, p_k - 1}$ *satisfy the inequality* $\sum\limits_{j=0}^{p_k-1} ||\Lambda_{ip_k}^{(j)}||_{m_k} < 1$ *for* $i = \overline{1, p_k}$. *If the sequence* $(x_m)_{m \geq 0} \subset \left(\mathbb{R}_+^{n_k}\right)^{p_k}$ *satisfies the inequality*

$$x_{m+1} \leq \bar{A} \cdot x_m + y_m, \forall m \in \mathbb{N},$$

where $(y_m)_{m \geq 0} \subset \left(\mathbb{R}_+^{n_k}\right)^{p_k}$, $\lim\limits_{m \to \infty} y_m = 0$ *and* $\bar{A} \in M_{p_k}\left(M_{n_k}(\mathbb{R}_+)\right)$ *such that* $\bar{A} = \begin{bmatrix} \Lambda_{1p_k}^{(0)} & \Lambda_{1p_k}^{(1)} & ... & \Lambda_{1p_k}^{(p_k-1)} \\ \Lambda_{2p_k}^{(0)} & \Lambda_{2p_k}^{(1)} & ... & \Lambda_{2p_k}^{(p_k-1)} \\ ... & ... & ... & ... \\ \Lambda_{p_kp_k}^{(0)} & \Lambda_{p_kp_k}^{(1)} & ... & \Lambda_{p_kp_k}^{(p_k-1)} \end{bmatrix}$, *then the sequence* $(x_m)_{m \geq 0}$ *is convergent to* 0.

Proof of the lemma. Let $|| \cdot ||_{n_k} : \mathbb{R}_+^{n_k} \to \mathbb{R}_+$ be a vector norm on $\mathbb{R}_+^{n_k}$ and $|| \cdot ||_{m_k} : M_{n_k}(\mathbb{R}_+) \to \mathbb{R}_+$ the subordinated matrix norm. We define

$||\cdot||_{np} : \left(\mathbb{R}_+^{n_k}\right)^{p_k} \to \mathbb{R}_+$ by

$$||x||_{np} = \max\left\{||x_i||_{n_k} \middle| x = (x_1, x_2, ..., x_{p_k}), x_i \in \mathbb{R}_+^{n_k}\right\}$$

and $||\cdot||_{mm} : M_{p_k}(M_{n_k}(\mathbb{R}_+)) \to \mathbb{R}_+$ by

$$||A||_{mm} = \max_{i=\overline{1,p_k}} \sum_{j=1}^{p_k} ||a_{ij}||_{m_k},$$

where $A = [a_{ij}]_{1\le i,j\le p_k}$ and $a_{ij} \in M_{n_k}(\mathbb{R}_+)$ for $1 \le i,j \le p_k$. With these notations we have the following properties:

(1) $||Ax||_{np} \le ||A||_{mm} \cdot ||x||_{np}$, $\forall\ x \in \left(\mathbb{R}_+^{n_k}\right)^{p_k}$ and $A \in M_{p_k}(M_{n_k}(\mathbb{R}_+))$;
(2) $||A \cdot B||_{mm} \le ||A||_{mm} \cdot ||B||_{mm}$, $\forall\ A, B \in M_{p_k}(M_{n_k}(\mathbb{R}_+))$;
(3) If $A \le B$ then $||A||mm \le ||B||_{mm}$.

From the given conditions we have $||\bar{A}||_{mm} = \max\limits_{i=\overline{1,p_k}} \sum\limits_{j=0}^{p_k-1} ||\Lambda_{ip_k}^{(j)}||_{m_k} < 1$, so the sequence $X_m = \sum\limits_{j=1}^{m} \bar{A}^j$ is convergent to a matrix $\underline{A}$. This implies that there exists $M \in \mathbb{R}_+$ such that $||\sum\limits_{j=0}^{p-1} \bar{A}^j||_{mm} < M$, $\forall\ p \in \mathbb{N}^*$ and for every $\epsilon > 0$ there exists $p(\epsilon) \in \mathbb{N}^*$ such that $||A^p||_{mm} < \frac{\epsilon}{M_1}$, $\forall\ p \ge p(\epsilon)$ where M_1 is a fixed constant. From the condition $\lim\limits_{m\to\infty} y_m = 0$ we deduce that for every $\epsilon > 0$ there exists $m(\epsilon) \in \mathbb{N}^*$ such that $||y_m|| \le \frac{\epsilon}{2M}$, $\forall\ m \ge m(\epsilon)$. By the other hand from the given inequality we deduce

$$\bar{A}^k \cdot x_{m+p-k} \le \bar{A}^{k+1} \cdot x_{m+p-k-1} + \bar{A}^k \cdot y_{m+p-k-1}, \quad k = \overline{0, p-1}.$$

Adding these inequalities term by term we obtain

$$x_{m+p} \le \bar{A}^p \cdot x_m + \sum_{j=0}^{p-1} \bar{A}^j \cdot y_{m+p-1-j}.$$

From this inequality we deduce

$$||x_{m_\epsilon+p}||_{np} \le ||\bar{A}^p||_{mm} \cdot ||x_{m_\epsilon}||_{np} + \frac{\epsilon}{2M} \cdot \sum_{j=0}^{p-1} ||\bar{A}^j||_{mm} \le ||\bar{A}||_{mm}^p \cdot M_1 + \frac{\epsilon}{2} \le \epsilon,$$

if $p \ge p(\epsilon)$. So there exists $n(\epsilon) = p(\epsilon) + m(\epsilon) \in \mathbb{N}^*$ such that

$$||x_n||_{np} \le \epsilon, \quad \forall n \ge n(\epsilon).$$

This implies that $\lim_{n\to\infty} x_n = 0$.

$\square$

Proof of the theorem. First we prove the theorem for $q = 1$ and then we proceed by induction on q. For $q = 1$ let's consider the sequences $(x_n^0)_{n\geq 0} \subset X_0$ and $(x_n^1)_{n\geq 0} \subset X_1$ defined by the relations

$$x_{n+1}^0 = A_0(x_n^0), \forall n \geq 0 \quad \text{and} \quad x_{n+1}^1 = A_1(x_n^0, x_n^1), \forall n \geq 0. \tag{2.1}$$

The sequence $(x_n^0)_{n\geq 0}$ is convergent to an element $x_0^* \in X_0$ because the operator A_0 is a weakly Picard operator. Due to the main result from [2] the operator $A_1(x_0^*, \cdot) : X_1 \to X_1$ is a Picard operator, so there exists an unique $x_1^* \in X_1$ such that $A_1(x_0^*, x_1^*) = x_1^*$. We prove that the sequence $(x_n^1)_{n\geq 0}$ is convergent to x_0^*.

$$d_1(x_{n+p_1}^1, x_1^*) = d_1(A_1(x_{n+p_1-1}^0, x_{n+p_1-1}^1), A_1(x_0^*, x_1^*)) \leq$$

$$\leq \sum_{j=1}^{p_1} d_1(A_1^{j-1}(A_1(x_{n+p_1-j}^0, x_{n+p_1-j}^1)), A_1^j(x_{n+p_1-j}^1)) + d_1(A_1^{p_1}(x_n^1), A_1^{p_1}(x_1^*)) \leq$$

$$\leq \sum_{j=1}^{p_1} d_1(A_1^{j-1}(A_1(x_{n+p_1-j}^0, x_{n+p_1-j}^1)), A_1^j(x_{n+p_1-j}^1)) +$$

$$+ \sum_{j=0}^{p_1-1} \Lambda_{p_1}^{(j)} \cdot d_1(A_1^j(x_n^1), A_1^j(x_1^*)),$$

where

$$A_1^j : X_1 \to X_1, \quad A_1^j(x) = \underbrace{A_1(x_0^*, A_1(x_0^*, ..., A_1}_{j}(x_0^*, x)...))$$

for $j = \overline{1, p_1}$ and $A_1^0(x) = x$, $\forall\, x \in X_1$. By the same technique we obtain

$$d_1(A_1^j(x_{n+p_1}^1), A_1^j(x_1^*)) \leq \sum_{l=1}^{p_1} d_1(A_1^{j+l-1}(A_1(x_{n+p_1-l}^0, x_{n+p_1-l}^1)), A_1^{j+l}(x_{n+p_1-l}^1)) +$$

$$+ \sum_{l=0}^{p_1-1} \Lambda_{p_1}^{(l)} \cdot d_1(A_1^{j+l}(x_n^1), A_1^{j+l}(x_1^*)),$$

for $j = \overline{1, p_1 - 1}$. By the other hand we can construct inductively the matrices

$\Lambda_{ip_1}^{(j)} \in M_{n_k}((R)_+)$ such that

$$\sum_{l=0}^{p_1-1} \Lambda_{p_1}^{(l)} \cdot d_1(A_1^{j+l}(x_n^1), A_1^{j+l}(x_1^*)) \leq \sum_{l=0}^{p_1-1} \Lambda_{ip_1}^{(l)} \cdot d_1(A_1^l(x_n^1), A_1^l(x_1^*)) \quad i = \overline{1, p_1}$$

and with this construction we have

$$\sum_{j=0}^{p_1-1} ||\Lambda_{ip_1}^{(j)}||_{m_1} < 1, \quad i = \overline{1, p_1}.$$

With these constructions we consider $\bar{A} = [\Lambda_{ip1}^{(j)}]_{i=\overline{1,p_1}, j=\overline{0,p_1-1}}$,

$$x_m = \begin{bmatrix} (d_1(x_{p\cdot m}^1, x_1^*) \\ d_1(A_1^1(x_{p\cdot m}^1), x_1^*) \\ d_1(A_1^2(x_{p\cdot m}^1), x_1^*) \\ \vdots \\ d_1(A_1^{p_1-1}(x_{p\cdot m}^1), x_1^*) \end{bmatrix}, \forall m \in \mathbb{N}. \tag{2.2}$$

and

$$y_m = \begin{bmatrix} \sum_{l=1}^{p_1} d_1(A_1^{l-1}(A_1(x_{n+p_1-l}^0, x_{n+p_1-l}^1)), A_1^l(x_{n+p_1-l}^1)) \\ \sum_{l=1}^{p_1} d_1(A_1^{1+l-1}(A_1(x_{n+p_1-l}^0, x_{n+p_1-l}^1)), A_1^{1+l}(x_{n+p_1-l}^1)) \\ \sum_{l=1}^{p_1} d_1(A_1^{2+l-1}(A_1(x_{n+p_1-l}^0, x_{n+p_1-l}^1)), A_1^{2+l}(x_{n+p_1-l}^1)) \\ \vdots \\ \sum_{l=1}^{p_1} d_1(A_1^{p_1+l-2}(A_1(x_{n+p_1-l}^0, x_{n+p_1-l}^1)), A_1^{p_1+l-1}(x_{n+p_1-l}^1)) \end{bmatrix}, \forall m \in \mathbb{N}. \tag{2.3}$$

From the previous inequalities, the properties of A_0 and the continuity of A_1 follows that the sequences $(x_m)_{m\geq 0}, (y_m)_{m\geq 0} \in (\mathbb{R}_+^{n_1})^{p_1}$ satisfy the conditions of lemma 2.1, so $\lim_{m\to\infty} d_1(x_{p\cdot m}^1, x_1^*) = 0$. From the continuity of A_1 we deduce $\lim_{m\to\infty} x_m^1 = x_1^*$.
If the theorem is proved for q we can prove it for $q+1$ by applying the case we have just proved (with $A_0 \to (A_0, A_1, ..., A_q)$ and $A_1 \to A_{q+1}$). □

3. Application

We give an example where theorem 1.1 or theorem 1.2 can not be applied without changing the norms. We mention that due to the theorem of Bessaga (which asserts that if (X, d) is a metric space and $A : X \to X$ is a Picard operator, than we can construct a metric $\rho : X \times X \to \mathbb{R}$ such that $A : (X, \rho) \to (X, \rho)$ became a contraction) whenever we can guarantee that $\lim\limits_{n\to\infty} A^n = 0$ with $A \in M_n(\mathbb{R})$, we can change the norm to have $||A|| < 1$. By the other hand if $\lim\limits_{n\to\infty} A^n = 0$, we can choose $p \in \mathbb{N}$ and $\alpha_1, ..., \alpha_p \in (0, 1)$ to have $||A^p|| \leq \sum\limits_{j=0}^{p-1} ||A^j||$, so we do not need to change the norm.
In what follows we denote

$$d((x_1, x_2), (y_1, y_2)) = \begin{bmatrix} |x_1 - y_1| \\ |x_2 - y_2| \end{bmatrix} \quad \forall\, x_1, x_2, y_1, y_2 \in \mathbb{R}$$

and

$$||A|| = \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\} \text{ if } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

For the matrix $A = \begin{bmatrix} \frac{5}{6} & \frac{1}{4} \\ \frac{1}{16} & \frac{5}{6} \end{bmatrix}$ we have $||A|| = \frac{13}{12}$, $||A^2|| = \frac{649}{576}$, $||A^3|| = \frac{465}{407}$, $||A^4|| = \frac{507}{445}$..., $||A^9|| = \frac{4211}{4210} > 1$ and $||A^{10}|| = \frac{1211}{1256} < 1$ so we have

$$0.99 \cdot ||A^{10}|| + \sum_{j=1}^{9} 0.001 \cdot ||A^j|| = \frac{762}{947} < 1. \tag{3.4}$$

Due to this property we can apply 2.1 in studying the following system:

$$\begin{cases} x_1(\lambda) = sin\left(\frac{5}{6}x_1(\lambda) + \frac{1}{4}x_2(\lambda) + \lambda\right) \\ x_2(\lambda) = cos\left(\frac{1}{16}x_1(\lambda) + \frac{5}{6}x_2(\lambda) + \lambda^2\right) \end{cases} \tag{3.5}$$

We have $A_0 : \mathbb{R}^2 \to \mathbb{R}^2$,

$$A_0(x_1, x_2) = \left(sin\left(\frac{5}{6}x_1(\lambda) + \frac{1}{4}x_2(\lambda) + \lambda\right), cos\left(\frac{1}{16}x_1(\lambda) + \frac{5}{6}x_2(\lambda) + \lambda^2\right)\right)$$

and $A_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $A_1(x_1, x_2, u_1, u_2) = (v_1, v_2)$, where

$$v_1 = \frac{5}{6} sin\left(\frac{5}{6}x_1(\lambda) + \frac{1}{4}x_2(\lambda) + \lambda\right) \cdot u_1 +$$

$$+\frac{1}{4}sin\left(\frac{5}{6}x_1(\lambda)+\frac{1}{4}x_2(\lambda)+\lambda\right)\cdot u_2+1,$$

$$v_2=\frac{1}{16}cos\left(\frac{1}{16}x_1(\lambda)+\frac{5}{6}x_2(\lambda)+\lambda^2\right)\cdot u_1+$$

$$+\frac{5}{6}cos\left(\frac{1}{16}x_1(\lambda)+\frac{5}{6}x_2(\lambda)+\lambda^2\right)\cdot u_2+2\lambda.$$

With these notations A_0 is a Picard operator because

$$d(A_0(x_1,x_2),A_0(y_1,y_2))\leq A\cdot d((x_1,x_2),(y_1,y_2))$$

and 3.4 (see [2]). By the other hand we have

$$d(A_1(x_1,x_2,u_1,u_2),A_1(x_1,x_2,v_1,v_2))\leq A\cdot d((u_1,u_2),(v_1,v_2)).$$

This implies

$$d(A_1^{(11)}(u_1,u_2),A_1^{(11)}(v_1,v_2))\leq A^j\cdot d(A_1^{(11-j)}(u_1,u_2),A_1^{(11-j)}(v_1,v_2)),$$

$j=\overline{1,10}$, where $A_1^{j+1}(u_1,u_2)=A_1^{(j)}(x_1,x_2,u_1,u_2)$ $\forall u_1,u_2\in\mathbb{R}$ with fixed $x_1,x_2\in\mathbb{R}$. So we have

$$d(A_1^{(11)}(u_1,u_2),A_1^{(11)}(v_1,v_2))\leq 0.99\cdot A^{10}\cdot d(A_1^{(1)}(u_1,u_2),A_1^{(1)}(v_1,v_2))+$$

$$+0.001\cdot\sum_{j=1}^{9}A^j\cdot(A_1^{(11-j)}(u_1,u_2),A_1^{(11-j)}(v_1,v_2)).$$

This inequality with 3.4 and theorem 2.1 guarantee the convergence of the sequences

$$(x_1^{(n+1)},x_2^{(n+1)})=A_0(x_1^{(n)},x_2^{(n)})$$

and

$$(u_1^{(n+1)},u_2^{(n+1)})=A_1(x_1^{(n)},x_2^{(n)},u_1^{(n)},u_2^{(n)}).$$

By choosing $x_1,x_2\in C^1[\lambda_1,\lambda_2]$, $u_1=\frac{\partial x_1}{\partial\lambda}$ and $u_2=\frac{\partial x_2}{\partial\lambda}$, we have $u_1^{(n)}=\frac{\partial x_1^{(n)}}{\partial\lambda}$ and $u_2^{(n)}=\frac{\partial x_2^{(n)}}{\partial\lambda}$, so from Weierstrass's theorem we obtain the continuous differentiability of the solution of 3.5 with respect to the parameter λ. Thus we have the following theorem:

Theorem 3.1. *The system 3.5 has an unique solution in $\mathbb{R}^2$ for every $\lambda\in[\lambda_1,\lambda_2]$ and the functions $\lambda\to x_1(\lambda)$ and $\lambda\to x_2(\lambda)$ are continuously differentiable with respect to λ.*

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