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SOME MAPS FOR WHICH PERIODIC AND FIXED POINTS COINCIDE

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Abstract. We show that a pair of maps satisfying a certain contractive condition has a common periodic point if and only if it has a unique common fixed point. As corollaries we obtain the same conclusion for a number of contractive conditions, as well as a result of [1].
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In 1977 the author [5] listed 125 contractive conditions for a single map and another 125 conditions for a pair of maps. For many of these either a fixed point theorem was established, a theorem was quoted from the literature, or examples were provided of maps which do not possess a fixed point. Additional such examples were obtained by Kinces and Totik [4] and Collaco [2], [3].

In 1990 [1] the authors characterized the existence of fixed points for some of the maps listed in [5] in terms of conditions on periodic points. In this paper it is shown that two maps satisfying definition (200) in [5] have a common periodic point if and only if they have a unique common fixed point. As one of the corollaries of this result we obtain Theorem 2 of [1]. For certain inequalities in this paper we use the numbering from [5].

Theorem. *Let f and g be selfmaps of a metric space (X, d) satisfying*

$$d(f^p x, g^q y) < \{d(x,y), d(x, f^p x), d(y, g^q y), d(x, g^q y), d(y, f^p x)\} \tag{200}$$

for each $x, y \in X$ for which the right hand side of (200) *is not zero, and where p, q are fixed positive integers. Then $u \in X$ is a common periodic point of f and g if and only if u is the unique common fixed point of f and g.*

Proof. Obviously any common fixed point is a periodic point. To prove the converse, define $S := f^p, T := g^q$. Then (200) becomes

$$d(Sx, Ty) < \max\{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\}. \tag{1}$$

Lemma 1. *Let S and T satisfy* (1)*. Then $p = Sp$ iff $p = Tp$.*

Proof. Suppose that $p = Sp$. If $p \neq Tp$, then from (1) we have

$$\begin{aligned} d(p, Tp) &= d(Sp, Tp) \\ &< \max\{0, 0, d(p, Tp), d(p, Tp), 0\} \\ &= d(p, Tp), \end{aligned}$$

a contradiction. Similarly, $p = Tp$ implies that $p = Sp$. □

Lemma 2. *Let f and g satisfy* (200)*. Then, if f^p and g^q have a coommon fixed point, it is unique.*

Proof. Suppose that $r = f^p r, s = g^q s$ and that $r \neq s$. Then, from (200),

$$\begin{aligned} d(r, s) &= d(f^p r, g^q s) \\ &< \max\{d(r, s), 0, 0, d(r, s), d(s, r)\} \\ &= d(r, s), \end{aligned}$$

a contradiction. □

Lemma 3. *Let f and g satisfy* (200)*. Then any fixed point of f^p is also a fixed point of g^q.*

Proof. Suppose that $r = f^p r$. If $r \neq g^q r$, then, from (200),

$$\begin{aligned} d(r, g^q r) &= d(f^p r, g^q r) \\ &< \max\{0, 0, d(r, g^q r), d(r, g^q r), 0\} \\ &= d(r, g^q r), \end{aligned}$$

a contradiction. □

Returning to the proof of the theorem, let u be a periodic point of f; i.e., there exists a positive integer m such that $u = f^n u = g^m u$. Since $u = f^m u$ and $u = g^m u, u = f^{pm} u = S^m u$ and $u = g^{qm} u = T^m u$. From (1),

$$\begin{aligned} d(u, Tu) &= d(S^m u, T^{m+1} u) \\ &< \max\{d(S^{m-1}u, T^m u), d(S^{m-1}, S^m u), d(T^m u, T^{m+1} u), \\ &\qquad d(S^{m-1}u, T^{m+1}u), d(T^m u, S^m u)\} \\ &= \max\{d(S^{m-1}u, u), d(u, Tu), d(S^{m-1}u, Tu)\} \\ &= \max\{d(S^{m-1}u, u), d(S^{m-1}u, Tu)\}. \end{aligned}$$

Suppose that $u \neq Tu$.

Case I. $d(u, Tu) < d(S^{m-1}u, u)$. Then we have

$$\begin{aligned} d(u, Tu) &= d(S^m u, T^{m+1} u) < d(S^{m-1}u, T^m u) \\ &< \cdots < d(u, Tu), \end{aligned}$$

a contradiction.

Case II. $d(u, Tu) < d(S^{m-1}u, Tu)$. Then we have

$$\begin{aligned} d(u, Tu) &= d(S^m u, Tu) < d(s^{m-1}u, Tu) \\ &< \cdots < d(u, Tu), \end{aligned}$$

a contradiction.

Therefore $u = Tu$. From Lemma 1, $u = Su$. We now have $u = f^p u = g^q u$. Thus $fu = f^p(fu)$. From Lemma 3, $fu = g^q(fu)$. Therefore fu is also a common fixed point of f^p and g^q. But, from Lemma 2, the common fixed point is unique. Therefore $u = fu$.

A similar argument shows that $u = gu$. □

Corollary 1. *Let f and g satisfy*

$$\begin{aligned} d(f^p x, g^p y) < \max\{&d(x, y), d(x, f^p x), d(y, g^p y), \\ &d(x, g^p y), d(y, f^p x)\} \end{aligned} \tag{175}$$

for all $x, y \in X$ for which the right hand side of (175) *is not zero, p a fixed positive integer. Then u is a common periodic point of f and g if and only if u is the unique common fixed point of f and g.*

In the Theorem set $q = p$.

Corollary 2. *Let f satisfy*

$$d(f^p x, f^q y) < \max\{d(x,y), d(x, f^p x), d(y, f^q y), \\ d(x, f^q y), d(y, f^p x)\} \quad (75)$$

for all $x, y \in X$ *for which the right hand side of* (75) *is not zero,* p, q *fixed positive integers. Then* u *is a common periodic point of* f *and* g *if and only if it is the unique common fixed point of* f *and* g.

In the Theorem set $g = f$.

Corollary 3. ([1], Theorem 2) *Let f satisfy*

$$d(f^p x, f^p y) < \max\{d(x,y), d(x, f^p x), d(y, f^p y), \\ d(x, f^p y), d(y, f^p x)\} \quad (50)$$

for all $x, \in X$ *for which the right hand side of* (50) *is not zero,* p *a fixed positive integer. Then* u *is a periodic point of* f *if and only if it is the unique fixed point of* f.

In Corollary 1 set $g = f$.

Since the definitions of this paper include a number of those in [5], one immediately obtains the fact that periodic and fixed points coincide for those contractive conditions also.

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