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ON THE COMPLEMENTARITY PROBLEM WITH RESPECT TO A NONCONVEX CONE IN A HILBERT SPACE

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Abstract. We consider in this paper the nonlinear complementarity problem and the ε-complementarity problem, both with respect to a nonconvex cone in a Hilbert space. We give two solvability theorems.
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1. Introduction

The Complementarity Theory is a relatively new chapter in Applied Mathematics and its main goal is the study of complementarity problems. Generally many complementarity problems are related to the study of equilibrium as it is defined in Physics, in Techniques and even in Economics.

Obviously, Complementarity Theory has many applications in Optimization, Economics, Engineering, Mechanics, Game Theory etc., [6], [8], [9], [15].

The classical Nonlinear Complementarity Problem is defined by a mapping and a closed convex cone, in the Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ or in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. It is possible to consider a complementarity problem with respect to an unbounded closed set, however, because of the generality of the set used in the definition, the study of the existence of solutions to this general complementarity problem is a hard problem. In the Euclidean space a such general complementarity problem was considered recently in our paper [18].

In this paper we will consider the Nonlinear Complementarity Problem with respect to a closed *cone, not necessarily convex*, in an arbitrary Hilbert space. A similar problem was considered in the Euclidean space [18] and an existence theorem was proved using the notion of *Exceptional Family of Elements* and an Alternative Theorem. The Alternative Theorem used in [18] is valid in an arbitrary Hilbert space but for *completely continuous fields*. The solvability of complementarity problem has been studied in many papers by using the notion of *Exceptional Family of Elements* introduced in [14] and the notion of completely continuous fields. (See [2], [3], [9]-[20], [23], [25]-[32]).

We will study the solvability of the Nonlinear Complementarity Problem, with respect to a closed cone not necessarily convex, in a Hilbert space, by two methods. The first method is based on the notion of Exceptional Family of Elements and a new Alternative Theorem deduced from a result proved recently in [4]. The second method is based on some results proved in Proximal Analysis [5]. Note that in the new Alternative Theorem used in this paper the operator is not completely continuous field. This fact is essential for the proof of our existence theorem. By this paper, we hope to open a new research direction in Complementarity Theory. This direction is the study of complementarity problems for nonconvex sets. The complementarity problems with respect to nonconvex sets can have applications to the study of new practical problems and to the study of sensitivity of classical complementarity problems.

2. Preliminaries

We will denote in this paper by $(H, \langle \cdot, \cdot \rangle)$ an arbitrary real Hilbert space. We recall that, a closed convex cone in H is a closed subset $\mathbf{K} \subset H$ such that the following properties are satisfied:

(k_1) $\mathbf{K} + \mathbf{K} \subseteq \mathbf{K}$,

(k_2) $\lambda \mathbf{K} \subseteq \mathbf{K}$ for all $\lambda \in \mathbb{R}_+$.

If we have also that

(k_3) $\mathbf{K} \cap (-\mathbf{K}) = \{0\}$

then in this case we say that $\mathbf{K}$ is a closed, pointed convex cone.

Given a closed convex cone, $\mathbf{K} \subset H$, the *dual* of $\mathbf{K}$ is, by definition

$$\mathbf{K}^* = \{y \in H | \ \langle x, y \rangle \geq 0, \text{ for all } x \in \mathbf{K}\}.$$

Obviously, $\mathbf{K}^*$ is a *closed convex cone* in H. More general, if $D \subset H$ is an arbitrary, closed, non-empty subset in H, the dual of D is also the set

$$D^* = \{y \in H | \ \langle x, y\rangle \geq 0, \text{ for all } x \in D\}.$$

It is easy to prove that D^* is a *closed convex cone* in H. We say that a closed pointed convex cone $\mathbf{K} \subset H$ is *well-based* if and only if there exists a convex set B such that $0 \notin \overline{B}$ and $\mathbf{K} = \bigcup_{\lambda \geq 0} \lambda B$.

It is known [9] that a pointed closed convex cone $\mathbf{K}$ is well-based if and only if $\mathbf{K}$ has a bounded base, that is there exists $\varphi \in \mathbf{K}^*$ with the property that $\varphi(x) > 0$ for any $x \in \mathbf{K} \setminus \{0\}$, the set $B_* = \{x \in \mathbf{K} | \ \varphi(x) = 1\}$ is bounded and it has the property that for any $x \in \mathbf{K}$, $x \neq 0$, there exist a unique $\lambda > 0$ and a unique $b \in B_*$ such that $x = \lambda b$ (see details in [9]). In this case, we say that B_* is a bounded base for $\mathbf{K}$. If B_* is not necessarily bounded we say that $\mathbf{K}$ has a base.

For this paper the following result is useful. *A closed, pointed convex cone* $\mathbf{K} \subset H$ *is locally compact, if and only if,* $\mathbf{K}$ *has a compact base.* (Klee's Theorem [9]).

We say that a set $B \subset H$ is *star-shaped* with respect to a convex set $A \subset B$ if and only if, $x \in B$ whenever $\lambda x + (1 - \lambda)y \in B$ for some $y \in A$ and any $\lambda \in [0, 1]$.

Let $\varepsilon > 0$, be a real number, eventually very small. We say that a non-empty subset $B \subset H$ is *ε-convex*, if and only if whenever $[x, y[\subset conv(B) \setminus B$, we have that $\|y - x\| < \varepsilon$. In this definition $[x, y[= \{\lambda y + (1 - \lambda)x | \ \lambda \in [0, 1[\}$. Let $D \subset H$ be a non-empty subset. We denote by $\mathbf{K}(D)$ the *smallest closed* convex cone such that $D \subseteq \mathbf{K}(D)$. We say that a non-empty subset $D \subset H$ is *locally compact pointed conical set* if the following properties are satisfied:

(c_1) for all $x \in D$ and all $\lambda \in \mathbb{R}_+$, we have $\lambda x \in D$,

(c_2) $\mathbf{K}(D) \cap (-\mathbf{K}(D)) = \{0\}$,

(c_3) $\mathbf{K}(D)$ is a locally compact convex cone.

From some practical problems the following examples are interesting.

(1) $D = \bigcup_{i \in I} \mathbf{K}_i$, where for every $i \in I$, $\mathbf{K}_i$ is pointed convex cone, or $\mathbf{K}_i$ is a polyhedral cone not necessarily convex.

(2) $D \cap B$ is a set, star-shaped with respect to a convex set $A \subset D \cap B$, where B is base of $\mathbf{K}(D)$.
(3) $D \cap B$ is an ε-convex set, with $\varepsilon > 0$ very small, where B is again a base of $\mathbf{K}(D)$.

We recall also that a mapping $T : H \to H$ is a *completely continuous* mapping if T is continuous and for any bounded set $A \subset H$, we have that $T(A)$ is relatively compact. A mapping $f : H \to H$ is said to be a *completely continuous field* if f has a representation of the form $f(x) = x - T(x)$, for any $x \in H$, where T is a completely continuous mapping.

The following notion is essential for this paper.

We call a mapping $f : H \to H$ *regular* if for each sequence $\{x_n\}_{n \in N} \subset H$ weakly convergent to an element $x_* \in H$ and such that the sequence $\{f(x_n)\}_{n \in N}$ is convergent in norm to an element $y_* \in H$ the equation $f(x_*) = y_*$ holds. This notion was used systematically by A. Carbone and P. P. Zabreiko in [4].

3. Complementarity problems

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbf{K} \subset H$, a closed pointed cone and $f : H \to H$ a mapping. The *Nonlinear Complementarity Problem* defined by f and $\mathbf{K}$ is:

$$NCP(f, \mathbf{K}) : \begin{cases} \text{find } x_* \in \mathbf{K} \text{ such that} \\ f(x_*) \in \mathbf{K}^* \text{ and } \langle x_*, f(x_*) \rangle = 0. \end{cases}$$

The $NCP(f, \mathbf{K})$ has been studied by many authors and it has many applications. In this sense, the reader is referred to [6], [8], [9], among others.

Consider an arbitrary non-empty subset $A \subset H$ and a mapping $f : H \to H$. *The nonlinear complementarity problem* defined by f and A is:

$$NCP(f, A) : \begin{cases} \text{find } x_* \in A \text{ such that} \\ f(x_*) \in A^* \text{ and } \langle x_*, f(x_*) \rangle = 0. \end{cases}$$

In our paper [18] we considered this general complementarity problem in $\mathbb{R}^n$, but under the name of *relational complementarity problem*. Obviously, it is not easy to find existence theorems for this general complementarity problem.

This paper focuses on $NCP(f, A)$, but in the particular case when $A = D$, where D is a conical set in an arbitrary Hilbert space. We note that in [18] we considered this problem, but only in the Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$.

4. Exceptional family of elements and the solvability of complementarity problem

Recently, in several of our papers [2], [3], [9]-[17] we studied the solvability of complementarity problems using a new topological method, based on the notion of *Exceptional Family of Elements*, denoted also by EFE. This notion was introduced in 1997, in our paper [14], by using the topological degree. Until now this notion has been used also in other papers such as [4], [7], [18], [19], [20], [23] and [25]-[32] among others.

We note that in several of our recent papers on complementarity problems or on variational inequalities we introduced the notion of EFE by Leray-Schauder type alternatives. It is known that the classical Leray-Schauder Alternative proved in 1934 in [22] is one of the most important theorem in Nonlinear Analysis.

Now we recall the notion of EFE introduced in [14].

Let $(H, \langle\cdot,\cdot\rangle)$ be again a Hilbert space, $\mathbf{K} \subset H$, a closed pointed convex cone and $f : H \to H$ a mapping.

Definition 1. [14] *We say that a family of element $\{x_r\}_{r>0} \subset \mathbf{K}$, is an EFE for f, with respect to $\mathbf{K}$, if and only is, for every real number $r > 0$ there exists a real number $\mu_r > 0$, such that the vector $u_r = f(x_r) + \mu_r x_r$ satisfies the following conditions:*

(1) $u_r \in \mathbf{K}^*$,

(2) $\langle x_r, u_r\rangle = 0$,

(3) $\|x_r\| \to +\infty$ *as* $r \to +\infty$.

In [13], by using the classical Leray-Schauder Alternative, we proved the following result.

Theorem 1. [13] *If $f : H \to H$ is a completely continuous field, and $\mathbf{K} \subset H$ is a closed convex cone, then there exists either a solution to the problem $NCP(f, \mathbf{K})$, or an EFE for f with respect to $\mathbf{K}$. Consequently, if f is a completely continuous field without EFE then the problem $NCP(f, \mathbf{K})$ has a solution.*

The proof of the main result in (*Theorem 3* of [18]) is strongly based on *Theorem 1.*

We note that, for the extension to the infinite dimensional case of *Theorem 3* proved in [18], the proof can not be based on *Theorem 1.* Therefore for the main result of this paper we need to replace *Theorem 1* by a similar alternative based on the notion of *regular mapping.* To obtain this useful alternative we use the main result proved recently by A. Carbone and P. P. Zabreico in [4]. This result is the following and its proof is based on the topological degree defined by Skrypnik in [24].

Theorem 2. [4] *Let* $f : \mathbf{K} \to H$ *be a regular, completely continuous mapping and* $0 < r < +\infty$.

Then either the $NCP((1-\lambda)I + \lambda f, \mathbf{K})$ *has a solution in the set* $S_r(\mathbf{K}) = \{x \in \mathbf{K} |\ \|x\| = r\}$ *for some* $\lambda \in]0,1[$, *or the* $NCP(f, \mathbf{K})$ *has a solution in the set* $B_r(\mathbf{K}) = \{x \in \mathbf{K} |\ \|x\| \leq r\}$.

From *Theorem 2*, we deduce the following alternative:

Theorem 3. *Let* $f : \mathbf{K} \to H$ *be a regular, completely continuous mapping. Then either the* $NCP(f, \mathbf{K})$ *has a solution or* f *has an EFE, in the sense of Definition 1, with respect to* $\mathbf{K}$.

Proof. Indeed, if the $NCP(f, \mathbf{K})$ has a solution, then the proof is finished. Suppose that the $NCP(f, \mathbf{K})$ has no solution. In this case for any $r > 0$ there exists $\lambda_r \in]0,1[$ and $x_r \in S_r(\mathbf{K})$ which is a solution of the problem $NCP((1-\lambda)I + \lambda f, \mathbf{K})$. Then we have $x_r \in \mathbf{K}$, $\|x_r\| = r$ and

$$\begin{cases} (1-\lambda_r)x_r + \lambda_r f(x_r) \in \mathbf{K}^* \text{ and} \\ \langle x_r, (1-\lambda_r)x_r + \lambda_r f(x_r) \rangle = 0. \end{cases} \tag{1}$$

Dividing both relations in (1) by λ_r, we obtain that $\{x_r\}_{r>0}$ is an EFE for f, in the sense of *Definition 1*, with respect to $\mathbf{K}$. $\square$

5. An existence theorem for complementarity problems with respect to nonconvex cones

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Suppose given a *non-empty, locally compact closed and pointed conical set.* The set D is supposed to be a nonconvex set. We say that the set $R(D; \mathbf{K}) = \mathbf{K}(D) \setminus D$ is the residual set of D with respect to $\mathbf{K}(D)$. Obviously, $0 \notin R(D; \mathbf{K})$ and $R(D; \mathbf{K})$ is empty if D is closed pointed convex cone.

We recall that $f : H \to H$ is called *bounded* if for any bounded set $B \subset H$, $f(B)$ is bounded.

A main result of this paper is the following theorem.

Theorem 4. *Let $D \subset H$ be a non-empty, locally compact, closed and pointed conical set. D is supposed to be nonconvex. Let $f : H \to H$ be a continuous bounded mapping. If there exists $\rho > 0$ such that the following assumptions are satisfied:*
(1) for every $x \in \mathbf{K}(D)$ with $\|x\| = \rho$, there exists $y \in \mathbf{K}(D)$ such that $\|y\| < \rho$ and $\langle f(x), x - y\rangle \geq 0$,
(2) for every $x \in R_\rho(D; \mathbf{K}) = \{z \in R(D; \mathbf{K})|\ \|z\| \leq \rho\}$, there exists $y \in \mathcal{K}(D)$ with $\|y\| < \|x\|$ such that $\langle f(x), x - y\rangle \geq 0$,
then the $NCP(f, D)$ has a solution x^* such that $\|x^*\| \leq \rho$.

Proof. Let $\varepsilon > 0$ be a real number. Consider the mapping

$$f_\varepsilon(x) := f(x) + \varepsilon x, \text{ for any } x \in H.$$

The mapping f_ε satisfies the following properties:
(i) f_ε is a continuous and bounded mapping,
(ii) for every $x \in \mathbf{K}(D)$, with $\|x\| = \rho$, there exists $y \in \mathbf{K}(D)$, such that $\|y\| < \rho$ and $\langle f_\varepsilon(x), x - y\rangle > 0$,
(iii) for every $x \in R_\rho(D; \mathbf{K})$, there exists $y \in \mathbf{K}(D)$, with $\|y\| < \|x\|$ such that $\langle f_\varepsilon(x), x - y\rangle > 0$.

Indeed, let $x \in \mathbf{K}(D)$, with $\|x\| = \rho$, (resp. $x \in R_\rho(D; \mathbf{K})$), then by assumption (1) (resp. (2)), there exists $y \in \mathbf{K}(D)$ such that $\|y\| < \rho$ (resp. $\|y\| < \|x\|$) and $\langle f(x), x - y\rangle \geq 0$. For such x and y we have

$$\langle f_\varepsilon(x), x - y\rangle = \langle f(x) + \varepsilon x, x - y\rangle = \langle f(x), x - y\rangle + \varepsilon[\|x^2\| - \langle x, y\rangle]$$

$$\geq \varepsilon[\|x\|^2 - \|x\|\|y\|] = \varepsilon\|x\|[\|x\| - \|y\|] > 0.$$

Therefore (ii) and (iii) are satisfied. Obviously (i) is also satisfied. For each $x \in \mathbf{K}(D)$, with $\|x\| > \rho$ we denote by $T_\rho(x)$ *the radial projection* onto $S_\rho^+ = \{x \in \mathbf{K}(D)|\ \|x\| = \rho\}$, i.e., $T_\rho(x) = \dfrac{\rho x}{\|x\|}$. Now, we consider the mapping $g_\varepsilon : \mathbf{K}(D) \to H$ defined by

$$g_\varepsilon(x) := \begin{cases} f_\varepsilon(x), & \text{if } \ \|x\| \leq \rho \\ f_\varepsilon(T_\rho(x)) + \|x - T_\rho(x)\|x, & \text{if } \ \|x\| > \rho. \end{cases}$$

For any $x \in \mathbf{K}(D)$ with $\|x\| > \rho$, there exists $\lambda_x > 0$ such that $x = \lambda_x T_\rho(x)$. (We note that $\lambda_x = \dfrac{\|x\|}{\rho}$). Since f_ε satisfies (ii), we have that for $T_\rho(x)$ there exists $u_\rho^x \in \mathbf{K}(D)$ with $\|u_\rho^x\| < \rho$ such that

$$\langle T_\rho(x) - u_\rho^x, f_\varepsilon(T_\rho(x))\rangle > 0. \tag{2}$$

Multiplying (2) by λ_x we obtain

$$\langle x - \lambda_x u_\rho^x, g_\varepsilon(x)\rangle > 0. \tag{3}$$

If for a give $x \in \mathbf{K}(D)$ with $\|x\| > \rho$, we let $y = \lambda_x u_\rho^x$, we obtain that the mapping g_ε satisfies the following property:
(iv) for every $x \in \mathbf{K}(D)$ with $\|x\| > \rho$, there exists $y \in \mathbf{K}(D)$ with $\|y\| < \|x\|$ such that $\langle x - y, g_\varepsilon(x)\rangle > 0$.

The mapping g_ε is completely continuous because it is continuous and for any bounded set $B \subset \mathbf{K}(D)$, we have that $g_\varepsilon(B)$ is relatively compact since $\mathbf{K}(D)$ being locally compact infers B is relatively compact. The mapping g_ε is also a regular mapping. Indeed, if $\{x_n\}_{n\in N}$ is a sequence in $\mathbf{K}(D)$, weakly convergent to an element $x_* \in \mathbf{K}(D)$, and $g_\varepsilon(x_n)$ is convergent in norm to an element v_*, then we must have $g_\varepsilon(x_n) = v_*$; the locally compactness of $\mathbf{K}(D)$ and the Eberlein-Šmulian Theorem, imply that $\{x_n\}_{n\in N}$ is also convergent in norm to the same element x_*.

(We used also the following classical result: "*if every sub-sequence of a sequence $\{x_n\}_{n\in N}$ has a sub-sequence which converges to x_*, then $\{x_n\}_{n\in N}$ is convergent also to x_*.*")

Now, we show that for any $\varepsilon > 0$, the mapping g_ε is without EFE with respect to $\mathbf{K}(D)$. Indeed, if we suppose that g_ε has an EFE, $\{x_r\}_{r>0} \subset \mathbf{K}(D)$, then we have

$$\begin{cases} u_r = \mu_r x_r + g_\varepsilon(x_r) \in (K(D))^*, \text{ for all } r > 0, \\ \langle x_r, u_r\rangle = 0, \text{ for all } r > 0 \text{ and} \\ \|x_r\| \to +\infty \text{ as } r \to +\infty. \end{cases}$$

Take $r > 0$ such that $\|x_r\| > \rho$. Since g_ε satisfies (iv), there exists $y_r \in \mathbf{K}(D)$ such that $\|y_r\| < \|x_r\|$ and $\langle x_r - y_r, g_\varepsilon(x_r)\rangle > 0$. We have

$$0 < \langle x_r - y_r, g_\varepsilon(x_r)\rangle = \langle x_r - y_r, u_r - \mu_r x_r\rangle$$

$$= \langle x_r - y_r, u_r\rangle - \mu_r\|x_r\|^2 + \mu_r\langle y_r, x_r\rangle \leq -\mu_r\|x_r\|[\|x_r\| - \|y_r\|] < 0$$

which is a contradiction.

Applying *Theorem 3*, we obtain that for any $\varepsilon > 0$ the classical $NCP(g_\varepsilon, \mathbf{K}(D))$ has a solution x^*_ε.

Because the mapping g_ε satisfies condition (iv), it is impossible to have $\|x^*_\varepsilon\| > \rho$. Hence, we must have $\|x^*_\varepsilon\| \leq \rho$ which implies that $g_\varepsilon(x^*_\varepsilon) = f_\varepsilon(x^*_\varepsilon)$ and we obtain the following result. For any $\varepsilon > 0$, the $NCP(g_\varepsilon, \mathbf{K}(D))$ has a solution x^*_ε such that $\|x^*_\varepsilon\| \leq \rho$. Considering the fact that f_ε satisfies (iii) we have that $x^*_\varepsilon \in \{x \in D|\ \|x\| \leq \rho\} = D_\rho$. If for any $n = 1, 2, \ldots,$ we take $\varepsilon_n = \dfrac{1}{n}$, we obtain a sequence $\left\{x^*_{\frac{1}{n}}\right\}_{n=1}^{\infty}$ such that $x^*_{\frac{1}{n}} \in D_\rho$, and for every $n \in N$, $x^*_{\frac{1}{n}}$ is a solution to the $NCP(f_{\varepsilon_n}, \mathbf{K}(D))$. Because D is a closed locally compact and pointed cone, D_ρ is compact and hence the sequence $\left\{x^*_{\frac{1}{n}}\right\}_{n=1}^{\infty}$ has a convergent sub-sequence $\left\{x^*_{\frac{1}{n_k}}\right\}_{k=1}^{\infty}$. As a consequence, $x^* = \lim\limits_{k\to\infty} x^*_{\frac{1}{n_k}}$ is an element of D. Considering the definition of f_ε and the fact $(\mathbf{K}(D))^* \subset D^*$ we obtain that x_* is a solution to the $NCP(f, D)$ and $\|x^*\| \leq \rho$. The proof is complete.

6. Proximal analysis and the ε-complementarity problem with respect to nonconvex cones

In this section we consider some recent results obtained in Proximal Analysis and we apply these results to the study of complementarity problems with respect to nonconvex cones. On the subject of Proximal Analysis the reader is referred to the recent book [5].

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $D \subset H$ be a non-empty subset. Let $x \in H$ such that $x \notin D$. Suppose that there exists a point $d \in D$ whose distance to x is minimal, i.e., $\|x - d\| \leq \|x - y\|$, for any $y \in D$.

In this case d is called *a projection* of x onto D. The set of all such closest points is denoted by $Proj_D(x)$. It is known that there exists sets D such D is nonconvex and $Proj_D(x)$ is a singleton [21]. If for any $x \in H$, the set $Prod_D(x)$ is a singleton, we denote this element by $P_D(x)$. Additionally, if $P_D(x)$ satisfies some continuity conditions, then the set D is a convex set. (See the results proved in [1]).

Let $D \subset H$ be closed, pointed and locally compact cone (not necessarily convex), that is D is a cone and there exists a neighborhood U of zero in H

such that $U \cap D$ is a compact set. In this case for any $x \in H$, $x \notin D$, the set $Proj_D(x)$ is a non-empty set. Indeed, let $0 \le \alpha = \inf\{\|x-y\| | \ y \in D\}$, and let $\{y_n\}_{n\in N}$ be a sequence in D such that $\lim_{n\to\infty} \|x - y_n\| = \alpha$. There exists $M > 0$ such that for any $n \in N$,

$$\|y_n\| = \|y_n - x + x\| \le \|x\| + \|x - y_n\| \le M.$$

Because D is a locally compact cone, the sequence $\{y_n\}_{n\in N}$ has a subsequence $\{y_{n_k}\}_{k\in N}$ convergent to an element $d \in D$ and we have $\|x-d\| = \alpha \le \|x - y\|$, for any $y \in D$ and, consequently $d \in Prod_D(x)$. The following result is useful for the main result of this section.

Theorem 5. *Let D be a non-empty subset of H and let $x \in H$, $d \in D$. The following are equivalent:*

(1) $d \in Proj_D(x)$,

(2) $\langle x - d, d' - d\rangle \le \frac{1}{2}\|d' - d\|$, *for all* $d' \in D$.

Proof. This theorem is a part of *Proposition 1.3* proved in [5]. □

Let $D \subset H$ be a closed, pointed and locally compact cone (not necessarily convex) and $f : H \to H$ a mapping. Given $\varepsilon > 0$, the ε-Complementarity Problem defined by f and D is the following:

$$\varepsilon - CP(f,D) : \begin{cases} \text{find } x_* \in D \text{ such that} \\ f(x_*) \in D^* \text{ and } |\langle f(x_*), x_*\rangle| \le \varepsilon. \end{cases}$$

We say that an element $x_* \in D$ is *regular* if for any $y \in D$ there exists $\varepsilon_0 > 0$ such that $x_* + \varepsilon' y \in D$ for any $\varepsilon' \in]0, \varepsilon_0[$.

Obviously, if D is a convex cone, then any $x_* \in D$ is regular.

Indeed, take an arbitrary $y \in D$ and $u = \lambda x_* + (1-\lambda)y$. We have that $u \in D$ which implies, $\frac{1}{\lambda}u \in D$. Hence $x_* + \frac{1-\lambda}{\lambda}y = \frac{1}{\lambda}y \in D$. If $\lambda \to 1$ then we obtain that $\varepsilon' = \frac{1-\lambda}{\lambda} \to 0$. Now, we can introduce the following result.

Theorem 6. *Let $f : H \to H$ be a mapping and $D \subset H$ a closed, pointed and locally compact cone (not necessarily convex). Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary real numbers (δ can be very big and ε small). If the set-valued mapping $x \to Proj_D[x - \alpha f(x)]$ with $\alpha \ge \frac{\delta^2}{4\varepsilon}$, has a regular fixed point $x_* \in D$ such that $\|x_*\| \le \delta$, then x_* is a solution to the $\varepsilon - CP(f,D)$.*

Proof. By assumption we have that

$$x_* \in Proj_D[x_* - \alpha f(x_*)].$$

In this case, by *Theorem 5* we have

$$\langle x_* - \alpha f(x_*) - x_*, d - x_* \rangle \le \frac{1}{2}\|d - x_*\|^2, \text{ for all } d \in D,$$

which implies

$$\langle -\alpha f(x_*), d - x_* \rangle \le \frac{1}{2}\|d - x_*\|^2, \text{ for all } d \in D. \tag{4}$$

If we take $d = \frac{1}{2}x_*$, we obtain

$$\left\langle -\alpha f(x_*), -\frac{1}{2}x_* \right\rangle \le \frac{1}{8}\|x_*\|^2,$$

or

$$\langle \alpha f(x_*), x_* \rangle \le \frac{1}{4}\|x_*\|^2. \tag{5}$$

If we take $d = \frac{3}{2}x_*$ in (4) we deduce

$$\left\langle -\alpha f(x_*), \frac{1}{3}x_* \right\rangle \le \frac{1}{8}\|x_*\|^2$$

or

$$\langle \alpha f(x_*), x_* \rangle \ge -\frac{1}{4}\|x_*\|^2. \tag{6}$$

From (5) and (6) we have

$$|\langle \alpha f(x_*), x_* \rangle| \le \frac{1}{4}\|x_*\|^2$$

which implies

$$|\langle f(x_*), x_* \rangle \le \frac{1}{4\alpha}\|x_*\|^2 \le \frac{1}{4\alpha}\delta^2 \le \varepsilon,$$

that is

$$|\langle f(x_*), x_* \rangle \le \varepsilon.$$

Because x_* is regular, then for any $y \in D$, there exists $\varepsilon_* > 0$ such that $d = \varepsilon' y + x_* \in D$, with $\varepsilon' \in]0, \varepsilon_*[$. If we let $d = \varepsilon' y + x_*$ in (4) we have

$$\langle -\alpha f(x_*), \varepsilon' y \rangle \le \frac{1}{2}\|\varepsilon' y\|^2, \text{ for any } \varepsilon' \in]0, \varepsilon_*[$$

or

$$\langle \alpha f(x_*), \varepsilon' y \rangle \ge -\frac{1}{2}\varepsilon'^2\|y\|, \text{ for any } \varepsilon' \in]0, \varepsilon_*[$$

which implies

$$\langle \alpha f(x_*), y \rangle \ge -\frac{1}{2}\varepsilon'\|y\|, \text{ for any } \varepsilon' \in]0, \varepsilon_*[$$

Taking $\varepsilon' \to 0$ we have $\langle f(x_*), y\rangle \geq 0$ for all $y \in D$, that is $f(x_*) \in D^*$ and therefore, x_* is a solution to the $\varepsilon - CP(f, D)$. $\square$

Remarks. 1. If D is an arbitrary closed convex cone, the mapping $x \to Proj_D[x - \alpha f(x)]$ is a single-valued mapping. If $x_* = Proj_D[x_* - \alpha f(x_*)]$, then x_* is a solution to the classical nonlinear complementarity problem $NCP(f, D)$ and we have $\langle f(x_*), x_*\rangle = 0$, that is x_* is a solution to the $\varepsilon - CP(f, D)$ for any $\varepsilon \geq 0$.

2. On closed locally compact cone, we have that *Theorem 6* reduces the study of the solvability of the $\varepsilon - CP(f, D)$ to the study of existence of fixed points of a set-valued mapping with respect to closed nonconvex sets. For practical problems, $\delta > 0$ is given and $\varepsilon > 0$ must be very small.

7. Comments

This paper may be considered as a starting point for the study of complementarity problems with respect to nonconvex cones and also for the study of fixed points of a set-valued mapping with respect to a nonconvex set.

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