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# REMARKS ON SOME EQUIVALENT CONDITIONS FOR NEARNESS

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**Abstract.** We first prove the equivalence between some conditions of nearness and accretivity for nonlinear operators. Then we show an easier proof of a result of Tarsia regarding the equivalence of Cordes and Campanato conditions.

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## 0. Introduction

The nearness of two operators means that they have similar algebraical or topological properties. We can mention the classical inverse function theorem, which says that if the differential at a point of a $C^1$ function is invertible, then the function itself is locally invertible.

In the last decade several notions of nearness between operators have been introduced [1, 2, 10, 12]. They were used to study the existence of solutions for nonlinear partial differential equations and to give estimates of their solutions. If we use the PDE point of view, the first nearness condition was the Cordes condition used by H. O. Cordes and G. Talenti [5, 11] to prove $C^{\alpha}$, $C^{1,\alpha}$ and $W^{2,2}$ estimates for the solutions of second order linear and symmetric partial differential equations in nondivergence form

$$\mathcal{A}u = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u\,,$$

where $A = (a_{ij}) \in L^\infty(\Omega, \mathbb{R}^{n\times n})$ is a symmetric matrix function. Later S. Campanato and A. Tarsia realized [2, 3, 12, 13] that, because of the Cordes conditions gives a nearness of the matrix $A$ to the identity matrix and hence a nearness of the operator $\mathcal{A}$ to the Laplacian $\Delta$ in the spaces $W_0^{2,2}(\Omega)$, it can be generalized to nonlinear operators. A. Buica and A. Domokos [1] observed that in general Banach spaces the nearness condition is related to the accretive condition.

In this paper we will prove first the equivalence of two notions of nearness in uniformly smooth Banach spaces, answering a question raised in [1]. As an immediate corollary of this result we give a characterization of strongly accretive and Lipschitzan mappings. Then we will give another, somewhat easier, proof of a result proved by Tarsia regarding the equivalence between two types of Campanato conditions [14]. Therefore, both of them are equivalent to the Cordes condition.

## 1. Nearness and accretivity

We begin by listing the nearness conditions and by proving results regarding the connections between them. Our starting point is the definition given by Robinson [9], which is the closest to the classical Fréchet differential of a mapping.

**Definition 1.1.** Let $X$ and $Z$ be Banach spaces, $x_0 \in X$ and $X_0$ a neighborhood of $x_0$. Let $\mathcal{A}, B : X_0 \to Z$. We say that $\mathcal{A}$ **strongly approximates** $\mathcal{B}$ at $x_0$ if for all $\varepsilon > 0$ there exists a neighborhood $X_\varepsilon \subset X_0$ of $x_0$ such that for all $x$, $y \in X_\varepsilon$

$$\|\mathcal{B}(x) - \mathcal{B}(y) - (\mathcal{A}(x) - \mathcal{A}(y))\| \le \varepsilon \|x - y\|\,. \tag{1.1}$$

The following notion of nearness is due to Campanato [2, 11].

**Definition 1.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be two mappings from a set $X$ into a Banach space $Z$. We say that $\mathcal{A}$ **is near** $\mathcal{B}$ if there exist $\alpha > 0$ and $0 \le c < 1$ such that for all $x, y \in X$ we have

$$||\mathcal{B}x - \mathcal{B}y - \alpha(\mathcal{A}x - \mathcal{A}y)|| \leq c||\mathcal{B}x - \mathcal{B}y|| \,. \tag{1.2}$$

A. Tarsia proved in [11] that if the restriction of $\mathcal{B}$ to a neighborhood $X_\varepsilon$ has a Lipschitz continuous inverse and if $\mathcal{A}$ strongly approximates $\mathcal{B}$ at $x_0$, then $\mathcal{A}$ is near $\mathcal{B}$ in a neighborhood of $x_0$. This shows that in problems where the invertibility of $\mathcal{B}$ is satisfied, the strong approximation implies nearness.

In a Banach space $Z$ the normalized duality mapping is a set-valued mapping $J : Z \rightsquigarrow Z^*$, defined by (see [8])

$$J(z) = \{z^* \in Z^* \ : \ \langle z^*, z\rangle = \|z^*\| \cdot \|z\| = \|z\|^2\} \,.$$

The semi-scalar product in a Banach space is defined by

$$\langle x, y\rangle_+ = \lim_{t \searrow 0} \frac{||x + ty||^2 - ||x||^2}{2t} \,.$$

In the case when the norm is Gâteaux differentiable outside of the origin we have that for $x \neq 0$

$$\langle x, y\rangle_+ = \lim_{t \searrow 0} \frac{||x - ty||^2 - ||x||^2}{-2t} = \sup_{t \searrow 0} \frac{||x - ty||^2 - ||x||^2}{-2t} \,.$$

The connection between the normalized duality mapping and the semi-scalar product is given by

$$\langle x, y\rangle_+ = \max\{\langle j(x), y\rangle \ : \ j(x) \in J(x)\} \,.$$

**Definition 1.3.** Let $\mathcal{A}$ and $\mathcal{B}$ be two mappings from a set $X$ into a Banach space $Z$. We say that $\mathcal{A}$ **is strongly accretive with respect to** $\mathcal{B}$ if there exists $0 < m < 1$ such that for all $x, y \in X$ there exists $j(\mathcal{B}x - \mathcal{B}y) \in J(\mathcal{B}x - \mathcal{B}y)$ such that

$$\langle j(\mathcal{B}x - \mathcal{B}y), \mathcal{A}x - \mathcal{A}y\rangle \geq m||\mathcal{B}x - \mathcal{B}y||^2 \,. \tag{1.3}$$

Using the semi-scalar product, condition (1.3) can be written as

$$\langle \mathcal{B}x - \mathcal{B}y, \mathcal{A}x - \mathcal{A}y\rangle_+ \geq m||\mathcal{B}x - \mathcal{B}y||^2 \,. \tag{1.4}$$

**Definition 1.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be two mappings from a set $X$ into a Banach space $Z$. We say that $\mathcal{A}$ **is Lipschitzian with respect to** $\mathcal{B}$ if there exists $L > 0$ such that for all $x, y \in X$

$$||\mathcal{A}x - \mathcal{A}y|| \leq L||\mathcal{B}x - \mathcal{B}y|| \,. \tag{1.5}$$

If $Z$ is a Hilbert space, then $\mathcal{A}$ is near $\mathcal{B}$ if and only if $\mathcal{A}$ is strongly accretive with respect to $\mathcal{B}$ and Lipschitzian with respect to $\mathcal{B}$ (see [1, 2, 12]).
Let us remind some facts about the differentiability of the norm in a Banach space.

**Definition 1.5.** We say that in a Banach space $Z$ the norm is uniformly Fréchet differentiable if for all $x \neq 0$ there exists the Fréchet differential $D(||x||) \in Z^*$ of the norm at $x$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in Z$ such that $||x|| = 1$ and $||y|| \leq \delta$ we have

$$\left| \frac{||x+y|| - ||x|| - D(||x||)(y)}{||y||} \right| \leq \varepsilon \, .$$

A Banach space $Z$ with its dual $Z^*$ uniformly convex has this property. Therefore the $L^p$ spaces have uniformly Fréchet differentiable norms for $1 < p < \infty$. Because of

$$D\left(\frac{||x||^2}{2}\right) = ||x|| D(||x||)$$

it follows that if the norm is uniformly Fréchet differentiable, then $||x||^2/2$ is also uniformly Fréchet differentiable.

We are now able to answer a conjecture about nearness in uniformly smooth Banach spaces.

**Theorem 1.1.** *Let $Z$ be a Banach space with uniformly Fréchet differentiable norm. Let $X$ be a Banach space and $\mathcal{A}, \mathcal{B} : X \to Z$. Then $\mathcal{A}$ is near $\mathcal{B}$ if and only if $\mathcal{A}$ is strongly accretive with respect to $\mathcal{B}$ and Lipschitzian with respect to $\mathcal{B}$.*

**Proof.** " $\Rightarrow$ "
The fact that if $\mathcal{A}$ is near $\mathcal{B}$ then $\mathcal{A}$ is strongly accretive with respect to $\mathcal{B}$ and Lipschitzian with respect to $\mathcal{B}$, was proved in [1]. We include the proof for the completeness of our treatment.
Let us suppose that $\mathcal{A}$ is near $\mathcal{B}$, which means that there exist $\alpha > 0$, $0 < c < 1$ such that for all $x, y \in X$

$$||\mathcal{B}x - \mathcal{B}y - \alpha(\mathcal{A}x - \mathcal{A}y)|| \leq c||\mathcal{B}x - \mathcal{B}y||.$$

Hence,

$$\langle \mathcal{B}x - \mathcal{B}y, \mathcal{B}x - \mathcal{B}y - \alpha(\mathcal{A}x - \mathcal{A}y) \rangle_+ \leq$$

$$\leq ||\mathcal{B}x - \mathcal{B}y||\ ||\mathcal{B}x - \mathcal{B}y - \alpha(\mathcal{A}x - \mathcal{A}y)|| \leq c||\mathcal{B}x - \mathcal{B}y||^2 \,.$$

Using the properties of the semi-scalar product we get

$$||\mathcal{B}x - \mathcal{B}y||^2 - \alpha\langle \mathcal{B}x - \mathcal{B}y, \mathcal{A}x - \mathcal{A}y\rangle_+ \leq c||\mathcal{B}x - \mathcal{B}y||^2,$$

and

$$\langle \mathcal{B}x - \mathcal{B}y, \mathcal{A}x - \mathcal{A}y\rangle_+ \geq \frac{1-c}{\alpha}||\mathcal{B}x - \mathcal{B}y||^2.$$

In this way we have proved that $A$ is strongly accretive with respect to $\mathcal{B}$.
In order to prove that $\mathcal{A}$ is Lipschitzian with respect to $\mathcal{B}$ it is enough to observe that

$$\alpha||\mathcal{A}x - \mathcal{A}y|| - ||\mathcal{B}x - \mathcal{B}y|| \leq$$

$$\leq ||\mathcal{B}x - \mathcal{B}y - \alpha(\mathcal{A}x - \mathcal{A}y)|| \leq c||\mathcal{B}x - \mathcal{B}y|| \,,$$

and conclude

$$||\mathcal{A}x - \mathcal{A}y|| \leq \frac{c+1}{\alpha}||\mathcal{B}x - \mathcal{B}y|| \,.$$

“ $\Leftarrow$ ” Let us suppose now that $\mathcal{A}$ is strongly accretive with respect to $\mathcal{B}$ and Lipschitzian with respect to $\mathcal{B}$. Therefore there exists $L \geq 0$ and $0 < m < 1$ such that (1.4) and (1.5) are true.
(1.5) implies that whenever $\mathcal{B}x = \mathcal{B}y$, then $\mathcal{A}x = \mathcal{A}y$ and the formula (1.2) for nearness is true for any constants.
Suppose now that $\mathcal{B}x \neq \mathcal{B}y$. The formula (1.4) implies that

$$\left\langle \frac{\mathcal{B}x - \mathcal{B}y}{||\mathcal{B}x - \mathcal{B}y||}, \frac{\mathcal{A}x - \mathcal{A}y}{||\mathcal{B}x - \mathcal{B}y||} \right\rangle_+ \geq m$$

and therefore

$$\lim_{t \searrow 0} \frac{\left\| \frac{\mathcal{B}x - \mathcal{B}y}{||\mathcal{B}x - \mathcal{B}y||} - t\frac{\mathcal{A}x - \mathcal{A}y}{||\mathcal{B}x - \mathcal{B}y||} \right\|^2 - \left\| \frac{\mathcal{B}x - \mathcal{B}y}{||\mathcal{B}x - \mathcal{B}y||} \right\|^2}{-2t} \geq m \,.$$

Formulas (1.4) and (1.5) implies that

$$m \leq \left\| \frac{\mathcal{A}x - \mathcal{A}y}{||\mathcal{B}x - \mathcal{B}y||} \right\| \leq L$$

so, by uniform Fréchet differentiability of the norm we get that for arbitrary $\varepsilon \in (0, m)$ there exists an $\alpha > 0$ such that $2\alpha(m - \varepsilon) < 1$ and for all $x, y \in Z$ such that $\mathcal{B}x \neq \mathcal{B}y$ we have

$$\frac{\left\| \frac{\mathcal{B}x - \mathcal{B}y}{||\mathcal{B}x - \mathcal{B}y||} - \alpha\frac{\mathcal{A}x - \mathcal{A}y}{||\mathcal{B}x - \mathcal{B}y||} \right\|^2 - \left\| \frac{\mathcal{B}x - \mathcal{B}y}{||\mathcal{B}x - \mathcal{B}y||} \right\|^2}{-2\alpha} \geq m - \varepsilon \,.$$

Therefore

$$||\mathcal{B}x - \mathcal{B}y - \alpha(\mathcal{A}x - \mathcal{A}y)||^2 \leq (1 - 2\alpha(m - \varepsilon))||\mathcal{B}x - \mathcal{B}y||^2 \,.$$

□

Theorem 1.1 gives the following characterization:

**Corollary 1.1.** *In a Banach space $Z$ with uniformly differentiable norm, a mapping $\mathcal{A} : Z \to Z$ is strongly accretive and Lipschitzian if and only if it is near the identity mapping of $Z$.*

## 2. Cordes and Campanato conditions

Let $\Omega \in \mathbb{R}^n$ be a bounded domain. Consider the matrix valued mapping $A : \Omega \to \mathcal{M}_n(\mathbb{R})$, defined by $A(x) = (a_{ij}(x))$, where $a_{ij} \in L^\infty(\Omega)$, and let

$$\mathcal{A}u = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}(u) \tag{2.1}$$

**Definition 2.1.** (Cordes condition $K_{\varepsilon,\sigma}$)
We say that $A$ satisfies condition $K_{\varepsilon,\sigma}$ if there exists $\varepsilon \in (0,1]$ and $\delta > 0$ such that

$$0 < \frac{1}{\sigma} \leq \sum_{i,j=1}^{n} a_{ij}^2(x) \leq \frac{1}{n-1+\varepsilon}\left(\sum_{i=1}^{n} a_{ii}(x)\right)^2 , \text{ a.e. } x \in \Omega \,. \tag{2.2}$$

*Remark* 2.1. The original condition $K_\varepsilon$ [5, 11, 14] corresponds to the $K_{\varepsilon,n}$, since the normalization condition $\sum_{i=1}^n a_{ii} = 1$ shows that $\sigma$ can be taken $n$.

We use the notations $\operatorname{trace} A = \sum_{i=1}^n a_{ii}$ for the trace of an $n \times n$ matrix $A$. Also, we denote by $I$ the identity $n \times n$ matrix, by $\langle A, B\rangle = \sum_{i,j=1}^n a_{ij}\, b_{ij}$ the inner product and by $||A|| = \sqrt{\sum_{i,j=1}^n a_{ij}^2}$ the norm in $\mathbb{R}^{n\times n}$.

Let us remark that condition $K_{\varepsilon,\delta}$ does not specify the sign of $\operatorname{trace} A$. In this section we need to impose a fixed sign for $\operatorname{trace} A$, therefore using (2.2), let us suppose that

$$\operatorname{trace} A \geq \sqrt{\frac{n-1+\varepsilon}{\sigma}} \,, \ \forall\, x \in \Omega' \,.$$

The Cordes condition $K_{\varepsilon,\sigma}$ implies that

$$\frac{\langle A(x), I\rangle^2}{||A(x)||^2} \geq n - (1-\varepsilon) \tag{2.3}$$

for all $x \in \Omega' \subset \Omega$, where the Lebesgue measure of $\Omega \setminus \Omega'$ is 0. Let be now $x \in \Omega'$ arbitrary, but fixed. Consider the quadratic polynomial

$$P(\alpha) = ||A(x)||^2\alpha^2 - 2\langle A(x), I\rangle\alpha + n - (1-\varepsilon)\,.$$

Inequality (2.3) shows that

$$\min_{\alpha\in\mathbb{R}} P(\alpha) = P\left(\frac{\langle A(x), I\rangle}{||A(x)||^2}\right) \leq 0\,. \tag{2.4}$$

Therefore there exists

$$\alpha(x) = \frac{\langle A(x), I\rangle}{||A(x)||^2} \tag{2.5}$$

such that $P(\alpha(x)) \leq 0$. Observing that

$$||I - \alpha(x)A(x)||^2 = ||A(x)||^2\alpha^2(x) - 2\langle A(x), I\rangle\alpha(x) + n$$

we get that (2.3) implies that

$$||I - \alpha(x)A(x)||^2 \leq 1 - \varepsilon\,,$$

which is equivalent to

$$|\langle I - \alpha(x)A(x), M\rangle| \leq \sqrt{1-\varepsilon}||M||\,,\ \forall\, M \in \mathcal{M}_n(\mathbb{R})\,. \tag{2.6}$$

Condition (2.6) can be written also as

$$\left|\sum_{i=1}^n m_{ii}\ -\ \alpha(x)\sum_{i,j=1}^n\ a_{ij}(x)m_{ij}\right|\ \leq\ \sqrt{1-\varepsilon}\left(\sum_{i,j=1}^n m_{ij}^2\right)^{1/2} \tag{2.7}$$

for all $M \in \mathcal{M}_n(\mathbb{R})$.

**Definition 2.2.** [14] Let $A \in L^\infty(\Omega, \mathcal{M}_n(\mathbb{R}))$. We say that $A$ satisfies the Campanato condition $C(\tau, \gamma, \delta)$ if there exist $\alpha \in L^\infty(\Omega)$ and $\tau > 0$, $\gamma > 0$, $\delta \geq 0$ such that $\gamma + \delta < 1$, $\alpha(x) \geq \tau$ and

$$\left|\sum_{i=1}^n m_{ii}\ -\ \alpha(x)\sum_{i,j=1}^n\ a_{ij}(x)m_{ij}\right|\ \leq\ \gamma\left(\sum_{i,j=1}^n m_{ij}^2\right)^{1/2}\ +\ \delta\left|\sum_{i=1}^n m_{ii}\right|\,, \tag{2.8}$$

for all $M = (m_{ij}) \in \mathcal{M}_n(\mathbb{R})$, and a.e. $x \in \Omega$.

We can summarize the introduction to this section as:

**Proposition 2.1.** [14] *The Cordes condition* $K_{\varepsilon,\sigma}$ *is equivalent to a Campanato condition* $C(\tau, \gamma, 0)$.

Tarsia [14] also proved that if $A$ satisfies the condition $C(\tau,\gamma,\delta)$, then $A$ satisfies also a condition $C(\tau',\gamma',0)$, possible with other constants $\tau'$ and $\gamma'$, and therefore satisfies the Cordes condition.

We show now a simpler version of his proof.

**Theorem 2.1.** *Suppose $A$ satisfies the Campanato condition $C(\tau,\gamma,\delta)$. Then there exists $\tau'>0$, $0<\gamma'<1$ such that $A$ satisfies the condition $C(\tau',\gamma',0)$.*

**Proof.** We want to prove that there exist $\tau'>0$, $0<\gamma'<1$, $\alpha'\in L^\infty(\Omega)$ such that for a.e $x\in\Omega$ we have $\alpha'(x)\geq\tau'$ and

$$\big|\langle I,M\rangle-\alpha'(x)\langle A(x),M\rangle\big|\leq\gamma'||M||\,,\ \forall\, M\in\mathcal{M}_n\,. \tag{2.9}$$

For the moment, suppose that for an $x\in\Omega$ inequality (2.9) is valid. Then (2.9) is equivalent to

$$||I-\alpha'(x)A(x)||\leq\gamma'<1$$

which means that $\alpha'(x)A(x)$ is included in a closed ball $B(I,\gamma')$ of radius $\gamma'$ around the identity matrix $I\in\mathbb{R}^{n\times n}$. Because of (2.8) and $A\in L^\infty(\Omega,\mathbb{R}^{n\times n})$, another equivalent way is to saying that $A(x)$ is included in the cone with vertex at 0 and generated by $B(I,\gamma')$.

Therefore, we have to prove that in the 2 dimensional subspace of $\mathbb{R}^{n\times n}$ spanned by $I$ and $A(x)$, the cosine of the angle between $I$ and $A(x)$ is bounded below by a positive number

$$c_0>\frac{\sqrt{n-1}}{\sqrt{n}} \tag{2.10}$$

that does not depend on $x$, i.e.

$$\frac{\langle I,A(x)\rangle}{\sqrt{n}||A(x)||}\geq c_0\,,\quad \text{a.e. } x\in\Omega\,.$$

In this subspace generated by $I$ and $A(x)$ let $E_1(x)$ and $E_2(x)$ two $n\times n$ matrices such that

(1) $E_1(x)$ is that tangent to the circle with center at $I$ and radius 1 which is the closest to $A(x)$.
(2) $\langle E_1(x),E_2(x)\rangle=0$.
(3) $I=E_1(x)+E_2(x)$.

We have that

$$\langle I,E_1(x)\rangle=n-1\ ,\ ||E_1(x)||=\sqrt{n-1}\ ,\ \langle I,E_2(x)\rangle=1\ ,\ ||E_2(x)||=1\,.$$

Let us write

$$A(x) = a_1(x)E_1(x) + a_2(x)E_2(x)$$

and then use $E_1(x)$ and $E_2(x)$ for $M$ in formula (2.8) to get

$$0 < \frac{1 - \frac{\gamma}{\sqrt{n-1}} - \delta}{\alpha(x)} \leq a_1(x) \leq \frac{1 + \frac{\gamma}{\sqrt{n-1}} + \delta}{\alpha(x)}$$

and

$$0 < \frac{1 - \gamma - \delta}{\alpha(x)} \leq a_2(x) \leq \frac{1 + \gamma + \delta}{\alpha(x)}\,.$$

Therefore, the angle between $I$ and $A(x)$ will be smaller then the angle between $I$ and

$$V(x) = \frac{1 + \frac{\gamma}{\sqrt{n-1}} + \delta}{\alpha(x)} E_1(x) + \frac{1 - \gamma - \delta}{\alpha(x)} E_2(x)\,.$$

The cosine of the angle between $I$ and $V(x)$ is given by

$$\frac{\langle I, V(x)\rangle}{\sqrt{n}||V(x)||} = \frac{\left(1 + \frac{\gamma}{\sqrt{n-1}} + \delta\right)(n-1) + 1 - \gamma - \delta}{\sqrt{n}\sqrt{\left(1 + \frac{\gamma}{\sqrt{n-1}} + \delta\right)^2 (n-1) + (1 - \gamma - \delta)^2}} = c_0$$

For $n \geq 2$ the inequality

$$1 + \frac{\gamma}{\sqrt{n-1}} + \delta > \frac{n-2}{2(n-1)}\left(1 - \gamma - \delta\right)$$

implies (2.10) and this finishes the proof. □

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