Fixed Point Theory, Volume 4, No. 2, 2003, 237-240
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

A NOTE ON A ČIRIČ'S FIXED POINT THEOREM

MIRCEA BALAJ AND SORIN MUREŞAN

Department of Mathematics, Oradea University
3700 Oradea, Romania
e-mail: mbalaj@uoradea.ro, e-mail: smuresan@uoradea.ro

Abstract. In this paper we give a new proof for a fixed point theorem due to Čirič.Our method permits the localization of the fixed point into a certain closed ball.
2000 Mathematics Subject Classification: 54H25, 54E50.
Key Words and Phrases: complete metric space, fixed point, Banach's principle.

Generally, if a mapping defined on a complete metric space with values into itself admits a fixed point, then the fixed point is obtained by the successive approximations method. We will use other method closed that used by Darbo in [3], to prove a fixed point theorem due to Čirič. Our method has the advantage that it permits the localization of the fixed point into a certain closed ball.

For a set Y in a metric space (X, d) we shall denote by $\delta(Y)$ and $\overline{Y}$ the diameter of Y and its closure, respectively. The closed ball with radius r and the center x will be denoted by $\overline{B}(x, r)$.

The following lemma will play a crucial role:

Lemma 1. *Let (X, d) be a complete metric space, Y a nonempty closed bounded subset of X and $f : Y \to Y$ a mapping. Put $Y_0 := Y$, $Y_1 := \overline{f(Y_0)}$,..., $Y_n := \overline{f(Y_{n-1})}$,... If $\delta(Y_n) \to 0$ for $n \to \infty$, then f has a unique fixed point.*

Proof. Let us observe that each set Y_n is nonempty, closed and $Y_n \supseteq Y_{n+1}$. Since X is complete and $\lim_{n\to\infty} \delta(Y_n) = 0$, it follows that the set $\bigcap_{n\geq 0} Y_n$ consists of a single point. Let it this be x_0. Since $x_0 \in Y_n$ for each $n \geq 0$ it follows that $f(x_0) \in Y_n$ for each $n \geq 1$. By $Y_1 \subseteq Y_0$ we infer $f(x_0) \in Y_0$, hence

$f(x_0) \in \bigcap_{n \geq 0} Y_n$. Thus $f(x_0) = x_0$. The uniqueness of the fixed point can be easily established by way of contradiction.

Using the previous lemma we give a new proof for the following well-known fixed point theorem due to Čirič [1]:

Theorem 2. *Let (X, d) be a complete metric space, $f : X \to X$ a mapping satisfying the condition*

$$d(f(x), f(y)) \leq \alpha_1(x, y)\, d(x, y) + \alpha_2(x, y)\, d(x, f(x)) + \alpha_3(x, y)\, d(y, f(y)) +$$

$$+\alpha_4(x, y)\, d(x, f(y)) + \alpha_5(x, y)\, d(y, f(x)), \tag{1}$$

for all $x, y \in X$, where $\alpha_i : X \times X \to [0, \infty)$, $i = \overline{1,5}$ and $\sum_{i=1}^{5} \alpha_i(x, y) \leq \alpha$ for each $x, y \in X$ and some $\alpha \in [0, 1)$. Then f has a unique fixed point.

Proof. Let $a \in X$ arbitrarily chosen. Supposing that we are unlucky, that is $f(a) \neq a$, let us put $k = d(a, f(a))$ and $r = \frac{2+\alpha}{2(1-\alpha)}k$. Clearly for all $x, y \in X$ we have

$$[2 + \alpha_2(x, y) + \alpha_3(x, y) + \alpha_4(x, y) + \alpha_5(x, y)]k \leq$$

$$\leq 2[1 - \alpha_1(x, y) - \alpha_2(x, y) - \alpha_3(x, y) - \alpha_4(x, y) - \alpha_5(x, y)]r. \tag{2}$$

We shall prove that

$$f\left(\overline{B}(a, r)\right) \subseteq \overline{B}(a, r). \tag{3}$$

For this purpose we use a symmetric form of (1) which is obtained evaluating also $d(f(y), f(x))$ and adding

$$d(f(x), f(y)) \leq \alpha_1(x, y)d(x, y) + \frac{\alpha_2(x, y) + \alpha_3(x, y)}{2}[d(x, f(x)) + d(y, f(y))]+$$

$$+\frac{\alpha_4(x, y) + \alpha_5(x, y)}{2}[d(x, f(y)) + d(y, f(x))]. \tag{4}$$

Consider an element $x \in \overline{B}(a, r)$ and estimate $d(a, f(x))$ taking into account (4) and (2). For sake of simplicity we denote by $\alpha_i := \alpha_i(a, x)$, $i = \overline{1,5}$.
$d(a, f(x)) \leq d(a, f(a)) + d(f(a), f(x)) \leq$

$$\leq d(a, f(a)) + \alpha_1 d(a, x) + \tfrac{\alpha_2+\alpha_3}{2}[d(a, f(a)) + d(x, a) + d(a, f(x))] +$$

$+\frac{\alpha_4+\alpha_5}{2}\left[d\left(a, f(x)+d\left(x, a\right)+d\left(a, f\left(x\right)\right)\right]\right.$.
Hence

$$\begin{aligned} d\left(a, f\left(x\right)\right) &\leq \frac{1}{1-\frac{\alpha_2+\alpha_3}{2}-\frac{\alpha_4+\alpha_5}{2}}\left[\left(1+\frac{\alpha_2+\alpha_3}{2}+\frac{\alpha_4+\alpha_5}{2}\right) d\left(a, f\left(a\right)\right)+\right. \\ &\quad \left.+\left(\alpha_1+\frac{\alpha_2+\alpha_3}{2}+\frac{\alpha_4+\alpha_5}{2}\right) d\left(a, x\right)\right]= \\ &= \frac{1}{2-\alpha_2-\alpha_3-\alpha_4-\alpha_5}\left[\left(2+\alpha_2+\alpha_3+\alpha_4+\alpha_5\right) k+\right. \\ &\quad \left.+\left(2\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\right) d\left(a, x\right)\right] \leq \\ &\leq \frac{2\left(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5\right) r+\left(2\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\right) r}{2-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5} \\ &= r. \end{aligned}$$

Thus, (3) is proved. Taking $Y_0=\overline{B}\left(a, r\right), Y_n=\overline{f\left(Y_{n-1}\right)}, n \geq 1$, from (1) we get $\delta\left(Y_n\right) \leq \alpha \delta\left(Y_{n-1}\right)$ and, since $\alpha \in[0,1)$, $\delta\left(Y_n\right) \rightarrow 0$, for $n \rightarrow \infty$. By Proposition 1 the restriction $f\Big|_{\overline{B}(a, r)}$ has a fixed point. The uniqueness of the fixed point follows easily from (1).

Remark 1. Under the hypothesis of Theorem 2 the following inequality

$$\delta\left(X\right) \leq \frac{2+\alpha}{1-\alpha} \sup \left\{d\left(x, f\left(x\right)\right): x \in X\right\}$$

holds.

Indeed, if x_0 is the fixed point of f let us observe that $x_0 \in \bigcap_{x \in X} \overline{B}\left(x, r_x\right)$, where $r_x=\frac{2+\alpha}{2(1-\alpha)} d\left(x, f\left(x\right)\right)$. Then

$$d\left(x, y\right) \leq d\left(x, x_0\right)+d\left(x_0, y\right) \leq \frac{2+\alpha}{1-\alpha} \sup \left\{d\left(x, f\left(x\right)\right): x \in X\right\},$$

for any $x, y \in X$. Consequently, if the metric space X is unbounded, then $\sup \left\{d\left(x, f\left(x\right)\right): x \in X\right\}=\infty$.

Remark 2. The radius of the ball $\overline{B}\left(a, r\right)$ where the fixed point must be sought is:

- $r=\frac{1}{1-\alpha} d\left(a, f\left(a\right)\right)$ for the Banach's contraction principle ($\alpha_1=\alpha \in [0,1)$, $\alpha_2=\alpha_3=\alpha_4=\alpha_5=0$);
- $r=\frac{1+\alpha}{1-2\alpha} d\left(a, f\left(a\right)\right)$ for the Kannan's fixed point theorem [5] ($\alpha_1= \alpha_4=\alpha_5=0$, $\alpha_2=\alpha_3=\alpha \in[0, \frac{1}{2})$);

- $r = \frac{1+\beta}{1-2\alpha\beta} d(a, f(a))$ for the Čirič-Reich-Rus fixed point theorem [6] ($\alpha_1 = \alpha$, $\alpha_2 = \alpha_3 = \beta$, $\alpha_4 = \alpha_5 = 0$, $\alpha + 2\beta \in [0, 1)$).
- $r = \frac{2+\alpha}{2(1-\alpha)} d(a, f(a))$ for the Hardy and Rogers fixed point theorem [4] (α_1, α_2, α_3, α_4, α_5 are positive constants such that $\alpha := \sum_{i=1}^{5} \alpha_i \in [0, 1)$.

Acknowledegment. The authors would like to thank professor I. A. Rus for his helpful suggestions and comments.

References

[1] L. B. Čirič, *Generalized contractions and fixed-point theorems*, Publ. L'Inst. Math., 12, **26**(1971), 19-26.

[2] L. B. Čirič, *On a family of contractive maps and fixed points*, Publ. L'Inst. Math., **17**(1974), 45-51.

[3] G. Darbo, *Punti uniti in transformazioni a codomenio noncompacto*, Rend. Sem. Mat. Univ. Padova, **24**(1955), 84-92.

[4] G. Hardy, T. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. **16**(1973), 201-206.

[5] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60**(1968), 71-76.

[6] I. A. Rus, *Metrical fixed point theorems*, University of Cluj-Napoca, Department of Mathematics, 1979.

[7] I.A. Rus, *Generalized contractions*, Presa Universitară Clujeană, 2001.