MAPPINGS WITH THE INTERSECTION PROPERTY AND A MEASURE OF NONCOMPACTNESS ON SEPARABLE BANACH SPACES

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Abstract. The aim of this note is to present examples of mappings with the intersection property. Also we shall give an explicit definition of an abstract measure of noncompactness on a separable Banach space and we shall prove that Hausdorff’s noncompactness measure is a mapping with the intersection property.

Keywords: measure of noncompactness, \((\theta, a)\)-contraction mapping, separable Banach space

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1 Mappings with the intersection property

Notations. Let \(X\) be a nonempty set and

\[
\begin{align*}
P(X) &= \{M \mid M \subset X\}, \\
P_b(X) &= \{M \subset X \mid M \text{ is bounded}\}, \\
P_{b,cl}(X) &= \{M \subset X \mid M \text{ is bounded and closed}\}, \\
P_{b,cl,cv}(X) &= \{M \subset X \mid M \text{ is bounded, closed, convex}\}, \\
P_{cp} &= \{M \subset X \mid M \text{ is compact}\}.
\end{align*}
\]

Definition 1.1. (\cite{4}) Let \(X\) be a nonempty set, \(Z \subset P(X)\), \(Z\) is nonempty.
A mapping \(\theta : Z \to \mathbb{R}\), has the intersection property if

\[
Y_{n+1} \subset Y_n, \quad Y_n \in Z, \quad n \in \mathbb{N}, \quad \lim_{n \to \infty} \theta(Y_n) = 0
\]
implies that \(Y_\infty := \bigcap_{n \in \mathbb{N}} Y_n\) is nonempty, \(Y_\infty \in Z\) and \(\theta(Y_\infty) = 0\). Let us illustrate this notion by

Example 1.1. \cite{5} Let \((X, |\cdot|)\) be a Banach space. A mapping \(\alpha : P_b(X) \to \mathbb{R}_+\) is said to be a Danes-Pasicki noncompactness measure if:

(i) \(\alpha(A) = 0\) implies \(\overline{A} \in P_{cp}(X)\);
(ii) \(A \subset B\) implies \(\alpha(A) \leq \alpha(B)\);
(iii) \(\alpha(A \cup \{x\}) = \alpha(A)\) for all \(A \in P_b(X)\) and \(x \in X\).

We denote such a mapping by \(\alpha_{DP}\).

Theorem 1.1. \(\alpha_{DP}\) has the intersection property.

Proof. Let \(Z = P_{b,cl}(X)\), \(\theta = \alpha_{DP}\), \(Y_{n+1} \subset Y_n\), \(Y_n \in Z\), \(n \in \mathbb{N}\), \(\lim_{n \to \infty} \theta(Y_n) = 0\) and \(a_n \in Y_n\), \(n \in \mathbb{N}\). Let \(B_n = \{a_n, a_{n+1}, \ldots\}\). Obviously \(B_{n+1} \subset B_n\) for every
\[ n \in N. \text{ Since } a_{n+p} \in Y_{n+p} \subset Y_n \text{ for } p \geq 1 \text{ we have } B_n \subset Y_n, n \in N \text{ and since } Y_0 \in P_{b,cl}(X) \text{ we obtain } B_n \in P_0(X) \text{ for every } n \in N. \text{ From (iii) we have}
\]
\[
\theta(B_0) = \theta(B_1) = \cdots = \theta(B_n).
\]

From (ii) we obtain \( 0 \leq \theta(B_n) \leq \theta(Y_n) \) and since \( \lim_{n \to \infty} \theta(Y_n) = 0 \), we have \( \lim_{n \to \infty} \theta(B_n) = 0 \). This relation and (1) implies \( \theta(B_n) = 0 \). Using (i) we obtain \( B_0 \in P_{cp}(X) \). Let \( a^* \in B_0 \). Then, there exists \( a_{n_i} \in B_0, a_{n_i} \to a^* \) as \( n_i \to \infty \). We have \( a_{n_i} \in Y_{n_i} \) and \( Y_{n_i} \in P_{cl}(X) \). This implies \( a^* \in Y_{n_i} \) for \( n_i \geq n^* \), \( n^* \in N \). But \( Y_{n_i} \subset Y_{n^*} \) for \( n_i \geq n^* \). It follows that \( a^* \in Y_n \) for every \( n \in N \), so \( a^* \in Y_\infty \) and \( Y_\infty \) is nonempty. From \( Y_\infty \subset Y_n \), we obtain \( \theta \leq \theta(Y_\infty) \leq \theta(Y_n) \). Since \( \lim_{n \to \infty} \theta(Y_n) = 0 \) we have \( \theta(Y_\infty) = 0 \). Now \( Y_n \in P_{b,cl}(X) \) for every \( n \in N \), implies \( Y_\infty = \bigcap_{n \in N} Y_n \in P_{b,cl}(X) \), so \( \alpha_{DP} \) has the intersection property.

The next lemma gives us an other example of mapping with the intersection property.

**Lemma 1.2.** Let \( X \) be a Banach space, \( Y \subset P(X), Y \) is nonempty and \( \theta_1, \theta_2 : Y \to R^+_+, \varphi : R^+_+ \to R^+_+ \). Let \( \theta : Y \to R^+_+ \) be given by \( \theta(A) = \varphi(\theta_1(A), \theta_2(A)) \) for \( A \in Y \). Suppose that:

(i) \( \varphi \in C(R^+_+, R^+_+) \);

(ii) \( Z_{\varphi(x,x)} = \{0\} \) for every \( x \in R^+_+ \);

(iii) \( \theta_1 \) and \( \theta_2 \) are increasing mappings and \( \theta_1 \) has the intersection property.

Here \( Z_{\varphi(x,x)} = \{y \in R^+_+ \mid \varphi(x,y) = 0\} \).

Then \( \theta = \varphi(\theta_1, \theta_2) \) has the intersection property.

**Proof.** Let \( A_n \in Y, A_{n+1} \subset A_n \) for \( n \in N, \theta(A_n) \to 0, n \to \infty \) \( (*) \). Since \( \theta_1 \) and \( \theta_2 \) are increasing, we have \( 0 \leq \theta_i(A_{n+1}) \leq \theta_i(A_n) \leq \theta_i(A_0), i = 1,2 \). This implies that the sequences \( (\theta_1(A_n))_n \) and \( (\theta_2(A_n))_n \) are decreasing and bounded. Then, there exists \( \lim_{n \to \infty} \theta_1(A_n) = a_1 \) and \( \lim_{n \to \infty} \theta_2(A_n) = a_2 \). Now, using \( (*) \), we have \( \lim_{n \to \infty} \varphi(\theta_1(A_n), \theta_2(A_n)) = 0 \) and since \( \varphi \) is continuous, we obtain \( \varphi(a_1, a_2) = 0 \). This together with (ii) implies \( a_1 = 0 \), so \( \theta_1(A_n) \to 0, n \to \infty \). Since \( \theta_1 \) has the intersection property it follows that \( A_\infty := \bigcap_{n \in N} A_n \) is nonempty, \( A_\infty \subset Y \) and \( \theta_1(A_\infty) = 0 \). Then, from (ii) we derive \( \theta(A_\infty) = \varphi(\theta_1(A_\infty), \theta_2(A_\infty)) = \varphi(0, \theta_2(A_\infty)) = 0 \). Thus the proof is complete.

Let \( X \) be a Banach space and \( (X, S(X), M) \) be the fixed point structure (see [5]), where \( S(X) = P_{b,cl,cv}(X) \) and \( M(Z) = \{f : Z \to Z \mid f \text{ is continuous}\} \) for each \( Z \in S(X) \). Consider \( Y = S(X) \). Let \( \eta : P(X) \to P(X), \eta(A) = \overline{A} \) for every \( A \in P(X) \). Obviously \( \eta(Y) = Y \). We consider a mapping \( \theta : Y \to R^+_+ \) as in the previous lemma. We have:

(1) \( \eta \) is a closure operator, \( S(X) = \eta(Y) = Y, \theta(\eta(A)) = \theta(A) \) for \( A \in Y \) since \( \eta(A) = A \) for \( A \in Y \).

(2) \( F_\eta \cap Z_\theta \subset S(X) \) since \( A \in F_\eta \cap Z_\theta \) implies \( A \in F_\eta \Rightarrow \eta(A) = A \Rightarrow \overline{A} = A \Rightarrow A \in S(X) \).
From (1) and (2) it follows that \((\theta, \eta)\) is a pair compatible with the fixed point structure. Let \(Z \in \eta(Y)\) and \(f \in M(Z)\). According to Lemma 1.2, we have:

(i') \(\theta/\eta(Y)\) has the intersection property.

In addition suppose:

(ii') \(f\) is \((\theta, a)\)-contraction.

**Theorem 1.3.** [4] Suppose (i'), (ii') and above fixed point structures. Then:

a) \(I(f) \cap S(X)\) is nonempty;

b) \(F_f\) is nonempty;

c) if \(F_f \in Y\), then \(\theta(F_f) = 0\).

Here \(I_f(G) = \{G \subset Z | f(G) \subset G\}\).

2 A measure of noncompactness on separable Banach space

Let \(E\) be a Banach space, \(P_{b,c}(E) = \{Y \subset E | Y\) is nonempty, bounded and countable\},

\(d(\cdot, \cdot) : E \times E \to \mathbb{R}_+\) be the metric induced by the norm of \(E\), and \(d(x, M) = \inf_{y \in M} d(x, y)\), \(M \subset E\).

**Definition 2.1.** We say that a mapping \(\beta : P_{b,c}(E) \to \mathbb{R}_+\) is an abstract measure

of noncompactness on \(E\) if:

\((C_1)\) \(A \subset B\) implies \(\beta(A) \leq \beta(B)\), for every \(A, B \in P_{b,c}(E)\);

\((C_2)\) \(\beta(A) = \beta(\overline{A})\) for every \(A \in P_{b,c}(E)\);

\((C_3)\) \(\beta(\text{co}A) = \beta(A)\) for every \(A \in P_{b,c}(E)\);

\((C_4)\) \(\beta(\{b\} \cup A) = \beta(A)\) for every \(b \in E\) and \(A \in P_{b,c}(E)\);

\((C_5)\) \(\beta(A) = 0\) if and only if \(A\) is relatively compact.

**Theorem 2.1.** Let \(E\) be a separable Banach space. Assume \(E = \bigcup_{n \in \mathbb{N}} E_n\), where \((E_n)_{n \in \mathbb{N}}\) is an increasing sequence of finite dimensional subspaces of \(E\). For any bounded countable set \(A = \{x_m | m \in \mathbb{N}\} \subset E\) let

\[\beta(A) = \lim_{n \to \infty} \lim_{m \to \infty} d(x_m, E_n)\).

Then \(\beta\) is a measure of noncompactness on \(E\) in the sense of Definition 2.1.

**Proof.** (\(C_1)\) Let \(A, B \in P_{b,c}(E)\). From the definition of upper limit, we have

\[\lim_{m \to \infty} d(x_m, E_n) \leq \lim_{m \to \infty} (y_m, E_n)\]

for \(x_m \in A\) and \(y_m \in B\). This implies that

\[\lim_{n \to \infty} \lim_{m \to \infty} d(x_m, E_n) \leq \lim_{n \to \infty} \lim_{m \to \infty} d(y_m, E_n)\]

for \(x_m \in A\) and \(y_m \in B\). It follows that \(\beta(A) \leq \beta(B)\).

\((C_2)\) We have \(A \subset \overline{A}\). Then, by \((C_1)\),

\[\beta(A) \leq \beta(\overline{A})\]
We have
\[ \mathcal{T} = \{ x_m \mid \exists y^m_p \in A, \exists p_0 \in N, \forall p \geq p_0, \forall \varepsilon > 0, \quad d(y^m_p, x_m) < \varepsilon \} \quad (*) \]

Let \( \varepsilon > 0 \) and \( p \geq m \). We have
\[ \beta(\mathcal{T}) = \lim_{n \to \infty} \lim_{m \to \infty} d(x_m, E_n) \leq \lim_{n \to \infty} \lim_{m \to \infty} d(x_m, y^m_p) + \lim_{n \to \infty} \lim_{m \to \infty} d(y^m_p, E_n). \]

But \( p \geq m \) and \( y^m_p \in A \). This implies (from \( (*) \))
\[ \lim_{n \to \infty} \lim_{m \to \infty} d(x_m, y^m_p) \leq \varepsilon \]
and
\[ \lim_{n \to \infty} \lim_{m \to \infty} d(y^m_p, E_n) = \beta(A). \]

That is \( \beta(\mathcal{T}) \leq \beta(A) + \varepsilon \), for every \( \varepsilon > 0 \). Letting \( \varepsilon \to 0 \), we obtain \( \beta(\mathcal{T}) \leq \beta(A) \) and from (1), one has \( \beta(A) = \beta(\mathcal{T}) \).

(C3) Let \( (coA)_Q = \{(1 - a_q)a_m + a_q b_m \mid a_m, b_m \in A, \ b_m \in A, \ a_q \in [0, 1] \cap Q \} \). Obviously, \( A \subset (coA)_Q \) and from (C1),
\[ \beta(A) \leq \beta((coA)_Q). \quad (2) \]

Now
\[ \beta((coA)_Q) = \lim_{n \to \infty} \lim_{m \to \infty} d((1 - a_q)a_m + a_q b_m, E_n) \leq \lim_{n \to \infty} \lim_{m \to \infty} d((1 - a_q)a_m, E_n) + \lim_{n \to \infty} \lim_{m \to \infty} d(a_q b_m, E_n) = \lim_{n \to \infty} \lim_{m \to \infty} d((1 - a_q)a_m, (1 - a_q)E_n) + \lim_{n \to \infty} \lim_{m \to \infty} d(a_q b_m, a_q E_n) = (1 - a_q) \lim_{n \to \infty} \lim_{m \to \infty} d(a_m, E_n) + a_q \lim_{n \to \infty} \lim_{m \to \infty} d(b_m, E_n) = (1 - a_q)\beta(A) + a_q \beta(A) = \beta(A). \]

This and (2) implies
\[ \beta(A) = \beta((coA)_Q) \quad (3) \]

Since \( \mathcal{Q} = R \), we have \( \overline{(coA)_Q} = coA \). From (3) and (C2) we obtain
\[ \beta(coA) = \beta((coA)_Q) = \beta((coA)_Q) = \beta(A). \]

That is (C3).

(C4) We have \( b \in E \) and \( E = \bigcup_{n \in N} E_n \). Then there exists \( (y_p)_{p \in N} \in \bigcup_{n \in N} E_n \) with \( d(y_p, b) < \varepsilon \), for every \( p \geq p_0 \) and every \( \varepsilon > 0 \). Since \( E_n \subset E_{n+1} \) for any \( n \in N \), we obtain that there exists \( n_0 \in N \) such that for every \( n \geq n_0 \) there is an \( y_p \in E_n \) with \( d(y_p, b) < \varepsilon \), for \( p \geq p_0 \) (**).
According to (**) we have
\[ \beta (\{ b \}) = \lim_{n \to \infty} \lim_{m \to \infty} d(b, E_n) = \lim_{n \to \infty} d(b, E_n) \leq \]
\[ \leq \lim_{n \to \infty} d(b, y_p) + \lim_{n \to \infty} d(y_p, E_n) \leq \epsilon + 0 = \epsilon, \]
for every \( \epsilon > 0 \). That is \( \beta (\{ b \}) = 0 \). It follows (\( C_4 \)).

(\( C_5 \)) Let be \( \beta (A) = 0 \). We have
\[ \lim_{n \to \infty} \lim_{m \to \infty} d(x_m, E_n) = 0. \]
Then there exists \( n_0 \in N \) such that for every \( n \geq n_0 \) and \( \epsilon > 0 \),
\[ \lim_{m \to \infty} d(x_m, E_n) < \epsilon. \]
It follows that for every \( n \geq n_0, \epsilon > 0 \) and \( x_m \in A \), there exists \( y_m \in E_n \) with
\[ d(x_m, y_m) < \epsilon. \]
Therefore, for every \( n \geq n_0 \), \( \{ y_m \} \subset E_n \) is an \( \epsilon \)-net for \( A \) (4).

Let \( z_n \in E_n \) be fixed. We have
\[ d(z_n, y_m) \leq d(y_m, x_m) + d(x_m, z_n) < \epsilon + d(x_m, z_n), \quad x_m \in A, \]
\( A \) is bounded. These imply that\( d(x_m, z_n) \leq c_n \). We obtain \( d(z_n, y_m) \leq \epsilon + c_n \).

Since \( z_n \) is fixed we have that \( \{ y_m \} \) is bounded. Then \( \{ y_m \} \) is relatively compact and, from (4) and Hausdorff’s theorem (see [3]), we have that \( A \) is relatively compact.

Now let \( A \) be a relatively compact set. From Hausdorff’s theorem, for every \( \epsilon > 0 \), there exists a finite net \( R_\epsilon \) for \( A \). This implies that for every \( x_m \in A \), there exists \( y_m \in R_\epsilon \), with \( d(x_m, y_m) < \epsilon \) for every \( \epsilon > 0 \). We have
\[ \beta (A) = \lim_{n \to \infty} \lim_{m \to \infty} d(x_m, E_n) \leq \]
\[ \leq \lim_{n \to \infty} \lim_{m \to \infty} d(x_m, y_m) + \lim_{n \to \infty} \lim_{m \to \infty} d(y_m, E_n) \leq \epsilon + \beta (R_\epsilon). \]
But \( R_\epsilon \) is a finite net and from (\( C_4 \)) we have \( \beta (R_\epsilon) = 0 \). Then \( \beta (A) \leq \epsilon \) for every \( \epsilon > 0 \). Letting \( \epsilon \to 0 \), we have \( \beta (A) = 0 \). The theorem is completely demonstrated.

**Remarks.** 1) The above definition of \( \beta \) has the advantage that it is given in terms of countable bounded sets. Such sets occur in fixed point theorems of Mönch type (see [2]). Recall that originally the Hausdorff measure of noncompactness is defined for all bounded sets of a Banach space.

2) We can replace (\( C_5 \)) with

(\( C'_5 \)) \( \beta (A) = 0 \) implies \( A \) is relatively compact. Notice (\( C'_5 \)) is more general than (\( C_5 \)).

3) The conditions (\( C_1 \)), (\( C_4 \)), (\( C'_5 \)) and Theorem 1.1 shows that \( \beta : P_{b,c}(E) \to R_+ \),
\[ \beta (A) = \lim_{n \to \infty} \lim_{m \to \infty} d(x_m, E_n) \]
where \( x_m \in A, E = \bigcup_{n \in N} E_n, E_n \subset E_{n+1}, n \in N, (E_n)_{n \in N} \) is an increasing sequence of finite dimensional Banach subspaces, is a mapping with the intersection property.
Theorem 2.2. Let $E$ be a Banach space, $E = \bigcup_{n \in \mathbb{N}} E_n$, where $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of finite dimensional subspaces of $E$. Let $f : E \to E$ be such that there exists $L > 0$ with $d(f(u), f(v)) \leq Ld(u, v)$, for every $u, v \in E$. Then $\beta(f(M)) \leq L\beta(M)$, for every $M \in P_{b,c}(E)$.

Proof. We have

$$\beta(f(M)) := \lim_{n \to \infty} \lim_{m \to \infty} d(f(x_m), E_n)$$

Since $E_n \subset E_{n+1}$ for $n \in \mathbb{N}$ we have that there exists $x_0 \in M$ such that $x_0 \in E_n,

f(x_0) \in E_n$ for $n \geq n_0$, for some one $n_0 \in \mathbb{N}$. (4)

We have

$$d(f(x_m), E_n) \leq d(f(x_m), f(x_0)) + d(f(x_0), E_n) \leq Ld(x_m, x_0) + d(f(x_0), E_n) \leq L[d(x_m, E_n) + d(x_0, E_n)] + d(f(x_0), E_n)$$

(5)

From (4) we have

$$\lim_{n \to \infty} \lim_{m \to \infty} d(x_0, E_n) = \lim_{n \to \infty} d(x_0, E_n) = 0$$

and

$$\lim_{n \to \infty} \lim_{m \to \infty} d(f(x_0), E_n) = \lim_{n \to \infty} d(f(x_0), E_n) = 0$$

and for $n \to \infty$ in (5) we obtain

$$\lim_{n \to \infty} \lim_{m \to \infty} d(f(x_m), E_n) \leq \lim_{n \to \infty} \lim_{m \to \infty} d(x_m, E_n).$$

That is $\beta(f(M)) \leq L\beta(M)$.

References


